GP-Stability of Linear Multistep Methods for Delay Retarded Differential Equations with Several Delays

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ABSTRACT

This paper modifies the numerical solution of initial value problems of the Delay Differential Equations (DDEs) by making it deals with Retarded Differential Equations (RDEs) with several delays

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_\mu)), \qquad t > 0$$
$$y(t) = \phi(t), \qquad t < 0$$

where $\tau_1, \tau_2, \ldots, \tau_\mu$ are positive constants, ϕ and f denote given vector-valued functions. The stability behaviors of linear multistep method for RDEs is studied and it is proved that the linear multistep method is GP-stable if it is A-stable for Ordinary Differential Equations (ODEs).

Keywords: GP-Stability, Linear Multistep Methods, Delay Retarded Differential Equations.

الحل العددي واستقرارية (GP) لمسائل القيم الابتدائية للمعادلات التفاضلية ذات التأخير الزمني من خلال اعتماد معادلات تفاضلية ذات تأخير زمني من النوع المعوق (RDEs) وبتأخير زمني متعدد

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الملخص

يطور هذا البحث الحل العددي لمسائل القيم الابتدائية للمعادلات التفاضلية ذات التأخير الزمني وذلك من خلال تناولها لمعادلات تفاضلية ذات تأخير زمني من النوع المعوق (RDEs) وبتأخير زمني متعدد ذات الصيغة:

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_\mu)) \qquad t > 0$$
$$y(t) = \phi(t) \qquad t < 0$$

حيث أن $\tau_1, \tau_2, \ldots, \tau_\mu$ هي ثوابت موجبة ، ϕ و f تمثل دوال ذات قيمة اتجاهية. يهتم هذا البحث بدراسة سلوك الاستقرارية لطرق المتعددة الخطوات الخطية لمسائل (RDEs). فلقد تم استنتاج أن طرق المتعددة الخطوات هي مستقرة − GP إذا كانت مستقرة − A للمعادلات التفاضلية الاعتيادية.

الكلمات المفتاحية: استقرارية (GP)، طرائق متعددة الخطوات الخطية، معادلات تفاضلية ذات تأخير زمني من النوع المعوق (RDEs).

1 - Introduction

In this paper, we are concerned with stability behaviors in numerical methods for the solution of initial-value problems for systems of Delay Differential Equations (DDEs) with several delays:

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_\mu)), \qquad t > 0$$
$$y(t) = \phi(t), \qquad t < 0 \qquad \ldots(1)$$

where $\tau_1, \tau_2, \ldots, \tau_\mu$ are positive constants, f and ϕ denote given vector-valued functions with

$f(t, x, y_1, y_2, \ldots, y_\mu) \in C^N$, (where $t \in R,\ y(t) \in C^N$)

$y(t-\tau_1)\in C^N,\ y(t-\tau_2)\in C^N,\ \dots,\ y(t-\tau_\mu)\in C^N),\ \phi(t)\in C^N,\ \tau_1,\tau_2,\dots,\tau_\mu>0$

and $y(t)\in C^N$ is unknown for $t>0$

In particular, stability properties of linear multistep methods (LMMs) will be investigated based on the following test system:

$$\begin{aligned} y'(t) &= Ay(t)+\sum_{i=1}^{\mu} B_i y(t-\tau_i), && t>0 \\ y(t) &= \phi(t), && t<0 \end{aligned} \qquad \dots(2)$$

Where A & $B_1, B_2, \dots, B_\mu$ denote complex constant $N\times N-$matrices and $\tau_1,\tau_2,\dots,\tau_\mu>0$. The solution of (2) is called asymptotically stable if

$$\lim_{t\to\infty} y(t)=0 \qquad \dots(3)$$

For any matrix X, denote its determinant by $\det[X]$, its spectrum by $\sigma[X]$ and its spectral radius by $\rho[X]$

2 - The Stability of The Tested Systems:

Define a function of two complex variables z, w by:

$$F(z,w)=\det[zI-(A+w^{-1}B)] \qquad \dots(4.a)$$

and for a system of several delays, the function becomes:

$$F(z,w)=\det[zI-(A+w^{-1}\sum_{i=1}^{\mu}B_i)] \qquad \dots(4.b)$$

Lemma 2.1: ([6] & [4])

Let $\|C\|\prec 1$, then all exact solutions to the system:

$$\begin{aligned} y'(t) &= Ay(t)+By(t-\tau)+Cy'(t-\tau), && t>0 \\ y(t) &= \phi(t), && t<0 \end{aligned} \qquad \dots(5)$$

are asymptotically stable if:

(C1) Every eigenvalue of the matrix A has negative real part

$(i.e.\ \lambda\in\sigma[\mathrm{A}] \ \Rightarrow\ \mathrm{Re}(\lambda)<0)$,

(C2) For any pure imaginary ξ, the spectral radius of $(\xi I-A)^{-1}(B+\xi C)$ is less than 1 in magnitude .

$(i.e.\ \sup_{\substack{\mathrm{Re}(\xi)=0\\ \xi\neq 0}} \rho[(\xi I-A)^{-1}(\xi C+B)]<1)$,

(C3) The spectrum of $A^{-1}B$ does not include (-1).

$(i.e.\ -1\notin\sigma[A^{-1}B])$

where $\|A\|=\sup_{\|x\|=1}\|Ax\|,\ \|x\|^2=\langle x,x\rangle,\ x\in C^N$ where $\langle x,x\rangle$ is the inner product.

Note That:

(1) The condition (C2) is readily verified to be equivalent to the condition that for the parameterized matrix $Q(\xi) = (I - \xi C)^{-1}(A + \xi B)$ we have:

(C_2^*) All the non-zero eigenvalues of the matrix $Q(\xi)$ have negative real part whenever $|\xi| \le 1$. [4]

(2) Under condition (C_1), the identity $w\, F_1(z, w)$:

$$F_1(z,w) = \det\left[zI - \left\{A + w^{-1}(zC + B)\right\}\right]$$
$$= \det\left[(zI - A) - w^{-1}(zC + B)\right]$$
$$= \det\left[(zI - A)\left\{I - w^{-1}(zI - A)^{-1}(zC + B)\right\}\right]$$
$$wF_1(z,w) = \det\left[(zI - A)\left\{wI - (zI - A)^{-1}(zC + B)\right\}\right]$$
$$= \det\left[(zI - A)\right]\det\left[wI - (zI - A)^{-1}(zC + B)\right]$$

holds for z such that $\mathrm{Re}(z) \ge 0$. [6]

The condition:

$(\tilde{C}_2)$ $F_1(z,w) \ne 0$ for any z $(\ne 0)$ and w which satisfy $\mathrm{Re}(z) \ge 0$ and $|w| \ge 1$,

Implies (C_2) if (C_1) satisfied.

where $F_1(z,w) = \det[zI - \{A + w^{-1}(zC + B)\}]$ is a function of two complex variables *z*, *w*. [6]

Lemma 2.2: [6]

Let $\|C\| \prec 1$, then all exact solution of equation (5) is asymptotically stable if (C_1), $(\tilde{C}_2)$, (C_3) are satisfied.

Lemmas, 2.1 and 2.2, deals with Neutral Differential Equations (NDEs) type systems. Putting $C = 0$ transforms the attention from NDEs systems to RDEs systems, the treatment will then be generalized to RDEs with one delay by making it deals with several delays. By these assumptions and the first two conditions only, it will be shown that the linear multistep method for the system (2) is GP-stable.

New Investigations:

Putting $C = 0 \Rightarrow \|C\| = 0 \prec 1$ in Lemmas 2.1 and 2.2, it will be shown that all exact solutions to (2) preserve the asymptotical stability if:

(S_1) $\forall \lambda \in \sigma[A] \Rightarrow \mathrm{Re}(\lambda) < 0$ …(6)

(S_2) $\sup\limits_{\mathrm{Re}(\xi)=0} \rho\left[(\xi I - A)^{-1}\sum_{i=1}^{\mu} B_i\right] < 1$ …(7)

or (S_2^*) $\forall\ \lambda \in \sigma[Q(\xi)],\ \lambda \ne 0 \Rightarrow \mathrm{Re}(\lambda) < 0$ whenever $|\xi| \le 1$,

where $Q(\xi) = A + \xi \sum_{i=1}^{\mu} B_i$.

where $\|A\| = \sup\limits_{\|x\|=1} \|Ax\|$, $\|x\|^2 = \langle x, x\rangle$, $x \in C^N$ where $\langle x, x\rangle$ is the inner product

Assumption 2.3:

Following condition (S_1) above, the identity:

$$F(z,w) = \det[zI - (A + w^{-1}\sum_{i=1}^{\mu} B_i)]$$

$$= \det[(zI - A) - w^{-1}\sum_{i=1}^{\mu} B_i]$$

$$= \det[(zI - A)\{I - w^{-1}(zI - A)^{-1}\sum_{i=1}^{\mu} B_i\}]$$

$$wF(z,w) = \det[(zI - A)\{wI - (zI - A)^{-1}\sum_{i=1}^{\mu} B_i\}]$$

$$= \det[(zI - A)]\det[wI - (zI - A)^{-1}\sum_{i=1}^{\mu} B_i]$$

Holds for z such as $\mathrm{Re}(z) \geq 0$.

Lemma 2.4:

The condition $(\tilde{S}_2)$ $F(z,w) \neq 0$ for any z $(\neq 0)$ and w which satisfy $\mathrm{Re}(z) \geq 0$ and $|w| \geq 1$, implies(S_2) if (S_1)is satisfied, where *F*(*z*, *w*)is defined in equation (4).

Proof:

To prove that where (S_1) is satisfied, then $(\tilde{S}_2) \Rightarrow (S_2)$. Assume that $(\tilde{S}_2)$ is satisfied, but (S_2) does not hold if (S_1) is satisfied, then whenever z_0 $(\neq 0)$, Re (z_0)=0 we have: $\rho\left[(z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i\right] \geq 1$

By the definition of spectral radius [3]:

$$\rho[(z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i] = \sup\left\{|w_0| : w_0 \in \sigma[(z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i\right\} \geq 1,$$

then Now since $w_0 \in \sigma[(z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i] \Rightarrow \det[w_0 I - (z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i] = 0$

By Assumption 2.3 above and under the condition (S_1) we have:

$$w_0 F(z_0, w_0) = \det[(z_0 I - A)]\det[w_0 I - (z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i]$$

$$= 0$$

Thus, either w_0=0 or F(z_0, w_0)=0. And since $w_0 \neq 0$, $(|w_0| \geq 1) \Rightarrow F(z_0, w_0) = 0$ which contradicts condition $(\tilde{S}_2)$

$$\rho\left[(z_0 I - A)^{-1}\sum_{i=1}^{\mu} B_i\right] < 1 \quad \text{whenever} \quad \mathrm{Re}(z_0) \neq 0 \ \& \ z_0 \neq 0. \quad \#$$

Lemma 2.5:

Let $\|C\| = 0$, then all exact solution of equation (5) with $C = 0$ and several delays (that is equation (2)) will preserve the asymptotical stability if (S_1) & $(\tilde{S}_2)$ are satisfied.

Proof:

According to Lemma 2.2, we have $\|C\| = 0 < 1 \Rightarrow C = 0$, so equation (5) with $C = 0$ is given by

$$y'(t) = Ay(t) + By(t-\tau), \qquad t > 0$$
$$y(t) = \phi(t), \qquad t < 0$$

and for several delays we get test system (2).

Thus, according to Lemma 2.2, this system preserves asymptotical stability if it is proven that $(\tilde{C}_2) \cong (\tilde{S}_2)$.

Suppose that $(\tilde{S}_2)$ is satisfied, but $(\tilde{C}_2)$ does not hold, then for any z_0 $(\neq 0)$ and w_0 which satisfy $\mathrm{Re}(z_0) \geq 0$ and $|w_0| \geq 1$, $F_1(z_0, w_0) = 0$

$$\Rightarrow \det[z_0 I - (A + w_0^{-1}(z_0 C + B))] = 0$$

Since we have the assumption $\|C\| = 0 \Rightarrow C = 0$, we get

$$\Rightarrow \det[z_0 I - (A + w_0^{-1} B)] = 0$$

and for several delays system we have

$$\Rightarrow \det[z_0 I - (A + w_0^{-1} \sum_{i=1}^{\mu} B_i)] = 0$$

$$\Rightarrow F(z_0, w_0) = 0$$

which is in contradiction to the condition $(\tilde{S}_2)$.

Conversely, Suppose that $(\tilde{C}_2)$ is satisfied, but $(\tilde{S}_2)$ does not hold, then for any $z^*(\neq 0)$ and w^* which satisfy $\mathrm{Re}(z^*) \geq 0$ and $|w^*| \geq 1$, $F(z^*, w^*) = 0$

$$\Rightarrow \det[z^* I - (A + w^{*-1} \sum_{i=1}^{\mu} B_i)] = 0$$

Also for one delay system we get:

$$\Rightarrow \det[z^* I - (A + w^{*-1} B)] = 0$$

And for equation (5) we have:

$$\Rightarrow \det[z^* I - (A + w^{*-1}(z^* C + B))] = 0$$

$$\Rightarrow F_1(z^*, w^*) = 0$$

which is in contradiction to the condition $(\tilde{C}_2)$. #

Theorem 2.6:

Let $\|C\| = 0$, then all exact solutions of equation (5) with $C = 0$ and several delays (that is equation (2)) will preserve the asymptotical stability if:

1 - (S1) and (S2) are satisfied. or

2 - (S_1) and (S_2^*) are satisfied.

Proof:

According to Lemma 2.5, one needs only to prove that:

1- $(\tilde{S}_2) \cong (S_2)$. or

2- $(\tilde{S}_2) \cong (S_2^*)$.

Proof (1): By lemma 2.4 it has been proved that $(\tilde{S}_2) \Rightarrow (S_2)$.

If (S_1) is satisfied, so it is sufficient to prove that $(S_2) \Rightarrow (\tilde{S}_2)$.

Assume that (S_2) is satisfied, but $(\tilde{S}_2)$ does not hold, then for any $z^*(\neq 0)$ and w^* which satisfy $\mathrm{Re}(z^*) \geq 0$ and $|w^*| \geq 1,$ we have $F(z^*, w^*) = 0$

$$\Rightarrow \det[z^* I - (A + w^{*-1} \sum_{i=1}^{\mu} B_i)] = 0$$

Again by assumption 2.3 we have:

$$w^* F(z^*, w^*) = \det[(z^* I - A)] \det[w^* I - (z^* I - A)^{-1} \sum_{i=1}^{\mu} B_i] = 0$$

$$\Rightarrow \text{ either } \det[z^* I - A] = 0 \Rightarrow \exists\, z^* \in \sigma[A] \rightarrow \mathrm{Re}(z^*) < 0 (\textit{not valied})$$

$$\text{or } \det[w^* I - (z^* I - A)^{-1} \sum_{i=1}^{\mu} B_i] = 0$$

$$\Rightarrow \exists\, w^* \in \sigma[(z^* I - A)^{-1} \sum_{i=1}^{\mu} B_i] \Rightarrow \sup\left\{ |w^*| : w^* \in \sigma[(z^* I - A)^{-1} \sum_{i=1}^{\mu} B_i] \right\} \geq 1$$

$$\Rightarrow \rho[(z^* I - A)^{-1} \sum_{i=1}^{\mu} B_i] \geq 1$$

which is in contradiction to the condition (S_2).

Proof (2): Suppose that $(\tilde{S}_2)$ is satisfied, but (S_2^*) does not hold,

$\Rightarrow \exists \xi_0 \in C$ with $|\xi_0| \leq 1 \;\&\; \lambda_0 \in \sigma[Q(\xi_0)]$ with $\lambda_0 \neq 0$ such that $\mathrm{Re}(\lambda_0) \geq 0$ where $Q(\xi_0) = A + \xi_0 \sum_{i=1}^{\mu} B_i$.

Let $z_0 = \lambda_0$ & $w_0 = \xi_0^{-1}$ $(\xi_0) \neq 0$. Thus $|w_0| \geq 1$ & $\because$ $z_0 \in \sigma[Q(w_0^{-1})],\ z_0 \neq 0$

$\Rightarrow \mathrm{Re}(z_0) \geq 0$ (whenever $|w_0^{-1}| \leq 1,$ *i.e.* $|w_0| \geq 1$)

$$F(z_0, w_0) = \det[z_0 I - (A + w_0^{-1} \sum_{i=1}^{\mu} B_i)]$$

$$= 0$$

and this is in contradiction to the condition $(\tilde{S}_2)$.

Conversely, Assume that (S_2^*) is satisfied, but $(\tilde{S}_2)$ does not hold, then for any $z^*(\neq 0)$ and w^* which satisfy $\mathrm{Re}(z^*) \geq 0$ and $|w^*| \geq 1,$ we have $F(z^*, w^*) = 0$

$$\Rightarrow \det[z^* I - (A + w^{*-1} \sum_{i=1}^{\mu} B_i)] = 0$$

$$\Rightarrow \exists z^* \in \sigma[Q(\xi^*)] \text{ with } z^* \neq 0 \Rightarrow \text{Re}(z^*) \geq 0 \text{ (where } \xi^* = w^{*^{-1}}\text{)}.$$

which is in contradiction to the condition (S_2^*), this completes the proof of theorem 2.6. #

Illustrative Example 2.7:

To illustrate the main theorem stated in this paper, we give an example of RDE. Recall equation:

$$y'(t) = Ay(t) + \sum_{i=1}^{\mu} B_i y(t - \tau_i) \qquad t > 0$$

This example, see [3], is the case for:

$$A = \begin{bmatrix} 0 & 1 \\ -100 & -10 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & -25 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{20} \end{bmatrix}$$

It easy to calculate the spectrum of A, that is $|\lambda I - A| = 0 \Rightarrow \lambda_j = -5 \mp 8.6603i$, $j = 1, 2$, so the spectrum of A is just (-5) $\Rightarrow \lambda_1$ & $\lambda_2 \in \sigma[A] \Rightarrow$ $\text{Re}(\lambda_1)$ & $\text{Re}(\lambda_2) < 0$.
Furthermore,

$$Q(\xi) = \begin{bmatrix} 0 & 1 \\ -100 & -10 \end{bmatrix} + \xi \left(\begin{bmatrix} 0 & 0 \\ 0 & -25 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{20} \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 \\ -100 & -(10 + 24.95\xi) \end{bmatrix}$$

$$|\lambda I - Q(\xi)| = 0 \Rightarrow \lambda^2 + (10 + 24.95\xi)\lambda + 100 = 0$$

whenever $|\xi| \leq 1$, we have

$\lambda_1 \leq -3.1331 + 0i$,

$\lambda_2 \leq -31.8059 + 0i$,

λ_1 & $\lambda_2 < 0$, have negative real part

$\Rightarrow \lambda_1$ & $\lambda_2 \in \sigma[Q(\xi)]$, λ_1 & $\lambda_2 \neq 0 \Rightarrow$ $\text{Re}(\lambda_1)$ & $\text{Re}(\lambda_2) < 0$

whenever $|\xi| \leq 1$, where $Q(\xi) = A + \xi \sum_{i=1}^{\mu} B_i$.

Henceforth, conditions (S_1) and (S_2^*) are satisfied, and the system (2) is analytically asymptotically stable. Now, Consider the ordinary differential equations:

$$x'(t) = f(t, x(t)), \qquad t > 0$$

$$x(0) = x_0,$$

where $x(t)$ and $f(t, x(t))$ are vector-valued functions. If $h > 0$ denotes a given stepsize, the gridpoint t_n is given by $t_n = nh$, and x_n denotes an approximation to $x(t_n)$. A linear multistep method can be written as

$$\sum_{j=0}^{k} \alpha_j x_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, x_{n-j}), \qquad (n = k+1, k+2, \ldots) \qquad \ldots(8)$$

Here α_j & $\beta_j (j = 0,1,\ldots,k)$ denote the coefficients of a LMM. Let $\rho(z)$ & $\sigma(z)$ be the usual characteristic polynomials:

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^{k-j}$$

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^{k-j}$$

Remark 2.8:

A linear multistep method (ρ, σ) is called A-stable if all roots z of $\rho(z) - \lambda\sigma(z) = 0$ satisfy $|z| \prec 1$ whenever $\mathrm{Re}(\lambda) \prec 0$. Then we easily obtain the following result.

Lemma 2.9: [7]

The linear multistep method is A-stable if and only if $\rho(z)I - \sigma(z)A$ is inevitable (whenever all eigenvalues λ of A satisfy $\mathrm{Re}(\lambda) < 0$ and $|z| \geq 1$).

In section 3, we will present adaptation of the linear multistep methods for the numerical solution of (1). In section 4, we investigate the stability of LMM and we will show that the LMM is GP-stable if it is A-stable for ODEs. Where GP stands for the general P-stability of DDEs.

3 - Linear Multistep Method for DDEs:

In order to make the linear multistep method (8) adapt to(1), we introduce unknowns $u_1(t), u_2(t), \ldots, u_\mu(t)$ as:

$$\begin{aligned} u_1(t) &= y(t - \tau_1) \\ u_2(t) &= y(t - \tau_2) \\ \vdots \quad & \quad \vdots \\ u_\mu(t) &= y(t - \tau_\mu) \end{aligned} \qquad \ldots(9)$$

Then (1) can be converted into the following form:

$$\begin{aligned} y'(t) &= f(t, y(t), u_1(t), u_2(t), \ldots, u_\mu(t)), & t > 0 \\ y(t) &= \phi(t), & t < 0 \end{aligned} \qquad \ldots(10)$$

Application of the linear multistep method (8) to (10) yields

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, y_{n-j}, u_{1_{n-j}}, u_{2_{n-j}}, \ldots, u_{\mu_{n-j}}) \qquad \ldots(11)$$

Let $\tau_i = (m_i - \delta_i)h > 0,\ i = 1, 2, \ldots, \mu.$

with the integer $m_i \geq 1$ & $\delta_i \in [0,1),\ i = 1, 2, \ldots, \mu.$ Then $(u_{i_{n-j}}, i = 1, 2, \ldots, \mu)$ can be computed by Lagrange interpolation [2] where the derivative and infinite integral of any polynomial are easy to determine and the results again polynomials so that $(u_{i_{n-j}}, i = 1,2,\ldots,\mu)$ can be computed by Lagrange interpolation as:

$$u_{1_{n-j}} = \sum_{k=-r}^{s} L_k(\delta_1) y_{n-j-m_1+k}$$
$$u_{2_{n-j}} = \sum_{k=-r}^{s} L_k(\delta_2) y_{n-j-m_2+k} \qquad \ldots(12)$$
$$\vdots \qquad \vdots$$
$$u_{\mu_{n-j}} = \sum_{k=-r}^{s} L_k(\delta_\mu) y_{n-j-m_\mu+k}$$

$(whenever \;\; n \geq m_i + 1, \;\; s+1 \leq m_i \leq -r, \quad i = 1,2,\ldots,\mu)$

Here r & s denote given non-negative integers, and

$$L_k(\delta_i) = \prod_{\substack{j=-r \\ j\neq k}}^{s} \frac{\delta_i - j}{k - j}, \;\; i = 1,2,\ldots,\mu \qquad \ldots(13)$$

To go into the stability behavior of (8) in the numerical solutions of the general equation (1), we apply (11) & (12) to the test system (2) and obtain

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = \sum_{j=0}^{k} \beta_j \bar{A}\, y_{n-j} + \sum_{j=0}^{k} \beta_j \bar{B}_1 u_{1_{n-j}} + \ldots + \sum_{j=0}^{k} \beta_j \bar{B}_\mu u_{\mu_{n-j}}.$$
$$= \sum_{j=0}^{k} \beta_j \bar{A} y_{n-j} + \sum_{j=0}^{k} \beta_j \bar{B}_1 \sum_{i=-r}^{s} L_i(\delta_1) y_{n-j-m_1+i} + \ldots + \sum_{j=0}^{k} \beta_j \bar{B}_\mu \sum_{i=-r}^{s} L_i(\delta_\mu) y_{n-j-m_\mu+i}$$
$$= \sum_{j=0}^{k} \beta_j \bar{A} y_{n-j} + \sum_{l=1}^{\mu} \sum_{j=0}^{k} \beta_j \bar{B}_l \sum_{i=-r}^{s} L_i(\delta_l) y_{n-j-m_l+i} \qquad \ldots(14)$$

where $\bar{A} = hA,\; \bar{B} = hB.$

4 - Numerical Stability of LMM:

Let

$$p_1(z;\delta_1) = (\sigma(z)\bar{B}_1)\alpha_1(z;\delta_1),$$
$$p_2(z;\delta_2) = (\sigma(z)\bar{B}_2)\alpha_2(z;\delta_2),$$
$$\vdots \qquad \vdots$$
$$p_\mu(z;\delta_\mu) = (\sigma(z)\bar{B}_\mu)\alpha_\mu(z;\delta_\mu),$$

i.e. $\displaystyle\sum_{k=1}^{\mu} p_k(z;\delta_k) = \sum_{k=1}^{\mu} (\sigma(z)\bar{B}_k)(\alpha_k(z;\delta_k)) = \sigma(z)\sum_{k=1}^{\mu} \bar{B}_k \alpha_k(z;\delta_k)$

$$Q(z) = \rho(z) I - \sigma(z)\bar{A}$$

$$\alpha_k = \sum_{i=-r}^{s} L_i(\delta_k) z^{i+r}, \;\; \mathrm{k} = 1,2,\ldots,\mu.$$

Then the characteristic equation of (14) can be written as:

$$\det\left[z^{m+r} Q(z) - \sum_{i=-r}^{s} p_k(z;\delta_k) \right] = 0.$$

In view of the assumption discussed in section 2 and definition 3 of reference [5], we introduce the set:

$$H = \{(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)\} \in C^{N\times N} \times C^{N\times N} \times \ldots \times C^{N\times N} (\mu+1 \text{ times})$$

$: (\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)$ satisfies:

(2) $\sup_{\mathrm{Re}(\xi)=0} \rho\left[(\xi I - \overline{A})^{-1} \sum_{i=1}^{\mu} \overline{B}_i\right] < 1$

or $\forall\ \lambda \in \sigma[Q(\xi)],\ \lambda \neq 0 \Rightarrow \mathrm{Re}(\lambda) < 0$ whenever $|\xi| \leq 1$, where $Q(\xi) = \overline{A} + \xi \sum_{i=1}^{\mu} \overline{B}_i$.

Definition 4.1:

Let $(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)$ be given and $\delta_i \in [0,1)$. Then the process (14) is $(\delta_1,\delta_2,\ldots,\delta_\mu)-$ stable at $(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)$ if and only if :

$\lim_{t\to\infty} y_n = 0$ whenever $s+1 \leq m_i \leq -r.$

The $(\delta_1,\delta_2,\ldots,\delta_\mu)-$stability region of the process (14) is defined by:

$$S_{\delta_i}(r,s) = \{(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu) \in C^{N\times N} \times C^{N\times N} \times \ldots \times C^{N\times N} \quad (\mu+1 \text{ times})$$

:the process (14) is δ_{i} - stable at $(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)$, $\mathrm{i} = 1,2,\ldots,\mu.\}$.

The stability region S of the process (14) is defined by:

$$S(r,s) = \bigcap_{0\leq\delta_1,\delta_2,\ldots,\delta_\mu \prec 1} S_{\delta_i}(r,s), \quad i = 1,2,\ldots,\mu$$

Definition 4.2:

The process (14) is GP-stable if $H \subseteq S(r,s)$.

Now as an equivalent to remark 2.8 for ODEs we can say that the process (14) is $(\delta_1,\delta_2,\ldots,\delta_\mu)-$stable at $(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)$ if and only if the characteristic polynomial of (14) satisfies:

$$\det\left[z^{m+r}Q(z) - \sum_{k=1}^{\mu} p_k(z;\delta_k)\right] = 0 \Rightarrow |z| < 1, \qquad \text{...(15)}$$

(whenever $\mathrm{z} \in \mathrm{C}, \mathrm{s}+1 \leq \mathrm{m_i} \leq -r$).

Simplification 4.3:

The process (14) is $(\delta_1,\delta_2,\ldots,\delta_\mu)-$ stable at $(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)$ whenever $\mathrm{s}+1 \leq \mathrm{m_i} \leq -r$ if and only if:

$Q(z)$ is inevitable (whenever $|z| \geq 1$) …(16)

$$\sup_{|z|=1} \rho\left[Q(z)^{-1} \sum_{k=1}^{\mu} p_k(z;\delta_k)\right] \leq 1 \qquad \text{…(17)}$$

$$\det\left[z^{m+r}Q(z) - \sum_{k=1}^{\mu} p_k(z;\delta_k)\right] \neq 0 \qquad \text{...(18)}$$

whenever $\mathrm{s}+1 \le \mathrm{m_i} \le -r,\ |z|=1,\ \ \rho\left[Q(z)^{-1}\sum_{k=1}^{\mu} p_k(z;\delta_k)\right]=1.$

Lemma 4.4: [7]

Assume the LMM (ρ,σ) is A-stable for ODEs, then $|z|\ge 1$ and $\sigma(z)\ne 0 \Rightarrow \operatorname{Re}\dfrac{\rho(z)}{\sigma(z)} \ge 0.$

Remark 4.5: [7] & [6]

In order to investigate the statement: $\sup\limits_{|z|=1} \rho\left[Q(z)^{-1} p(z;\delta)\right] \le 1,$

We consider the regions Σ & Γ given respectively by:

$$\Sigma=\left\{\xi:\xi\in C,\left|\frac{1+\xi/2}{1-\xi/2}\right|<1\right\} \qquad \ldots(19)$$

$$\Gamma=\left\{\xi:\xi\in C,\left|\frac{1+\xi/2}{1-\xi/2}\right|=1\right\} \qquad \ldots(20)$$

Lemma 4.6: [7]

Assume $\sigma(\overline{A})\subseteq\Sigma$. Then: $\rho[(\xi I-A)^{-1}\overline{B}] \le \sup\limits_{\xi\in\Gamma} \rho[(\xi I-A)^{-1}\overline{B}],$ whenever $\xi\notin\Sigma.$

Lemma 4.7: [6,4]

For polynomial α, consider the condition: $|\alpha(z;\delta)|\le 1,$ whenever $|\mathrm{z}|=1$ & $0\le\delta<1.$

The following Theorem forms the main result of the paper.

Theorem 4.8:

If the linear multistep method (ρ,σ) is A-stable for ODEs, then the process (14) is GP-stable.

Proof:

In order to proof that the process (14) is GP-stable we have to show that: $H\subseteq S(r,s).$

Now assume that the method (ρ,σ) is A- stable for ODEs. Let) is inevitable (whenever z($Q$from lemma 2.9 it follows that $(\overline{A},\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_\mu)\in H,$

Hence, (16) is fulfilled .)$|z|\ge 1$

(i) If $\sigma(z)=0$ for some z with $|z|=1,$ then:

$$\rho\left[Q(z)^{-1}\sum_{k=1}^{\mu} p_k(z;\delta_k)\right]=\rho[0]=0<1$$

(*ii*) If $\sigma(z) \neq 0$ with $|z| = 1$, then it follows from lemma 4.4 that:

$$\operatorname{Re} \frac{\rho(z)}{\sigma(z)} \geq 0.$$

Then from the definition of the set of H, we have:

(1) When $\operatorname{Re} \frac{\rho(z)}{\sigma(z)} = 0$, $\rho\left[Q(z)^{-1} \sum_{k=1}^{\mu} p_k(z;\delta_k) \right] = \rho\left[\left(\frac{\rho(z)}{\sigma(z)} I - \overline{A} \right)^{-1} \sum_{k=1}^{\mu} \overline{B}_k \right] < 1$

(2) When $\operatorname{Re} \frac{\rho(z)}{\sigma(z)} > 0$, then $\frac{\rho(z)}{\sigma(z)} \notin \Sigma$.

from lemma 4.6 and $\sigma(\overline{A}) \subseteq \Sigma$, we arrive at:

$$\rho\left[Q(z)^{-1} \sum_{k=1}^{\mu} p_k(z;\delta_k) \right] = \rho\left[\left(\frac{\rho(z)}{\sigma(z)} I - \overline{A} \right)^{-1} \sum_{k=1}^{\mu} \overline{B}_k \right]$$

$$\leq \sup_{\xi \in \Gamma} \rho\left[(\xi I - \overline{A})^{-1} \sum_{i=1}^{\mu} \overline{B}_i \right] < 1$$

Hence, $\rho\left[Q(z)^{-1} \sum_{k=1}^{\mu} p_k(z;\delta_k) \right] \prec 1$. From simplification 4.3 we conclude that $H \subseteq S(r,s)$.

Thus if LMM is A-stable for ODEs $\Rightarrow$ process (14) is GP-stable for (2). And this completes the proof of this theorem. #

5 - General Conclusions:

In this paper two major properties have been proved theoretically for DDEs, namely

(*i*) The NDEs has been successfully transformed into RDEs with several delays.

(*ii*) It has been proved that the LMM is GP- stable for DDEs if it was A- stable for ODEs.

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