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The n-Hosoya Polynomials of the Square of a Path and of a Cycle

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ABSTRACT

The n-Hosoya polynomial of a connected graph G of order t is defined by: $H_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k$, Where, $C_n(G,k)$ is the number of pairs (v,S), in which $|S|=n-1$, $3\le n\le t$, $v\in V(G)$, $S\subseteq V(G)$, such that $d_n(v,S)=k$, for each $0\le k\le \delta_n = diam_n(G)$.

In this paper, we find the n-Hosoya polynomial of the square of a path and of the square of a cycle. Also, the n-diameter and n-Wiener index of each of the two graphs are determined.

Keyword: n-diameter, n-Hosoya polynomial, n-Wiener index, path square and cycle square.

متعددات حدود هوسويا−n لمربعي الدرب والدارة

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الملخص

تعرف متعددة حدود هوسويا−n لبيان متصل G من الرتبة t على أنها: $H_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k$، حيث أن $C_n(G,k)$ يمثل عدد الأزواج (v,S) والتي لكل منها المسافة الصغرى بين الرأس v ($v\in V(G)$) والمجموعة S ($|S|=n-1$، $S\subseteq V$) هي k، أي أن ($d_n(v,S)=k$)، لكل $3\le n\le t$ و $0\le k\le \delta_n = diam_n(G)$.

في هذا البحث تم إيجاد متعددة حدود هوسويا-n لكل من مربع الدرب ومربع الدارة، وكذلك تم تحديد القطر−n ودليل وينر−n لكل منهما.

الكلمات المفتاحية: القطر −n ، متعددات حدود هوسويا − n، دليل وينر − n ،مربع الدرب ، مربع الدارة.

1. Introduction:

The **n-distance** [1] in a connected graph $G=(V,E)$ of order t is the minimum distance from a singleton , $v\in V$ to an (n-1)-subset S , $S\subseteq V$, $3\le n\le t$, that is ,
$d_n(v,S)=\min\{d(v,u):\ u\in S\}$, $3\le n\le t$.
It is clear that
$d_n(v,S)=0$; when $v\in S$,
$d_n(v,S)\ge 1$; when $v\notin S$.

The <u>n-Wiener index</u> of a connected graph $G=(V,E)$ is the sum of the minimum distances of all pairs (v,S) in the graph G , that is:

$$W_n(G)=\sum_{\substack{(v,S),|S|=n-1\\ v\in V,S\subseteq V}} d_n(v,S)\ ,\ 3\le n\le t.$$

The <u>n-diameter of G</u> is defined by:

$$\mathrm{diam}_n G=\max\{d_n(v,S):v\in V(G),|S|=n-1,S\subseteq V(G)\}.$$

Now, let $C_n(G,k)$ be the number of pairs (v,S) , $|S|=n-1$, $3\le n\le t$, $v\in V$, $S\subseteq V$, such that $d_n(v,S)=k$, for each $0\le k\le \delta_n=\mathrm{diam}_n(G)$, then the <u>n-Hosoya polynomial of G</u> is defined by:

$$H_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k.$$

We can obtain the n-Wiener index of G from the n-Hosoya polynomial of G as follows:

$$W_n(G)=\frac{d}{dx}H_n(G;x)\Big|_{x=1}=\sum_{k=1}^{\delta_n} k\,C_n(G,k).$$

For a vertex v of a connected graph G, let $C_n(v,G,k)$ be <u>the number of (n-1)-subsets S</u> of vertices of G such that $d_n(v,S)=k$, for $n\ge 3$, $0\le k\le \delta_n$. The <u>n-Hosoya polynomial of the vertex v</u>, denoted by $H_n(v,G;x)$, is defined as:

$$H_n(v,G;x)=\sum_{k\ge 0} C_n(v,G,k)x^k.$$

It is clear that for all $k\ge 0$,

$$\sum_{v\in V(G)} C_n(v,G,k)=C_n(G,k),$$

and

$$\sum_{v\in V(G)} H_n(v,G;x)=H_n(G;x).$$

For more information about these concepts, see the References [1, 2, 5, 6].

The next lemma will be used in proving our results.

Lemma 1.1:[1] Let v be any vertex of a connected graph G. If there are r vertices of distance $k\ge 1$ from v, and there are s vertices of distance more than k from v, then, for $n\ge 3$,

$$C_n(v,G,k) = \binom{r+s}{n-1} - \binom{s}{n-1}. \quad ...(1.1)$$

Definition 1.2: Let G be a connected non-trivial graph . The square G^2 of the graph G , introduced by Harary and Ross [7], has $V(G^2) = V(G)$ with u, v adjacent in G^2, whenever $1 \le d_G(u,v) \le 2$.

Notice that the square of complete graph, star graph, wheel graph, complete bipartite graph are complete graphs.

In [1,2,3,4] , the n-Hosoya polynomials for many special graphs and many compound graphs are obtained . In this paper , we continue such works by obtaining the n-Hosoya polynomials of the square of paths and cycles.

2. The n-Hosoya Polynomial of the Square of a Path:

In this section , we obtained the n-Hosoya polynomial of the square P_t^2 of a path P_t of order t . We shall consider two main cases of P_t^2 according to the parity of t.

First Case : Even t , $t = 2r$, $r \ge 2$.

Let $P_t : u_1, u_2, u_3, \ldots, u_t$, then P_t^2 is shown in Fig.2.1, and by relabeling its vertices, we have Fig. 2.2 for P_{2r}^2 .

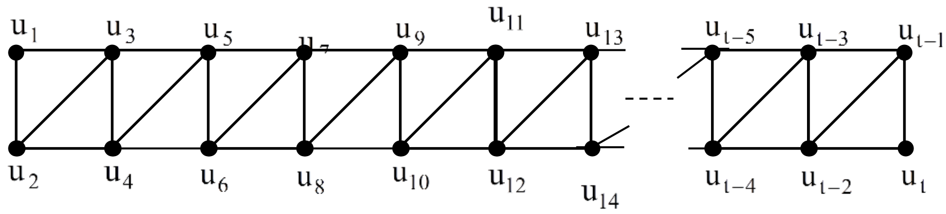

Fig. (2.1). The Path Square P_t^2

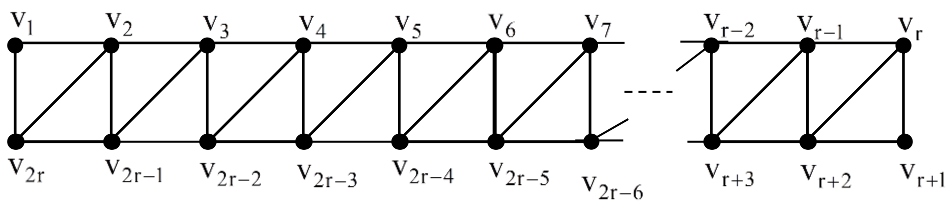

Fig. (2.2). The Path Square P_{2r}^2

Second Case : If **t is odd** , then there exists an integer r such that $t = 2r + 1$. The graph P_t^2 is shown in Fig.2-3.

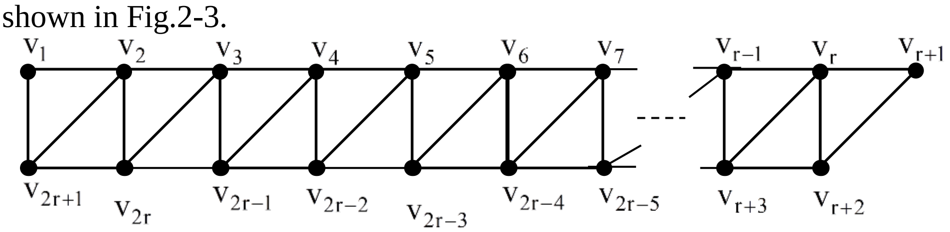

Fig. (2.3). The Path Square P_{2r+1}^2 .

Theorem 2.1: For $t \geq 5$ and $n \geq 2$, let $r = \left\lfloor \frac{t}{2} \right\rfloor$, then ,

$$\mathrm{diam}_n(P_t^2) = \begin{cases} r+1-\left\lceil \frac{n}{2} \right\rceil & , \text{ for even } t , \\ r+1-\left\lfloor \frac{n}{2} \right\rfloor & , \text{ for odd } t . \end{cases}$$

Proof:

(1). Let t be even , then $t = 2r$.

From Fig.2.2 , we notice that $\mathrm{diam}(P_{2r}^2) = d(v_1, v_{r+1}) = r$, then $\mathrm{diam}_n(P_{2r}^2) = d_n(v_1, S)$, $n \geq 2$, where S consists of the first $n-1$ vertices from the sequence $\{v_{r+1}; v_r, v_{r+2}; v_{r-1}, v_{r+3}; v_{r-2}, v_{r+4}; \ ... \ ; v_2, v_{2r}\}$.

Thus , if n is even , then

$S = \{v_{r+1}\}$, $n = 2$,

$S = \{v_{r+1}, v_r, v_{r+2}; v_{r-1}, v_{r+3}; v_{r-2}, v_{r+4}; \ ... \ ; v_{r+2-\frac{n}{2}}, v_{r+\frac{n}{2}}\}$, $n = 4,6,8, \ ... \ ,2r$.

So , $d_n(v_1, S) = r + 1 - \frac{n}{2}$.

If n is odd , then

$S = \{v_{r+1}, v_r; v_{r+2}, v_{r-1}; v_{r+3}; v_{r-2}; \ ... \ ; v_{r+\frac{n-1}{2}}, v_{r+1-\frac{n-1}{2}}\}$, $n = 3,5,7, \ ... \ ,2r-1$.

So , $d_n(v_1, S) = r + 1 - \frac{n+1}{2}$.

Therefore , $\mathrm{diam}_n(P_{2r}^2) = r + 1 - \left\lceil \frac{n}{2} \right\rceil$, for all $n \geq 2$.

(2). Let t be odd , then $t = 2r+1$.

From Fig.2.3 , we notice that $\mathrm{diam}(P_{2r+1}^2) = d(v_1, v_{r+1}) = r$,(or $d(v_1, v_{r+2})$, or $d(v_{2r+1}, v_{r+1})$), then $\mathrm{diam}_n(P_{2r+1}^2) = d_n(v_1, S)$, $|S| = n-1, n \geq 2$, where S consists of the first $n-1$ vertices from the sequence $\{v_{r+1}, v_{r+2}; v_r, v_{r+3}; v_{r-1}, v_{r+4}; ...; v_2, v_{2r+1}\}$.

Thus , if n is odd , then ,

$S = \{v_{r+1}, v_{r+2}; v_r, v_{r+3}; v_{r-1}, v_{r+4}; \ ... \ ; v_{r+2-\frac{n-1}{2}}, v_{r+1+\frac{n-1}{2}}\}$, $n = 3,5,7, \ ... \ ,2r+1$.

So , $d_n(v_1, S) = d(v_1, v_{r+2-\frac{n-1}{2}}) = r + 1 - \frac{n-1}{2}$.

If n is even , then ,

$S = \{v_{r+1}, v_{r+2}; v_r, v_{r+3}; v_{r-1}, v_{r+4}; \ ... \ ; v_{r+2-\frac{n}{2}}, v_{r+1+\frac{n}{2}}\}, n = 4,6,8, \ ... \ ,2r$.

So , $d_n(v_1, S) = d(v_1, v_{r+2-\frac{n}{2}}) = r + 1 - \frac{n}{2}$.

Therefore ,

$\mathrm{diam}_n(P_{2r+1}^2) = r + 1 - \left\lfloor \frac{n}{2} \right\rfloor$, for all $n \geq 2$. #

Remark : Throughout this work , we assume that $\binom{a}{b}=0$, if $a<b$.

Theorem 2.2: For any $n\ge 3$, the n-Hosoya polynomial of P_t^2 , $t\ge 6$, is given by:

$$H_n(P_t^2;x)=\sum_{k=0}^{\delta_n} C_n(P_t^2,k)x^k ,$$

Where , $\delta_n = \mathrm{diam}_n(P_t^2)$,

$$C_n(P_t^2,0)=t\binom{t-1}{n-2} , \qquad \ldots(2.2.1)$$

$$C_n(P_t^2,1)=t\binom{t-1}{n-1}-2\left[\binom{t-3}{n-1}+\binom{t-4}{n-1}\right]-(t-4)\binom{t-5}{n-1}, \qquad \ldots(2.2.2)$$

$$C_n(P_t^2,k)=2\left[\binom{t-2k}{n-1}+\binom{t-2k+1}{n-1}\right]+(t-4k+2)\binom{t-4k+3}{n-1}$$

$$-2\sum_{i=0}^{2}\binom{t-4k+i}{n-1}-(t-4k)\binom{t-4k-1}{n-1},\ 2\le k\le\left\lfloor\frac{\delta_n}{2}\right\rfloor, \qquad \ldots(2.2.3)$$

$$C_n(P_t^2,k)=2\left[\binom{t-2k}{n-1}+\binom{t-2k+1}{n-1}\right] ,\ \left\lfloor\frac{\delta_n}{2}\right\rfloor+1\le k\le\delta_n . \qquad \ldots(2.2.4)$$

Proof: It is clear that $C_n(P_t^2,0)=t\binom{t-1}{n-2}$.

From Fig.2.2, we notice that in P_t^2 , there are two vertices of degree 2, two vertices of degree 3, and $t-4$ vertices of degree 4. Thus, using formula (1.4.5) in [1], we obtain (2.2.2).

For each vertex w and given k , let

$S_1(w,k)=\{v\in V: d(w,v)=k\}$,

$S_2(w,k)=\{v\in V: d(w,v)>k\}$.

First , we shall prove (2.2.3) and (2.2.4) for **even t**, assuming $t=2r$, $r\ge 4$. It is clear, from Fig. 2.2, that for $n\ge 3$,

$$C_n(v_i,P_t^2,k)=C_n(v_{i+r},P_t^2,k), \qquad \ldots(2.2.5)$$

for $i=1,2,\ ...\ ,r$. Therefore, for $2\le k\le\delta_n$,

$$C_n(P_{2r}^2,k)=2\sum_{i=1}^{r} C_n(v_i,P_{2r}^2,k) . \qquad \ldots(2.2.6)$$

Now , let $2\le k\le\left\lfloor\frac{\delta_n}{2}\right\rfloor$, in which δ_n is determined by Theorem 2.1, that is

$\delta_n = r+1-\left\lceil\frac{n}{2}\right\rceil$.

Since, $n\ge 3$, then $\delta_n\le r-1$, for $r\ge 4$.

But, in proving (2.2.3), we assume that $\delta_n \geq 4$.

According to the given value of k, we partition $\{v_1, v_2, \ldots, v_r\}$ into the following **four cases**:

(1). For $i = 1, 2, \ldots, k$, we notice, from Fig. 2.2, that:

$S_1(v_i, k) = \{v_{i+k}, v_{2r+2-i-k}\}$,

$S_2(v_i, k) = V(P_{2r}^2) - \{v_1, v_2, \ldots, v_{i+k}, v_{2r+2-i-k}, v_{2r+3-i-k}, \ldots, v_{2r}\}$.

Thus,

$|S_1(v_i, k)| = 2$, $|S_2(v_i, k)| = t + 1 - 2k - 2i$.

So, by Lemma 1.1, we have , for $i = 1, 2, \ldots, k$,

$$C_n(v_i, P_t^2, k) = \binom{t+3-2k-2i}{n-1} - \binom{t+1-2k-2i}{n-1}. \qquad \ldots\text{(c1)}$$

(2). For $i = 1, 2, \ldots, k-1$, we obtain, from Fig. 2.2,

$S_1(v_{r+1-i}, k) = \{v_{r-k-i+1}, v_{r+k+i}\}$,

$S_2(v_{r+1-i}, k) = V(P_t^2) - \{v_{r-k-i+1}, v_{r-k-i+2}, \ldots, v_r, v_{r+1}, \ldots, v_{r+k+i}\}$.

Thus,

$|S_1(v_{r+1-i}, k)| = 2$, $|S_2(v_{r+1-i}, k)| = t - 2k - 2i$.

So, using Lemma 1.1, we obtain , for $i = 1, 2, \ldots, k-1$,

$$C_n(v_{r+1-i}, P_t^2, k) = \binom{t+2-2k-2i}{n-1} - \binom{t-2k-2i}{n-1}. \qquad \ldots\text{(c2)}$$

(3). For v_{r-k+1}, we have

$S_1(v_{r-k+1}, k) = \{v_{r+1}, v_{2k+r},, v_{r+1-2k}\}$,

$S_2(v_{r-k+1}, k) = V(P_t^2) - \{v_{r-2k+1}, v_{r-2k+2}, \ldots, v_r, v_{r+1}, \ldots, v_{r+2k}\}$.

Thus,

$|S_1(v_{r-k+1}, k)| = 3$, $|S_2(v_{r-k+1}, k)| = t - 4k$.

So , using Lemma 1.1 , we get ,

$$C_n(v_{r-k+1}, P_t^2, k) = \binom{t+3-4k}{n-1} - \binom{t-4k}{n-1}. \qquad \ldots\text{(c3)}$$

(4). For $i = k+1, k+2, \ldots, r-k$,

$S_1(v_i, k) = \{v_{i-k}, v_{i+k}, v_{2r+k-i+1}, v_{2r-k-i+2}\}$,

$S_2(v_i, k) = V(P_t^2) - \{v_{i-k}, v_{i-k+1}, \ldots, v_{i+k}, v_{2r-k-i+2}, v_{2r-k-i+3}, \ldots, v_{2r+k-i+1}\}$.

Thus,

$|S_1(v_i, k)| = 4$, $|S_2(v_i, k)| = t - 4k - 1$.

Therefore, using Lemma 1.1, we get , for $i = k+1, k+2, \ldots, r-k$,

$$C_n(v_i, P_t^2, k) = \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1}. \qquad \ldots\text{(c4)}$$

Thus, from (2.2.6) and summing up the formulas (c1)-(c4) we get for $2 \leq k \leq \left\lfloor \frac{\delta_n}{2} \right\rfloor$,

$$C_n(P_t^2,k)=2\left\{\sum_{i=1}^{k}\left[\binom{t+3-2k-2i}{n-1}-\binom{t+1-2k-2i}{n-1}\right]\right.$$
$$+\sum_{i=1}^{k-1}\left[\binom{t+2-2k-2i}{n-1}-\binom{t-2k-2i}{n-1}\right]$$
$$\left.+\binom{t-4k+3}{n-1}-\binom{t-4k}{n-1}+(r-2k)\left[\binom{t-4k+3}{n-1}-\binom{t-4k-1}{n-1}\right]\right\}.$$

$$=2\left\{\left[\binom{t-2k+1}{n-1}-\binom{t-4k+1}{n-1}\right]+\left[\binom{t-2k}{n-1}-\binom{t-4k+2}{n-1}\right]\right.$$
$$\left.+\binom{t-4k+3}{n-1}-\binom{t-4k}{n-1}+(r-2k)\left[\binom{t-4k+3}{n-1}-\binom{t-4k-1}{n-1}\right]\right\}.$$
$$=2\left[\binom{t-2k}{n-1}+\binom{t-2k+1}{n-1}\right]+(t-4k+2)\binom{t-4k+3}{n-1}$$
$$-2\sum_{j=0}^{2}\binom{t-4k+j}{n-1}-(t-4k)\binom{t-4k-1}{n-1}.$$

Now, we give the proof of (2.2.4) for $\left\lfloor\frac{\delta_n}{2}\right\rfloor+1\le k\le\delta_n$. Here, we have **t<u>wo cases</u>**:

(a). For $i=1,2,\ \dots\ ,r-k$,

$S_1(v_i,k)=\{v_{i+k},v_{2r+2-i-k}\}$,

$S_2(v_i,k)=V(P_t^2)-\{v_1,v_2,\ \dots\ ,v_{i+k},v_{2r+2-i-k},v_{2r+3-i-k},\ \dots\ ,v_{2r}\}$.

Thus,

$|S_1(v_i,k)|=2$, $|S_2(v_i,k)|=t+1-2k-2i$.

So, by Lemma 1.1, we have, for $i=1,2,\ \dots\ ,r-k$,

$$C_n(v_i,P_t^2,k)=\binom{t-2k-2i+3}{n-1}-\binom{t-2k-2i+1}{n-1}. \qquad \dots(d1)$$

(b). For v_{r+1-i} , $i=1,2,\ \dots\ ,r-k$, we have

$S_1(v_{r+1-i},k)=\{v_{r-k-i+1},v_{r+k+i}\}$,

$S_2(v_{r+1-i},k)=V(P_t^2)-\{v_{r-k-i+1},v_{r-k-i+2},\ \dots\ ,v_r,v_{r+1},\ \dots\ ,v_{r+k+i}\}$.

Thus,

$|S_1(v_{r+1-i},k)|=2$, $|S_2(v_{r+1-i},k)|=t-2k-2i$.

So, by Lemma 1.1, we have , for $i=1,2,\ \dots\ ,r-k$,

$$C_n(v_{r+1-i},P_t^2,k)=\binom{t-2k-2i+2}{n-1}-\binom{t-2k-2i}{n-1}. \qquad \dots(d2)$$

Therefore, using (2.2.6) and summing up (d1) and (d2), we get for $\left\lfloor\frac{\delta_n}{2}\right\rfloor+1\le k\le\delta_n$,

$$C_n(P_t^2,k)=2\left\{\sum_{i=1}^{r-k}\left[\binom{t-2k-2i+3}{n-1}-\binom{t-2k-2i+1}{n-1}\right]\right.$$

$$\left.+\sum_{i=1}^{r-k}\left[\binom{t-2k-2i+2}{n-1}-\binom{t-2k-2i}{n-1}\right]\right\}$$

$$=2\left\{\left[\binom{t-2k+1}{n-1}-\binom{t-2r+1}{n-1}\right]+\left[\binom{t-2k}{n-1}-\binom{t-2r}{n-1}\right]\right\}$$

$$=2\left[\binom{t-2k+1}{n-1}+\binom{t-2k}{n-1}\right],\text{ because } n\ge 3.$$

<u>Second</u>, the proofs of (2.2.3) and (2.2.4) for **odd t**, $t=2r+1$, $r\ge 3$, are similar to the proofs of (2.2.3) and (2.2.4) for even t.
Hence, the proof of the Theorem is completed. #

Corollary 2.3: The n-Wiener index of P_t^2 is given by:

$$W_n(P_t^2)=t\binom{t-1}{n-1}-2\left[\binom{t-3}{n-1}+\binom{t-4}{n-1}\right]-(t-4)\binom{t-5}{n-1}+\sum_{k=2}^{\delta_n}kC_n(P_t^2,k)\ ,$$

in which

$$C_n(P_t^2,k)=2\left[\binom{t-2k}{n-1}+\binom{t-2k+1}{n-1}\right]+(t-4m+2)\binom{t-4k+3}{n-1}$$
$$-2\sum_{i=0}^{2}\binom{t-4k+i}{n-1}-(t-4k)\binom{t-4k-1}{n-1},\ 2\le k\le\left\lfloor\frac{\delta_n}{2}\right\rfloor,$$

$$C_n(P_t^2,k)=2\left[\binom{t-2k}{n-1}+\binom{t-2k+1}{n-1}\right]\ ,\ \left\lfloor\frac{\delta_n}{2}\right\rfloor+1\le k\le\delta_n\ .\quad \#$$

3. The n-Hosoya Polynomial of the Square of a Cycle :

There are many classes of connected graphs G in which for each k, $1\le k\le\delta_n$, $C_n(v,G,k)$ is the same for every vertex $v\in V(G)$; such graphs are called [2] **vertex-n-distance regular graphs**, and for the given value of n, $2\le n\le t$, $H_n(G;x)=t\,H_n(v,G;x)$, where v is any vertex of G and t is the order of G.
The graph C_t^2 is the square of a cycle of order t, shown in Fig. 3.1. We shall find the n-diameter, n-Hosoya polynomial, and n-Wiener index of C_t^2.

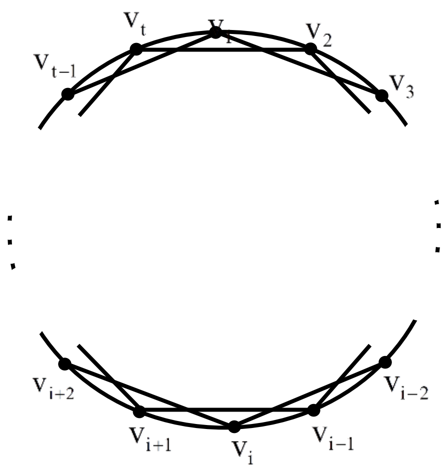

Fig. (3.1) The Cycle Square C_t^2 , $t \geq 6$.

Lemma 3.1: $\mathrm{diam}_n(C_t^2) = \delta_n = 1 + \left\lfloor \frac{t-n}{4} \right\rfloor$, $n \geq 2$, $t \geq 6$.

Proof: Let $m = \left\lfloor \frac{t}{4} \right\rfloor$, then $t = 4m + r$, $r = 0,1,2,3$.

For $r = 2$, C_t^2 is redrawn in Fig. 3.2.

Since, C_t^2 is vertex n-distance regular graph , then $\mathrm{diam}_n(C_t^2) = e_n(v_1)$.

To find the n-eccentricity of v_1, we partition $V(C_t^2) - \{v_1\}$ into $S_1, S_2, \ldots, S_{m+1}$, where

$S_1 = \{v_2, v_3, v_t, v_{t-1}\}$,

$S_2 = \{v_4, v_5, v_{t-2}, v_{t-3}\}$,

$S_3 = \{v_6, v_7, v_{t-4}, v_{t-5}\}$,

.

.

.

$S_j = \{v_{2j}, v_{2j+1}, v_{t-2(j-1)}, v_{t-2j+1}\}$,

.

.

.

$S_m = \{v_{2m}, v_{2m+1}, v_{t-2m+2}, v_{t-2m+1}\}$,

$S_{m+1} = V(C_t^2) - (\bigcup_{j=1}^{m} S_j \bigcup \{v_1\})$.

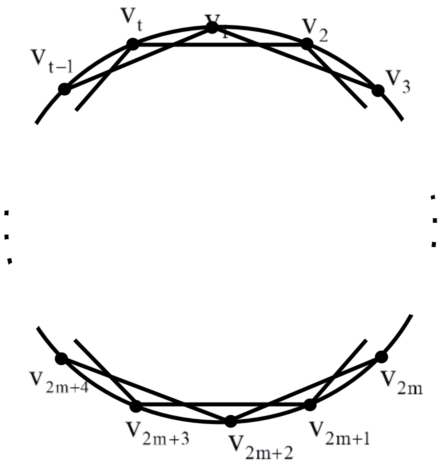

Fig. (3.2). The Cycle Square C_t^2 , $t = 4m + 2$, $m \ge 1$.

It is clear that each vertex of S_j , $1 \le j \le m$, is of (standard) distance j from v_1; and each of the other vertices (if exists) of C_t^2 (here in Fig. 3.2, we have $\{v_{\frac{t}{2}+1}\} = \{v_{2m+2}\} = S_{m+1}$) is of the distance $m+1$ from v_1. Notice that if $t = 4m+1$, then S_{m+1} is empty , and if $t = 4m$ then, S_{m+1} is empty and S_m consists of three elements; if $t = 4m+2$, $t = 4m+3$, then S_{m+1} consists of one, respectively two, elements.

Let k be the greatest positive integer such that the set $\bigcup_{i=k}^{m+1} S_i$ consists of at least (n-1) vertices. Therefore, since $|S_i| \le 4$.

$4(k-1)+1+(n-1) \le t$,

$4k \le t - n + 4$,

$k \le \frac{t-n}{4} + 1$.

Therefore, $\text{diam}_n(C_t^2) = k = 1 + \left\lfloor \frac{t-n}{4} \right\rfloor$, ($\because k$ is positive integer). #

Theorem 3.2: For any $n \ge 3$, the n-Hosoya polynomial of C_t^2 , $t \ge 6$ is given by:

$$H_n(C_t^2; x) = t\binom{t-1}{n-2} + \sum_{k=1}^{\delta_n - 1} t\left[\binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1}\right]x^k + C_n(C_t^2, \delta_n)x^{\delta_n},$$

Where , $C_n(C_t^2, \delta_n)$ is determined in Remark 3.3, and δ_n is determined by Lemma 3.1.

Proof: Let S be a set of (n-1) vertices of $V(C_t^2)$ such that $v_1 \notin S$, $v_1 \in V(C_t^2)$ and $d_n(v_1, S) = k$, $2 \le k \le \delta_n - 1$. Hence , S does not contain any vertex from $\{v_{t-2k+3}, \ldots, v_{t-1}, v_t, v_1, v_2, v_3, \ldots, v_{2k-1}\}$, (see Fig. 3.1) , but S must contain , at least , one vertex of $\{v_{2k}, v_{2k+1}, v_{t-2k+2}, v_{t-2k+1}\}$. Then, the number of vertices in C_t^2 of distance more than k from v_1 is $(t-4k-1)$ and there are four vertices in C_t^2 of distance k from v_1. Hence, by Lemma 1.1,

$$C_n(v_1, C_t^2, k) = \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1}, \text{ for } 2 \le k \le \delta_n - 1.$$

Moreover, it is clear that

$$C_n(v_1, C_t^2, 1) == \binom{t-1}{n-1} - \binom{t-5}{n-1}.$$

Since $C_n(v_1, C_t^2, k) = C_n(v_i, C_t^2, k)$, $2 \le i \le t$, then

$$C_n(C_t^2, k) = t\left[\binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1}\right], \text{ for } 1 \le k \le \delta_n - 1. \quad \#$$

Remark 3.3: From Fig. 3-2 , we can easily obtain $C_n(C_t^2, \delta_n)$, for $n \ge 3$.

1. If $t = 4m+3$, then,

$$C_n(C_t^2, \delta_n) = \begin{cases} t & ; \ n = 3 \\ t\left[\binom{t-4\delta_n+3}{n-1} - \binom{t-4\delta_n-1}{n-1}\right] & ; \ n \ge 4. \end{cases}$$

2. If $t = 4m+2$, $4m+1$, then,

$$C_n(C_t^2, \delta_n) = t\left[\binom{t-4\delta_n+3}{n-1} - \binom{t-4\delta_n-1}{n-1}\right] ; \ n \ge 3 .$$

3. If $t = 4m$, then,

$$C_n(C_t^2, \delta_n) = \begin{cases} t\binom{3}{n-1} & ; \ n = 3 , \ 4 \\ t\left[\binom{t-4\delta_n+3}{n-1} - \binom{t-4\delta_n-1}{n-1}\right] & ; \ n \ge 5. \end{cases}$$

Corollary 3.4: The n-Wiener index of C_t^2 is given by:

$W_n(C_t^2) = \sum_{k=1}^{\delta_n} k\, C_n(C_t^2, k)$, where $C_n(C_t^2, k)$, $1 \le k \le \delta_n$ is given in Theorem 3.2 and Remark 3.3 . #

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