Rings in which Every Simple Right R-Module is Flat

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ABSTRACT

The objective of this paper is to initiate the study of rings in which each simple right R-module is flat, such rings will be called right SF-rings. Some important properties of right SF-rings are obtained. Among other results we prove that: If R is a semi prime ERT right SF-ring with zero socle, then R is a strongly regular ring.

Keywords: strongly regular rings, SF-ring, reduced rings.

الحلقات التي فيها كل مقاس بسيط ايمن مسطح

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الملخص

يهدف هذا البحث الى دراسة الحلقات التي يكون فيها كل مقياس ايمن مسطحاً وتسمى هذه الحلقات بالحلقات منت النمط SF.وحصلنا على بعض الصفات المهمة للحلقات من النمط SF.ومن النتائج الأخرى التي حصلنا عليها "اذا كانتR شبه أولية ومن النمط ERT, SFوان SOR=0 فأن R حلقة منتظمة بقوة ".

الكلمات المفتاحية: الحلقات المنتظمة بقوة، الحلقات من النمط SF، حلقات مختزلة.

1. Introduction:

Throughout this paper, R denotes associative ring with identity and all modules are unitary. J(R) and Y(R) denote the Jacobson radical and the singular right ideal of R, respectively. For any nonempty subset X of a ring R, the right (resp. left) annihilater of X will hdbenoted by r(X) (resp. 1(X)). Recall that:

(1) A ring R is ERT if every essential right ideal of R is a - sided,[6]

(2) R is said to be von Nuumann regular (or just regular) if, $a \in aRa$,for every $a \in$ R, and R is called strongly regular if $a \in a^2$ R.

In [3] Ming asked the following question:

Is a semi prime right SF-ring, all of whose essential right ideals are two-sided von Neumann regular ?

In this paper, we give conditions for a semi prime right SF-ring all of whose essential right ideals are two sided to be von Neumann regular.

2. Basic Properties:

Following [5], a ring R is called a right (left) SF-ring, if every simple right (left) R- module is flat.

The following lemma which is due to [7], plays a central role in several of our proofs: Lemma 2.1:

Let I be a right ideal of R. Then R/I is a flat R-module if and only if for each ae I, there exists be I such that a=ba.

We shall begin with the following result:

Proposition 2.2:

If R is a right SF-ring. Then

(1) Any reduced principal right ideal of R is a direct summand.

(2) Every left or right R-module is divisible.

Proof (1):

Let I=aR be a reduced principal right ideal of R and let aR+r(a) # R. Then there exists a maximal right ideal M of R containing aR+r(a).

Now, since R/M is flat, then a=ba, for some b in R. Whence l-b$\in$l(a)=r(a)$\subseteq$M. Yielding l e M which contradicts M*R. In particular

ar+c=l, for some r e R and c$\in$r(a), whence a^2 r=a. If we set d=ar^2 $\in$l, then
a=a^2d. Clearly, $(a\text{-}ada)^2=0$ implies a=ada and hence I=eR, where e=ad, is idempotent element. Thus I is a direct summand.

Proof (2):

It is sufficient to prove that any non-zero divisor c of R is invertible. For then, if dc=cd=l, any right R-module M satisfies M=Mdc$\subseteq$Mc$\subseteq$M, whence M=Mc (similarly, any left R-module is divisible). Suppose that cR$\neq$R. Let K be a maximal right ideal containing cR. Since R/K is flat, there exist ueK, such that c=uc. Now, r(c) =1 (c)=0 implies u=l, contradicting K≠R. This proves that cd=l for some d∈R and hence dc=l.

Proposition 2.3:

Let R be a right SF-ring. Then either r(M)=0 or M is a direct summand.

Proof:

Suppose that r(M)≠0 and let b∈M∩r(M). Then b∈M and Mb=0. Since R/M is flat then there exists a∈M such that b=ab. Now b=ab∈Mb=0, so b=0. Thus M∩r(M)=0, this means that M not can be essential and hence M is a direct summand. Therefore r(M)⊕M=R.

3. The Connection Between SF-Rings and Other Rings:

In this section we study the connection between SF-rings, biregular rings and strongly regular rings.

Recall that the right (left) socle of a ring R is defined to be the sum of all minimal right (left) ideals of R. It is well know that in a semi prime ring R, the right and left socles of R coincide, which will be denoted by socR[8].

Following [4], a ring R is biregular if RaR is generated by a central idempotent for each a∈R.

Theorem 3.1:

Let R be an ERT SF-ring with right zero socle and for every $a \in R$, RaR is a principal right of R. Then R is biregular.

Proof:

For any $a \in R$, set M=RaR+l(RaR). Since M is a maximal right ideal, then M is a direct summand or essential. If M is a direct summand of R, then its complement is a minimal right ideal. This implies that R has a no-zero socle, which is a contradiction. So every maximal right ideal is essential and hence two-sided. By hypothesis R/M is flat. Also RaR=bR for some $b \in R$. Since $b \in M$, b=db for some $d \in M$. Then $1-d \in l(b) c M$ which yields $1 \in M$. Therefore l=bc+v, $c \in R$, $v \in l(b)$ this implies that b=bcb. Therefore RaR=bR=eR where e=bc is idempotent. R is therefore semi-prime and hence e is central in R. Thus R is biregular.

Theorem 3.2:

If R is a reduced ring and every maximal right ideal of R is either a right annihilator or flat, then R is strongly regular.

Proof:

Let $b \in R$. we claim first $bR+r(b)=R$. If not, there exists a maximal right ideal L containing $bR+r(b)$. In case R/L is flat, since $b \in L$, there exists $c \in L$ such that $b=cb$. Then $l\text{-}c \in l(b)=r(b)cL$, whence it follows $1 \in L$, a contradiction. On the other hand, in case $L=r(t)$ with some $0 \neq t \in R$, we have $t \in !(bR+r(b)) \subseteq l(b)=r(b) \subseteq L=r(t)$. Then $t^2=0$, a contradiction. Therefore let $bR+r(b)=R$, and hence R, is strongly regular.

Now , we give under what condition the answer of the question of ring is affirmative.

Proposition 3.2:

Let R be a semi-prime ERT right SF-ring with zero socle. Then R is strongly regular.

Proof;

Let M be a maximal right ideal of R. Then M is either a direct summand of R or an essential right ideal of R. If M is a direct summand of R, then its complement is a minimal right ideal. This implies that R has a nonzero socle, which is a contradiction. So every maximal right ideal is essential and hence two-sided.

Since R is semi-prime ERT, by applying [2,Lemma 2.1], we see that R is right non-singular, and J(R)=0. So R is isomorphic to a sub direct sum of division rings, which implies that R has no non-zero nilpotent elements. Therefore R is strongly regular.f 1, proposition 1].

Theorem 3.3:

Let R be a prime ERT and right SF-ring. Then R has non-zero socle.

Proof:

Let R be a prime ERT and right SF-ring. If SocR=0. By the above Theorem (3.2), R is strongly regular. Hence R is a division ring, and SocR=R contradicting our assumption. Therefore SocR$\neq$ (0).

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