A Note on Non – Singular Rings

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ABSTRACT

In this paper several new properties of singular ideals and non-singular rings are obtained, a connection between a singular ideal and the Jacobson radical is considered , and a sufficient condition for a non-singular ring to be reduced is given.

Keywords: Singular Ideals, Duo Ring, Jacobson Radical

ملاحظة حول الحلقات غير المنفردة

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الملخص

في هذا البحث درسنا المثاليات المنفردة والحلقات غير المنفردة وأعطينا خواص جديدة لها ، ثم أعطينا العلاقة بين المثاليات المنفردة وجذر جاكوبسون. و أخيرا أعطينا الشرط اللازم لكي تكون الحلقات غير المنفردة مختزلة .

الكلمات المفتاحية : المثاليات المنفردة ، حلقة ديو ، جذر جاكوبسون

1. Introduction:

Throughout this work all rings are assumed to be associative rings with identity. Recall that: 1- A ring R is said to be reduced if R contains no non- zero nilpotent element ; 2- A ring R is said to be a duo- ring if every right and left ideal is a two-sided ideal; 3- J(R) and N will stand respectively for the Jacobson radical ideal of R and the set of all nilpotent elements; 4- The right and left annihilators of a in R will be denoted by r(a) and l(a), respectively ; 5- An ideal I is said to be a right (left) pure if , for every $a \in I$, there exists $b \in I$ such that a=ab(ba).

2. The Singular Ideals:

In this section, new properties of singular ideals are given , and a connection between singular ideals and the Jacobson radical is obtained .

Definition 2.1:

A non-zero elements a of R is said to be right singular if r(a) is an essential right ideal of R . The set of all right singular elements of R is denoted by Y(R). The set of all left singular elements will be denoted by Z(R).

Clearly Y(R) and Z(R) are ideals of R .

We shall begin this section with the following lemma .

Lemma 2.2:

If a $\in$Y(R) , then r(1-a) = 0 .

Proof: let a be a non- zero element in Y(R) , then r(a) is a non-zero essential right ideal of R. Let $x \in r(a) \cap r(1-a)$, then a .x = 0 and (1-a).x = 0 , this implies that x = 0 Therefore , r(a) $\cap$r(1-a) = 0 ; since r(a) is a non-zero essential right ideal of R , then r(1-a) = 0 .

Corollary 2.3: Let R be a ring and Y(R) = 0 , the only idempotent element in Y(R) is zero .

Proof: Let $0 \neq a \in Y(R)$, and $a = a^2$, this implies that a.(1-a) = 0 .

Hence $a \in r$ (1-a). By [Lemma 2.2.] ; a = 0.

Next, we give the following result .

Proposition 2.4 :

Let R be a ring with every right non–zero element is invertible, then Y(R) $\subseteq$ J(R) .

Proof: let $0 \neq a \in Y(R)$. Then by Lemma 2.2 r(1-a) = 0 , and hence 1-a is invertable. Whence a $\in$J(R) .

Recall the following result of Ferreno and Puczylowski in [1].

Lemma 2.5:

Let I be a sem – prime ideal of R , then :

1- Y(I) = I$\cap$ Y(R).

2- Z(I) = I$\cap$ Z(R).

In view of the above lemma , we have the following :

Proposition 2.6:

Let I and J be semi – prime ideals of R , then :

If I $\subseteq$ J , then Y(I) $\subseteq$ Y(J).
If I $\subseteq$ J , then Z(I) $\subseteq$ Z(J).

Proof: Let I $\subseteq$ J , and let a $\in$Y(I) , then by Lemma 2.5. Y(I) = I$\bigcap$Y(R) , this implies that a $\in$I and a $\in$Y(R) , and hence a $\in$J and Y(R) . Whence a $\in$ J$\bigcap$Y(R) = Y(J).

3. Non – Singular Rings:

This section is devoted to study non-singular rings , we give condition for non–singular rings to be reduced , and we characterize non-singular rings in terms of maximal pure ideals and essential right ideals of R.

Definition 3.1:

A ring R is said to be a right (left) non–singular if Y(R) = 0,(Z(R) = 0).
A ring R is said to be non – singular if Y(R) = Z(R) = 0 .

Example:

The ring of integers module 6 , Z_6 is a non–singular ring .
Recall the following result of Ming in [2].

Lemma 3.2:

If Y(R) $\neq$ 0 , then there exists y$\in$Y(R) such that y^2 = 0 .

The following result characterizes non–singular rings in terms of maximal pure ideals .

Theorem 3.3:

Let R be a ring,such that for every nilpotent element y of R , there exists a maximal pure right ideal M such that r(y)$\subseteq$M . Then R is a right non–singular ring.

Proof: Let Y(R) $\neq$ 0 , then by Lemma 3.2. , there exists a non–zero element y in Y(R) such that y^2 = 0, then by the hypothesis there exists a maximal pure right ideal M of R such that r(y)$\subseteq$M . Since y^2 = 0 , then y $\in$ r(y) $\subseteq$ M , and since M is a right pure , there exists m $\in$ M such that y=ym.
So, y(1-m)=0. This implies that 1-m $\in$r(y) $\subseteq$M , which means that 1 $\in$ M , a contradiction . Therefore, Y(R) = 0.

Next, we give another condition for R to be a non-singular ring.

Proposition 3.4:

Let Y(R) be a left pure ideal , then R is a right non – singular ring.

Proof: Let Y(R) be a non- zero left pure ideal , and let a be a non – zero element in Y(R) , then there exists $b \in Y(R)$ such that a = ba . Since $b \in Y(R)$, then r(b) is essential right ideal of R .We claim that $r(b) \cap aR = 0$ .Let $x \in r(b) \cap aR$, then b.x = 0 and x = a.r for some r $\in$ R , hence ba.r = 0 .But ba = a , then we have a.r = x = 0 . Now, since r(b) is an essential right ideal of R , then aR=0 ; and hence a = 0. Therefore Y(R) = 0 .

It is well–known that if R is a reduced ring ,then R is a non–singular ring. However, the converse is not true, as the following example shows:

Example :

Let R be the ring of 2X2 upper triangular matrices with entries in Z_2 , where Z_2 is the ring of integers modulo 2 , that is :

R={ $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where : $a,b,c \in Z_2$ }.

Then by direct calculation ,we observe that R is a non–singular ring but it is not reduced.

The following result gives a sufficient condition for non-singular rings to be reduced

Theorem 3.5:

Let R be a right non–singular ring with $l(a) \subseteq r(a)$ for every a $\in$ R . Then R is a reduced ring .

Proof: Let a $\in$ R , and let $a^2 = 0$. We shall prove first that r(a) is an essential right ideal of R ; if not , then there exists a right ideal I of R such that $r(a) \cap I = 0$. Since a $\in$ l(a) , then i. a $\in$ l(a) for all i $\in$ I , which implies that $I.a \subseteq l(a) \subseteq r(a)$. So $I.a \subseteq I \cap r(a) = 0$; this gives I . a = 0 , therefore $I \subseteq l(a) \subseteq r(a)$, and hence I=0. Therefore r(a) is an essential right ideal of R. This implies that $a \in Y(R) = 0$. Thus R is reduced.

Recall that an element a is said to be regular (in the sense of Von Neumann) if a $\in$ aRa .

Before closing this section, we present two additional results.

Theorem 3.6:

Let R be duo right singular ring , then any nilpotent element of R is regular .

Proof: Let a $\in$R , such that $a^n = 0$ for some positive integer n ,and let $s = a^{n-1} \neq 0$.

If aR is not an essential right ideal of R , then there exists a non-trivial right ideal K of R such that aR$\oplus$K is an essential right ideal of R .Suppose that a is not a regular element in R, then aR $\oplus$ K $\neq$R.Observe that sK $\subseteq$K$\bigcap$aR = 0.

This implies that K $\subseteq$ r(s) .

Now, since $a^n = 0$, then s.a = 0 , hence a $\in$ r(s) , this gives aR $\subseteq$ r(s).This means that r(s) is an essential right ideal of R . Whence it follows that s $\in$ Y(R) = 0 . This is a contradicition. Therefore, a is a regular element of R .

Theorem 3.7:

A ring R is right non–singular, if and only if L (I) = 0 for every essential right ideal I of R.

Proof: Suppose that R is a right non–singular ring , then Y(R) = 0 . Let I be an essential right ideal of R , such that L(I) $\neq$ 0 , then there exists a non–zero element a in L(I) .This implies that a.I = 0 , and hence I $\subseteq$ r(a) . Since Y(R) = 0 , then r(a) is not essential right ideal of R , and hence there exists a non- trivial right ideal K of R , such that $r(a) \cap K = 0$. Since I$\subseteq$ r(a) and $K.I \subseteq K \cap I$ then $K.I \subseteq K \cap I \subseteq K \cap r(a) = 0$, which means that K $\bigcap$I = 0, a contradiction.

Conversely, assume that I is an essential right ideal of R, and let L(I) = 0. Suppose that Y(R) $\neq$ 0 , and let a be a non – zero element in Y(R) , then r(a) is an essential right ideal of R. Since L(I) = 0, then L (r(a)) = 0 , then for every y$\in$r(a), a.y. = 0 , hence a$\in$ L(y) = 0 , a contradiction . Therefore Y(R) = 0.

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