Continuity as a Galaxy of Hyperreal Functions

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Received on: 19/8/2007 **Accepted on: 4/11/2007**

ABSTRACT

In the present paper, the problem of defining continuity and s-continuity as a galaxy of hyperreal function is discussed. Our attempt is based on the fact that monads are subsets of some galaxies. New results are obtained, with nonstandard variables, related to a new extension of the continuity notion.

Keywords: continuity, s- continuity, monads, galaxy.

الاستمرارية كمجرة لدوال حقيقية فوقية

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تاريخ استلام البحث: 2007/8/19 **تاريخ قبول البحث: 2007/11/4**

الملخص

في هذا البحث تم تناول مسالة تعريف الاستمرارية و س- الاستمرارية كمجرة لدوال الحقيقية الفوقية. محاولتنا بنيت على حقيقة كون موناد هو مجموعة جزئية من بعض المجرات، وقد حصلنا على نتائج جديدة في دوال المعرفة على منطلقات غير قياسية وتم تعميم مفهوم الاستمرارية ليشمل جميع حالات الاستمرارية لدالة معرفة على منطلقات قياسية وغير قياسية.

الكلمات المفتاحية: استمرارية، س-استمرارية، هالة، مجرة.

1- Introduction: -

The following definitions of nonstandard analysis will be needed in this paper.

Every set or elements defined in a classical mathematics is called a standard. [2]

Any set or formula which does not involve a new predicates “standard, infinitesimals, limited, unlimited…etc” is called **internal**, otherwise is called **external**. [2] ,[4]

A real number x is called **unlimited** if $|x| > r$ for all real $r > 0$ otherwise called **limited**.[2]

The set of all unlimited real numbers is denoted by $\overline{\mathbf{R}}$, and the set of all limited real numbers is denoted by $\underline{\mathbf{R}}$ [2]

A real number x is called **infinitesimal** if $|x| < r$ for all positive standard real number r [4].

A real number x is called **appreciable** if it is neither unlimited nor infinitesimal, and the set of all positive appreciable numbers is denoted by A^+.[4]

Two real numbers x and y are said to be **infinitely close** if $x - y$ is infinitesimal and is denoted by $x \cong y$.[4]

If x is a limited number in $\mathbf{R}$, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x or **shadow** of x denoted by $st(x)$ or ${}^0x$, in this case we say that x is nearly standard [2][6].

Assume that f is a real valued function then:

1- A standard function f is continuous at a standard point x_o if for all x $x \cong x_o$ then $f(x) \cong f(x_o)$.[4]

2- f is called s-continuous at x_0 if for all x , $x \cong x_0$ then $f(x) \cong (x_0)$.

A standard function is **bounded** or **limited** if there exists a standard real constant number k such that $|f(x)| < k$.

$monad(x) = \{y \in \mathbf{R} : x \cong y\}$ for limited x, and is denoted by $m(x)$.[2]

$\alpha - monad(x) = \{y \in \mathbf{R} : \frac{y-x}{\alpha} \cong 0 \}$, and is denoted by $\alpha - m(x)$.[2]

$\alpha - micromonad(x) = \{y \ : y - x < \varepsilon^n \ \ \forall \ \text{standadd}\, n \}$ [2]

$galaxy(x) = \{y \in \mathbf{R} : y - x \ \ \text{limited} \}$ and is denoted by $gal(x)$ [2]

$\alpha - galaxy(x) = \{y \in \mathbf{R} : \frac{y-x}{\alpha} \ \ \text{limited} \}$ [2]

A function $f : A \rightarrow B$ is said to be **internal** if A is internal [1].
Let f be an internal function, we say that f satisfies ***k-Lipschitz*** condition if $|f(x) - f(y)| < k\,|x - y|$ for some standard constant $k \in \mathbf{N}$.[5]

If E is a standard metric space, then E is **complete** if for each $x \in E$ either x is nearly standard or there exists a standard number $r > 0$, such that the ball $B(x, r)$ contains no standard elements. [2]

Remark 1.1:

1. From the definition of monad and galaxy we have: $\forall t \in \mathbf{R}, \quad m(t) \subset gal(t)$.

2. Using the definition of monad we obtain:
A standard function f is continuous at a standard point x_o if for all x, $x \in m(x_0)$ then $f(x) \in m(f(x_0))$.

2- Galaxy Continuity

With nonstandard analysis the region of tangible elements is larger than that of standard analysis, therefore all problems that deal with such unusual elements take its frame space in nonstandard analysis, so the study of the behaviour of a function and its properties in nonstandard analysis give us a real phase and precision of the nature of it, which can never be imagined classically.

For functions that deal with only unlimited values or its values does not lie in the monad of any other points, the conventional and nonstandard definitions of continuity are meaningless and we are unable to say any thing about its continuity. In this paper we treat such problem by presenting a new version of continuity which includes all possible standard and nonstandard cases.

Definition 2.1:

Let $f : X \rightarrow Y$ be an internal function then we say that f is **galaxicaly continuous** at a point $x_0 \in X$ if for all x, $x \in m(x_0)$ then $f(x) \in gal(f(x_0))$, and is denoted by **g-continuous**.

The following examples give some relations between continuity, s-continuity, and g-continuity.

Examples 2.2

i) Let $f(x) = \frac{1}{x} \quad x \neq 0$, then we have:

1- For all standard x , f is continuous, and neither s-continuous nor g-continuous.

2- If $x \cong x_o$ such that $x \in x \bullet x_o - gal(x_o)$, then f is g-continuous at x_o, and otherwise it is not g-continuous.

3- f is not s-continuous, take $x_0 = \frac{1}{\omega}$, and $x = \frac{1}{a\omega}$ where ω is unlimited and $a > 1$ is a standard real number we have $x \in m(x_0)$ while $f(x) \notin m(f(x_0))$ also $f(x) \notin gal(f(x_0))$.

ii) $$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \in (-\varepsilon, 0) \cup (0, \varepsilon) \\ 0 & x = 0 \end{cases},$$

where $\varepsilon \cong 0$,
is not continuous at 0 but it is g-continuous.

iii) $f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$ is not continuous at any real point while it is g-continuous everywhere.

iv) $f(x) = e^{\frac{x}{\varepsilon}}$, $x \in [0, n\varepsilon]$, where n is standard positive integer, and $\varepsilon \cong 0$ is g-continuous but not s-continuous.

3- Main results

Theorem 3.1

1. Let f , and x_o be standard, if f is continuous at x_o then f is g-continuous at x_o.
2. If f is s-continuous then f is g-continuous.
3. If f satisfies ***k*-Lipschitz** condition then f is g-continuous.

Proof:

1. Let f be a standard continuous function at a standard point x_o then from the definition of continuity we have:

for all x $x \cong x_o$ then $f(x) \cong f(x_o)$,

use definition of monad we get

for all x, $x \in m(x_0)$ then $f(x) \in m(f(x_0))$.

Since $\forall t \in \mathbf{R}, \quad m(t) \subset gal(t)$, then we get:

for all if for all $x, x \in m(x_0)$ then $f(x) \in gal(f(x_o))$,

thus f is g-continuous.

2. It's proof is similar to the proof of (**1**).

3. Let f be an internal function (standard or nonstandard) such that f satisfies ***k*-Lipschitz** condition and $x, y \in \mathbf{R}$ such that $x \cong y$, to prove that $f(x) \in gal(f(y))$,

Since f satisfies ***k*-Lipschitz** condition then we get

$|f(x) - f(y)| < k|x - y|$ for some standard constant $k \in \mathbf{N}$,

but $x \cong y$, so $|f(x) - f(y)|$ is limited,

therefore $f(x) \in gal(f(y))$.

Remark: -

The converse of the above theorem is not true, for **1.** and **2.**see examples **2.2(iii)** and **2.2(ii)** respectively.

For the third statement let $f(x) = \begin{cases} x & x > 0 \\ 5 & x \le 0 \end{cases}$, its clear that f is g-continuous and to show that it does not satisfy ***k*-Lipschitz** condition;

Let $x = 0$ and $y = \varepsilon$ where ε is an infinitesimal number, then $x \cong y$ and $|x - y| \cong 0$, but there is no standard number k satisfies the inequality $|f(x) - f(y)| < k|x - y|$.

Theorem 3.2

Let $f(x) \in \varepsilon - gal(f(y))$ for all $x \cong y$, where ε is an infinitesimal number, then f is g-continuous if and only if f is uniformly s- continuous.

Proof:

let f be g-continuous, then foe all x, $x \cong y$ we have

$$f(x) \in gal(f(y)).$$

That is

$$f(x) - f(y) \text{ is limited.}$$

Since $\quad f(x) \in \varepsilon - gal(f(y))$,

then

$$\frac{f(x) - f(y)}{\varepsilon} \text{ is limited}$$

That is $\quad f(x) - f(y) = k\varepsilon^n$ for some standard numbers k and n.

Thus $f(x)-f(y)$ is infinitesimal

Therefore f is uniformly s- continuous.

The converse is obvious.

Theorem 3.3

Every limited function is uniform g-continuous.

Proof:

Assume that f is a limited function

Let $x \cong y$, since f is limited, then

$\forall x \in D_f$ we have $|f(x)| < k$ for some standard positive integer k .

Since

$$|f(x)-f(y)| \le |f(x)|+|f(y)| < 2k ,$$

So that $f(x)-f(y)$ is limited.

That is $f(x) \in gal(f(y))$,

then f is uniform g-continuous.

Remark: -

The converse of **Theorem 3.3** is not true in general (remembering that classically every continuous functions defined on a closed interval is limited). See **example 2.2(i)** on the bounded interval (0,1]

Corollary 3.4

Let $f:[a,b] \to \mathbf{R}$ be a standard continuous function, and h be a function such that $h(x) < f(x)$ for all x, $x \in [a,b]$, then h is g-continuous.

Proof:

Use **Theorem 2.8** [3] to get that f is limited.

Therefore h is limited,

Now, use **Theorem 3.3** to obtain the result.

Theorem 3.5

Let f be an infinitesimal valued function, then f is s-continuous function if and only if f is g-continuous.

Proof:

The forward way is direct.

For the backward direction, let f be g-continuous function,

Therefore $f(x) \in gal(f(y)) \quad \forall x, \quad x \cong y$, That is $f(x)-f(y)$ is limited

but $f(x)\cong 0$,

so $\forall x\in D_f$ we have $f(x)-f(y)$ is infinitesimal.

Thus f is s-continuous.

Theorem 3.6

Let f be an internal function, then f is g-continuous if and only if $f(x)\in\alpha-gal\big(f(y)\big)$ for all x, $x\cong y$ and α is appreciable.

Proof:

Let f be g-continuous, to prove that $f(x)\in\alpha-gal\big(f(y)\big)$ for all x, $x\cong y$, we have to prove that $\frac{f(x)-f(y)}{\alpha}$ is limited.

Since f is g-continuous then,

$f(x)-f(y)$ is limited so the result is obtained.

Conversely,

let $x\cong y$, since $f(x)\in\alpha-gal\big(f(y)\big)$ and α is appreciable,

Therefore

$$\frac{f(x)-f(y)}{\alpha} \text{ is limited,}$$

Thus $f(x)-f(y)$ is limited, and then f is g-continuous.

Theorem 3.7

Let X and Y be two metric spaces, and A be a complete subset of X. If f is s-continuous function from A^c into Y, then there exist a function $h:X\to Y$ which is g-continuous.

Proof:

Let $x\in X$,

if $x\in A^c$ then define $h(x)=f(x)$,

since s-continuity implies g-continuity, therefore h is g-continuous.

Now, for every $x\in A$,

We have to find a function h such that h is a g-continuous function.

Since A is complete then for all $x\in A$, either x is nearly standard or there exists a standard $r>0$, such that the ball $B(x,r)$ contain no standard element.

In the first case, there exist ${}^{o}x$ in X such that $x \cong {}^{o}x$.
Therefore, for all $x \in A$, define $h(x)={}^{o}x$ and assume that $x \cong y$, then
$h(x)={}^{o}x \cong x \cong y \cong {}^{o}y = h(y)$.
Thus h is g-continuous.
In the second case, there exists a standard $r>0$, such that for all $y \in B(x,r)$, then y is not standard.
Therefore we have the following concerning the value of y :

i. y is infinitesimal.

ii. y is appreciable.

iii. y is unlimited.

For the first two possibilities define $h(x)={}^{o}x$,
then $h(x)-h(y)$ is limited, therefore h is g-continuous.
For the third possibility, if $x \cong y$, then x is also an unlimited number, if $x \ncong y$ we get a contradiction to our assumption.
In this case, define $h(x)=x+t_x$, where t_x is any limited real number related with the variable x
Therefore

$$h(x)-h(y)=x-y+(t_x-t_y),$$
$$\cong (t_x-t_y),$$

which is limited.
Thus, the function

$$h(x)=\begin{cases} f(x) & \text{if } x \in A^c \\ {}^{o}x & \text{if } x \text{ is nearly standard} \\ x+t_x & \text{if } x \text{ is is unlimited} \end{cases}$$

is g-continuous.

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