Curtain Coating Flow of an Inclined Thin Liquid Films

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ABSTRACT

The mechanism of thin liquid films on solid surfaces is fundamental to a wide variety of phenomena such as surface coatings in paint. A mathematical model is constructed to describe the two dimensions of steady thin liquid films flow on an inclined plane with the use of lubrication approximation, we have applied Navier-Stokes equations in two dimensional coordinates for flow of incompressible fluid with the specified boundary conditions, and the solution of the film thickness equation has been drawn for flow for several inclination angles which modify the shape of the emerging patterns and also we derived the third order differential equations $\frac{d^3h}{dx^3}+\frac{\rho g}{\sigma}\sin(\theta)\frac{d^2h}{dx^2}-3Ca\frac{\cos(\theta)}{\sin(\theta)}\frac{dh}{dx}+3\frac{Ca}{h_0}h=-3\frac{Ca}{\varepsilon}$ that govern such flow. Finally the equations have been solved analytically.

Keywords: Navier-Stokes equations, continuity equation, Lubrication approximation, Integral momentum balance, Differential equations.

جريان طبقة الطلاء لأغشية سائلة رقيقة مائلة

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الملخص

إن ميكانيكية الأغشية السائلة الرقيقة على السطوح الصلبة هي اساس الاختلافات الكبيرة لظواهر عديدة كطلى السطوح بالالوان, وقد تم انشاء نموذج رياضي لوصف جريان الاغشية السائلة الرقيقة الثابتة , على سطح مائل مزيت تقريباً اذ طبقنا معادلات نافير-ستوكس ذات البعدين وبشروط حدودية لجريان السائل غير المضغوط, وكذلك رسمنا المعادلة التي تمثل سمك الغشاء الرقيق عند جريان السائل الى الاسفل و زوايا عديدة والتي تحدد لنا شكل النموذج وتمكنا أيضا من اشتقاق المعادلات التفاضلية ذات الرتبة الثالثة $\frac{d^3h}{dx^3}+\frac{\rho g}{\sigma}\sin(\theta)\frac{d^2h}{dx^2}-3Ca\frac{\cos(\theta)}{\sin(\theta)}\frac{dh}{dx}+3\frac{Ca}{h_0}h=-3\frac{Ca}{\varepsilon}$ التي تتحكم في مثل هذا الجريان, وقد تم حل هذه المعادلات بطريقة تحليلية.

الكلمـات المفتاحيـة: معـادلات نـافير–سـتوكس، معادلـة الاسـتمرارية، مزيـت تقريبـا، تـوازن الـزخم التكاملي، معادلات تفاضلية.

1. Introduction:

Curtain coating, in its precision model is a parameter coating process that has been used to manufacture single layer and hot fuel element surfaces in nuclear reactors thin films form a crucial element in many other applications such as industrial coating processes. A thin sheet of viscous liquid flowing between two vertical guide wires is an integral process called curtain coating. Experimentally the general behavior of liquid sheets in the context of curtain coating studied by Brown (1961) [1]. In the model curtain coating that would be investigated in this study is the thin liquid films flow at region on the inclined slide lip where the liquid changes its direction, curtain flow region beyond the lip where the falling liquid experiences uniaxial extensional deformation by gravity force, and take away region where liquid attains fully developed plug flow with the substrate speed (2004) [5]. The dynamic of the thin layer which flows steadily between two vertical guide wires was investigated but with zero shear stress at their bounding surfaces where the gravity has no significant effect on the liquid film Faraidun (2005) [4]. Cryse (1987) [2] obtained an analytic solution to a falling liquid curtain but with negligible effect of surface curvature, Diez (2002) [3] studied the linear stability analysis for flow of two incompressible viscous flow on an inclined channel. The objective of the present analysis is to apply the Navier-Stokes equation to a falling liquid curtain coating and present the derivation of the differential equation that governs the flow of the liquid curtain flow on an inclined solid and to obtain a solution of this equation which is valid for thin liquid film.

2. The Mathematical model :

2.1 Governing Equations:

To consider the two- dimensional inclined thin liquid films describe the flow of a slide flow, the Cartesian coordinates x and y , and the flow is predominantly in the two directions. Figure (2.1.1) shows the model to flow geometries of the slide flow and the curtain flow .

Let $u(x, y)$, $v(x, y)$ be the corresponding velocity vector component in x and y directions, respectively .

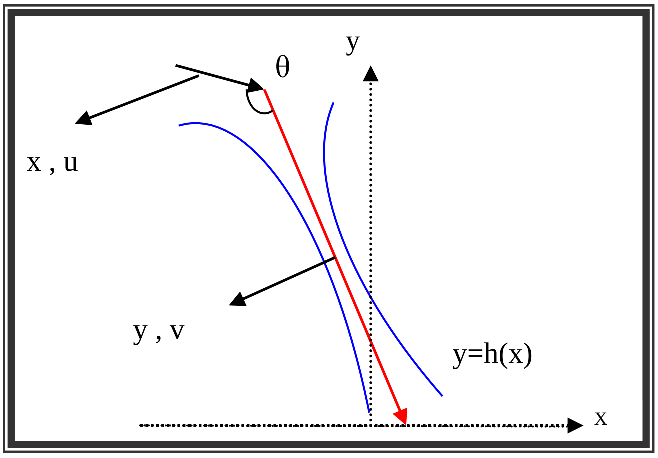

Figure (2.1.1): Flow geometries of a free surface- liquid film curtain flow

Normally in thin liquid films, the film thickness is much smaller than the width, and therefore we assume two-dimensional incompressible flow.

The steady two dimensional incompressible fluid flows governed by the following equations of motion and dimensionless in the slide flow of the curtain coating flow:

Equation of continuity:

The continuity equation for the flow of an incompressible fluid in two dimensions has the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{…….. (2.1.1)}$$

Navier-Stoke's equations [6] have the following form:
in x-direction:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+g\sin(\theta) \qquad \text{…….. (2.1.2)}$$

in y-direction:

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-g\cos(\theta) \qquad \text{…….. (2.1.3)}$$

where μ is the coefficient of viscosity of liquid, and u is component velocity, the density ρ is assumed to be constant throughout the process and g is a gravity.

Let $y = h(x)$ represent the thickness of the liquid film at a point x
The body force vector

$f = \sin(\theta)i - \cos(\theta)j$

The stress boundary condition at free surface, the Cartesian components of the unit normal vector n are given by:

$$\vec{n} = in_x + jn_y$$

where

$$\left.\begin{aligned} n_x &= -\frac{\partial h}{\partial x}\left\{1+\left(\frac{\partial h}{\partial x}\right)^2\right\}^{-1/2} \\ &\text{and} \\ n_y &= \left\{1+\left(\frac{\partial h}{\partial x}\right)^2\right\}^{-1/2} \end{aligned}\right\} \qquad \text{……..(2.1.4)}$$

The curvature of the liquid film is given by:

$$K = \frac{\partial n_i}{\partial x_i} = \frac{\partial^2 h}{\partial x^2}\left(1+\left(\frac{\partial h}{\partial x}\right)^2\right)^{\frac{-3}{2}} \qquad \text{……… (2.1.5)}$$

and since we restrict attention to the case when $\frac{\partial h}{\partial x}$ is very small, then the curvature (2.1.5) can be simplified since the term $h'^2(x)$ is very small over the domain x under consideration, the curvature becomes:

$$K = h''(x) = \frac{d^2 h}{dx^2} \qquad \text{……… (2.1.6)}$$

The no slip boundary condition is when $u = 0$, $v = 0$ at $y = 0$

The dimensionless parameters are as follows:

for continuity equation $\nabla.U = 0$ where $U = ui + vj$, $h_0 = \left(\frac{3\mu q}{\rho g \sin(\theta)}\right)^{\frac{1}{3}}$,

$\alpha = \frac{\rho g h_0^3}{\mu q}$ is stokes number, the Capillary number is $Ca = \frac{\mu q}{\sigma h_0}$, the Reynolds number is $\mathrm{Re} = \frac{\rho q}{\mu}$ and calculate forces in term of stress and substitute into the equations $f = i\sin(\theta) - j\cos(\theta)$ where force is normal to the thin film and shear component are acting on the inclined plane, with a thin films, h is film thickness and h_0 fully developed film thickness flowing down the slide

[5], μ is a liquid viscosity, ρ is a liquid density, q volumetric flow, i and j tangentional and normal stress at the slide respectively and θ is slide inclination angle from the horizontal line.

Thus results in the following boundary conditions:

From Material derivative, when $F(x,t) = y - h(x)$ we have

$$\frac{DF}{Dt} = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0 \qquad \text{.........} \ (2.1.7)$$

which gives $v = u\frac{dh}{dx}$ at $y = h(x)$ ……… (2.1.8)

The continuity equation (2.1.1) can be integrated over the film thickness, have $0 < y < h$ and the liquid film is symmetric, we have:

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dy = 0 \quad \to \int_0^h \frac{\partial u}{\partial x}\,dy + \int_0^h \frac{\partial v}{\partial y}\,dy = 0$$

$$\to \frac{\partial u}{\partial x}h + v(x,h(x)) - v(x,0) = 0$$

since the liquid film is symmetric, so $v(x,0) = 0$

$$\to \frac{\partial u}{\partial x}h + v(x,h(x)) = 0 \quad \to \quad d(hu) = 0$$

Integrate with respect to x, obtain $V = hu$ ……… (2.1.9)

From [4] where V is a constant representing the volumetric mass flow, it can be assumed that:

V=1 $\longrightarrow hu = 1$ ………(2.1.10)

The governing equations in the slide and curtain flow of (2.1.2) and (2.1.3) can be simplified by satisfying all boundary conditions and dimensionless parameters so the Naviers equations of momentum becomes

x-direction

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\mathrm{Re}}\left(-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\sin(\theta)\right) \qquad \text{.........} (2.1.11)$$

y-direction

$$-\frac{\partial p}{\partial y} - \alpha\cos(\theta) = 0 \qquad \text{.........} (2.1.12)$$

from the normal stress [5], we have

$$p + \frac{K}{Ca} = 0 \qquad at \ \ y = h \qquad \text{.........} \ (2.1.13)$$

Integrate the y-direction momentum equation over the film thickness $0 < y < h$, we have

$$\int_0^h \left(-\frac{\partial p}{\partial y} - \alpha\cos(\theta) \right) dy = \int_0^h -\frac{\partial p}{\partial y} dy - \int_0^h \alpha\cos(\theta) dy = 0$$

$$\rightarrow\ p = -\alpha\cos(\theta) h \qquad \text{……….(2.1.14)}$$

Also integrate the x-direction momentum equation over the film thickness, have $0 < y < h$ and substitute (2.1.14), the normal stress boundary condition and by the lubrication approximation to expand h in power series assuming $h_0 = \varepsilon L$ is so small and L is a length decompose the velocity u and the unknown h as

$$u(x, y) = u_0(x) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + ... \qquad \text{……. (2.1.15)}$$

and

$$h = h(x, y, \varepsilon) = h_0 + \varepsilon h_1(x, y) + \varepsilon^2 h_2(x, y) + ... \qquad \text{……. (2.1.16)}$$

here ε is related to h'. The functions $u_0(x)$ and h are unknown at this point and will be derived later in the analysis

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \int_0^h \frac{1}{\mathrm{Re}} \left(-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\sin(\theta) \right) dy \qquad \text{………. (2.1.17)}$$

Now we integrate the momentum equation over the film thickness in the same manners. The integrals of the non-linear inertia terms in the x-component of the momentum equation (2.1.11) are

$$\int_0^h u\frac{\partial u}{\partial x} dy = \int_0^h (u_0 u'_0 + \varepsilon u_1 u'_0 + \varepsilon^2 u_2 u'_0 + \varepsilon u_0 u'_1 + \varepsilon^2 u_1 u'_1$$

$$+ \varepsilon^3 u_2 u'_1 + \varepsilon^2 u_0 u'_2 + ...) dy$$

$$= u_0 u'_0\, h + O(\varepsilon)$$

To the first order, we get

$$\int_0^h u\frac{\partial u}{\partial x} dy = u_0 u'_0\, h$$

Omitting the subscripts, we have

$$\int_0^h u\frac{\partial u}{\partial x} dy = uu'h \qquad \text{………. (2.1.18)}$$

Since thin liquid films are symmetric so the second term of (2.1.11) can be written as

$$\int_0^h v\frac{\partial u}{\partial y} dy = v\, u\, h \qquad \text{……….(2.1.19)}$$

Also after some simplifications we obtain

$$\int_0^h \frac{\partial^2 u}{\partial y^2} dy = -3h\frac{\partial^2 u}{\partial x^2} + h\frac{\partial p}{\partial x} \qquad \text{……….(2.1.20)}$$

Now substitute all equations (2.1.10), (2.1.13), 2.1.14), (2.1.19) and (2.1.20) in the equation (2.1.17) to obtain

$$\mathrm{Re}\left(u u' h + u^2 h'\right) = \frac{1}{Ca} h \frac{\partial K}{\partial x} + \int_0^h \frac{\partial^2 u}{\partial y^2} dy + \alpha h \sin(\theta)$$

$$= \frac{1}{Ca} h \frac{dK}{dx} - 3h\frac{\partial^2 u}{\partial x^2} - \alpha h \cos(\theta)\, h' + \alpha h \sin(\theta)$$

where $K = h''(x) = \dfrac{d^2 h}{dx^2}$, we have to note here that both of u and h are functions for x and thus we have

$$\mathrm{Re}\left(u \frac{du}{dx} h + u^2 \frac{dh}{dx}\right) = \frac{1}{Ca} h \frac{d^3 h}{dx^3} + -3h\frac{d^2 u}{dx^2} - \alpha \cos(\theta)\frac{dh}{dx} + \alpha h \sin(\theta)$$

since $\longrightarrow hu = 1 \quad \longrightarrow u = \dfrac{1}{h}$ which gives

$$\longrightarrow \frac{1}{Ca} h \frac{d^3 h}{dx^3} + 3h\frac{d^2 h}{h^2 dx^2} - \alpha\, h \cos(\theta)\frac{dh}{dx} + \alpha h \sin(\theta) = 0$$

now use the lubrication approximation for the linearzing film thickness as (2.1.16) with $h_0 + \varepsilon h_1$, where ε represents tiny perturbation from h_0, we obtain

$$\frac{1}{Ca}(h_0 + \varepsilon h_1)\frac{d^3(h_0 + \varepsilon h_1)}{dx^3} + 3(h_0 + \varepsilon h_1)\frac{d^2(h_0 + \varepsilon h_1)}{(h_0 + \varepsilon h_1)^2 dx^2} -$$

$$\alpha (h_0 + \varepsilon h_1)\cos(\theta)\frac{d(h_0 + \varepsilon h_1)}{dx} + \alpha (h_0 + \varepsilon h_1)\sin(\theta) = 0$$

or

$$\frac{1}{Ca}\frac{d^3 h_1}{dx^3} + 3\frac{1}{h_0^2}\frac{d^2 h_1}{dx^2} - \alpha \cos(\theta)\frac{dh_1}{dx} + \frac{\alpha}{\varepsilon}\sin(\theta) + \alpha\frac{h_1}{h_0}\sin(\theta) = 0$$

or

$$\frac{d^3 h_1}{dx^3} + 3\frac{\mu q}{\sigma h_0}\frac{1}{h_0^2}\frac{d^2 h_1}{dx^2} - \alpha\frac{\mu q}{\sigma h_0}\cos(\theta)\frac{dh_1}{dx} + \frac{\mu q}{\sigma h_0}\alpha\frac{h_1}{h_0}\sin(\theta) = -\frac{\mu q}{\sigma h_0}\frac{\alpha}{\varepsilon}\sin(\theta)$$

substitute $h_0 = \left(\dfrac{3\mu q}{\rho g \sin(\theta)}\right)^{\frac{1}{3}}$ and $\alpha = \dfrac{\rho g h_0^3}{\mu q}$, we obtain

$$\frac{d^3h_1}{dx^3}+\frac{\rho g}{\sigma}\sin(\theta)\frac{d^2h_1}{dx^2}-3Ca\frac{\cos(\theta)}{\sin(\theta)}\frac{dh_1}{dx}+3Ca\frac{h_1}{h_0}=-3\frac{Ca}{\varepsilon} \qquad \text{……… (2.1.21)}$$

This is the third order differential equations and solving (2.1.21) analytically, the solution of this linearzed equation is the combination of three exponential functions obtains from the algebraic cubic characteristic equation, by comparing the equation (2.1.21) we see that it is more general than [5]. Equation (2.1.21) has a fundamental solution of the form

$$h_1(x)=e^{mx}$$

substitute in (2.1.21) to obtain, for the homogenous part

$$(m^3+\frac{\rho g}{\sigma}\sin(\theta)m^2-3Ca\frac{\cos(\theta)}{\sin(\theta)}m+3Ca\frac{1}{h_0})e^{mx}=0$$

since $e^{mx}\neq 0$

$$\rightarrow\ m^3+\frac{\rho g}{\sigma}\sin(\theta)m^2-3Ca\frac{\cos(\theta)}{\sin(\theta)}m+3Ca\frac{1}{h_0}=0 \qquad \text{……….(2.1.22)}$$

it has been proved that exponents have three roots of m one negative real and one complex conjugate pair whose real part is positive, changes by the value of Capillary numbers and the inclinations angle for any liquid to be used.

The roots m in (2.1.22) represents a balance among viscosity, density, inertia, Capillary numbers and inclination angle.

Some time the solution curves of equation (2.1.21) in (x,h_1) plane effects of process conditions on the curtain profile. The following figures how inertia or Capillary numbers affects the film profile by the different value of the inclination angles and Capillary numbers. In the case gravity force is comparable to viscous shear in slide flow region and viscous tensile force and inertia force in the curtain flow region, and some of the solution curves are drown and it shows that, thickness of the liquid film increases when the inclination angle decreases while the velocity of liquid increases as the inclination angle increases.

We can depict how to find the roots of equation (2.1.22) also find particular solution of (2.1.21) after that we can found the solutions by using the Brown's experiment [1], and drown the solution curves by maltab program [7] for difference liquid, we consider the values of $\sigma=72$, $\mu=0.01$, $g=980, \rho=1$ *and* $\varepsilon=0.001$, in the following figure.

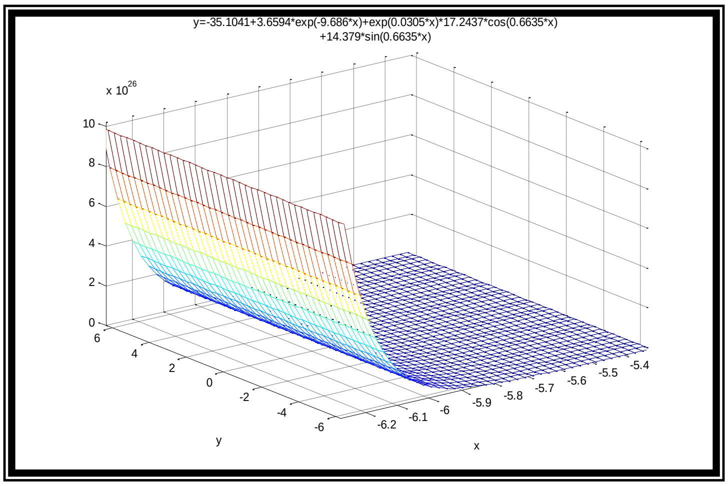

Figure (2.1.2) solution curve in (x, h_1) **plane for** $Ca = 0.05$, $\theta = 45$

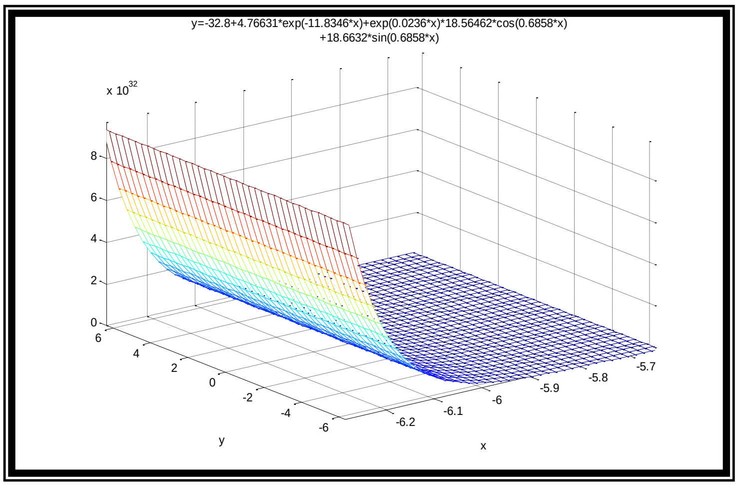

Figure (2.1.3) solution curve in (x, h_1) **plane for** $Ca = 0.05$, $\theta = 60$

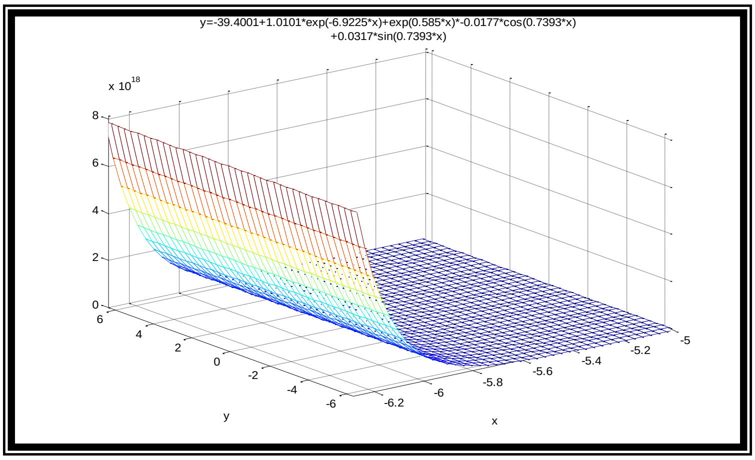

Figure (2.1.4) solution curve in (x, h_1) **plane for** $Ca = 0.05 \ , \theta = 30$

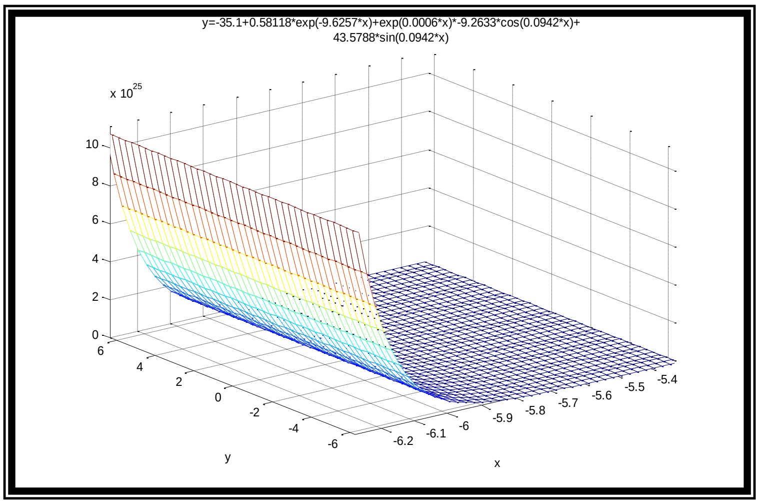

Figure (2.1.5) solution curve in (x, h_1) **plane for** $Ca = 0.001 \ , \theta = 45$

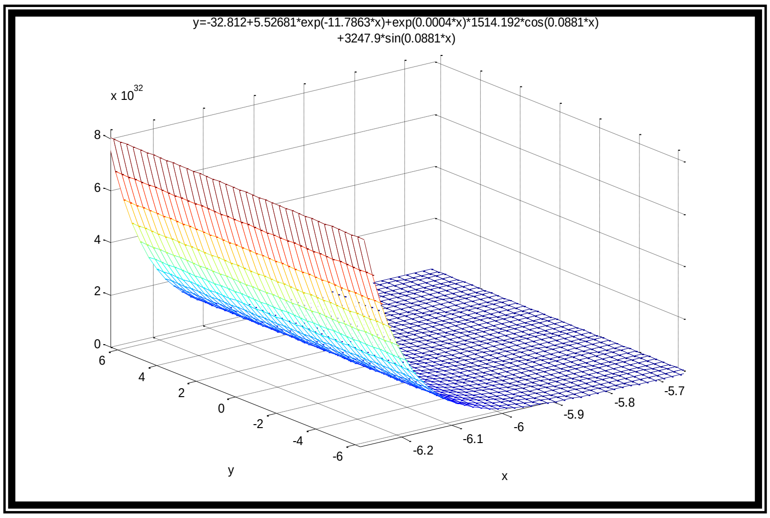

Figure (2.1.6) solution curve in (x,h_1) **plane for** $Ca = 0.001 \ ,\theta = 60$

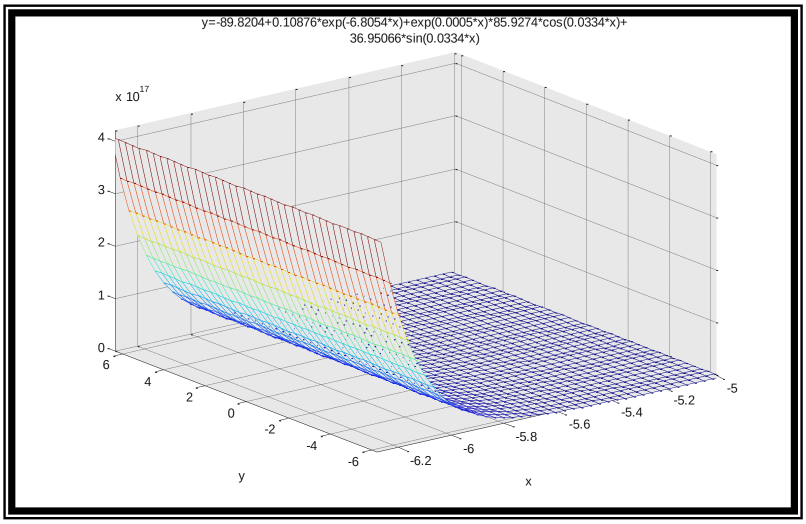

Figure (2.1.7) solution curve in (x,h_1) **plane for** $Ca = 0.001 \ ,\theta = 30$

Conclusion:

The theoretical and Mathematical models for curtain coating flow developed in this paper reproduces many of the features of this process that have been observed in experiments. The approximate governing equations for both slide and curtain flow have been successfully derived by thin film and integral the Navier's stocks equations, the equation has been solved analytically and used the Brown's experiment to obtain a solution of third order differential equation and the results by using the simplified models qualitatively agreeing with full theory and experimental observations. We show that how process conditions such as inertia, surface tension, density and inclination angle of the slide and some solutions curves are drawn and it shows that the thickness of the liquid film increases when the inclination angle decreases.

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