On Almost Weakly np – injective Rings

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ABSTRACT

The ring R is called right almost weakly np –injective, if for any $a \in N_2(R)$, there exists a positive integer n and a left ideal X_{a^n} of R such that $lr(a^n) = Ra^n \oplus X_{a^n}$. In this paper, we give some characterization and properties of almost weakly np – injective rings. And we study the regularity of right almost weakly np – injective ring and in the same time, when every simple (simple singular) right R – module is almost weakly np – injective, we also give some properties of an R.

Key words : np – injective ring ,reduced ,regular rings

حول الحلقات الغامرة بضعف من النمط - np تقريباً

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الملخص

يقال للحلقة R بأنها غامرة يمنى بضعف من النمط –npتقريباً, إذا كان لكل $a \in N_2(R)$, يوجد عدد صحيح موجب n و مثالي أيسر X_{a^n} في R بحيث أن $lr(a^n) = Ra^n \oplus X_{a^n}$. في هذا البحث أعطينا بعض مميزات وخواص الحلقة الغامرة اليمنى بضعف من النمط –npتقريباً وفي نفس الوقت, عندما كل مقاس بسيط (بسيط منفرد) أيمن عليها غامر بضعف من النمط –np, وكذلك أعطينا بعض خواصها في R.

الكلمات المفتاحية: الحلقات الغامرة من النمط – np ، مختزلة، حلقات منتظمة.

1 . Introduction

Throughout this paper, an R will be an associative ring with identity and all modules are unitary right R – modules. For right R – module M, S=End (M_R) denotes the endomorphism ring of an M. For $a \in R, r(a)(l(a))$ denote the right (left) annihilator of a. We write $Y(R)(Z(R)), N(R), N_2(R)$,
and $J(R)$ for the right (left) singular ideal, the set of all nilpotent elements, the set of all non – nilpotents and the Jacobson radical of R, respectively.

Generalization of injectivity has been discussed in many papers (See [3], [4], and [8]). A right R – module M is called generalized principally injective (GP – injective)if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any R – homomorphism of a^nR into M extends to one of R into M. A ring R is called right GP – injective if R_R is GP – injective, or equivalently $l_R r_R(a^n) = Ra^n$ for all $a \in R$ [9].

In [8], Yue Chi Ming first introduced and characterized a right np – injective ring, and gave many properties. A ring R is called right np – injective ring, if for any $c \in N_2(R)$, any right R – homomorphism $g: cR \to R$, there exists $b \in R$ such that $g(ca) = bca$ for all $a \in R$. They also continued to study rings with some other kinds of injective, namely, weakly np – injective rings [6]. A ring R is called right weakly np – injective, if for any $a \in N_2(R)$, there exists a positive integer n such that $lr(a^n) = Ra^n$ [6]. In [3], Page and Zhou introduced an almost principally injective (or AP – injective) module. Let M be a right R – module with S=End (MR). The module M is called AP – injective, if for any $a \in R$, there exists an S – Sub module X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S – modules, they also studied right AP – injective rings and gave some characterization and properties which generalized results of [10].

A right R-module M is said to be right weakly principally small injective (or WPSI), if for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any R-homomorphism from $a^n R \to M$ can be extended to R→M. A ring R is called a right WPSI ring, if RR is a right WPSI [5].

A ring R is called right weakly PP if, for each $a \in N_2(R)$, R is projective as right R – module, or equivalent $r(a) = eR$ for some $e^2 = e \in R$ [6]. R is regular (strongly regular), if for every $a \in R$, there exists $b \in R$ such that $a = aba$ ($a = a^2 b$) [1].

In this paper, we consider rings which are more general than weakly np – injective rings an idea parallel to the notion of AP – injective rings.

In the second section, we give some characterizations of right almost weakly np – injective rings, for example: Let R be a right almost weakly np – injective ring. (1) Any non zero divisor of R is left invertible. (2) If R is right WPSI, then $Y(R) = J(R)$. Also, we study regularity of right almost weakly np – injective rings. For example, if R is a right quasi duo, the following conditions are equivalent for a ring R. (1) Every right R – module is almost weakly np – injective. (2) Every cyclic right R – module is almost weakly np – injective. (3) Every simple right R – module is almost weakly np – injective. (4) Every element of $N_2(R)$ is strongly regular. (5) R is W – regular.

2. Right Almost Weakly np-ingective Rings

In this section, we consider rings which are more general than weakly np-injective rings, an idea parallel to the notions of AP-injective rings.

Definition 2.1:

A module MR with S=End (MR), is said to be right almost weakly np-injective, if for any $a \in N2(R)$, there exists $n \geq 1$ and an S-submoduleXan of M such that lMrR (an)= Man ⊕Xan as left S-module. If RR is almost weakly np – injective, then we call R is a right almost weakly np-injective ring. Clearly, right weakly np – injective rings are almost weakly np – injective.

Example

1- Let $R= \begin{bmatrix} 0 & z_2 \\ 0 & z_2 \end{bmatrix}$, $N_2(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

Let $a \in N_2(R)$. Then $lr(a) = Ra \oplus R$. So R is right almost weakly np-injective.

In this section, the following lemma, which is due to Zhao and Xianneng in [11], plays a central role in several of our proofs.

Lemma 2.2 :

Suppose M is a right R-module with S=End (M_R). If $l_M r_R(a^n) = Ma^n \oplus X_a$, where X_a is a left S-submodule of M_R. Set f: $a^n R \to M$ be a right R-homomorphism, then $f(a^n) = ma^n + x$ withm$\in$M, $x \in X_a$.

We have the following theorem.

Theorem 2.3:

Let R be a right almost weakly np-injective ring. Then,

1) Any right regular element of an R is left invertible.
2) $Y(R) \subseteq J(R)$.
3) If P is a reduced principal right ideal of R, then P = eR

Proof :

(1) Let $a \in R$ such that $r(a^n) = 0$ for some a positive integer n. Then, $a \in N_2(R)$. Since, R is right almost weakly np-injective, then $lr(a^n) = Ra^n \oplus X_a^n$, where X_a^n is a left ideal of R. Hence, $R = l(0) = Ra^n \oplus X_a^n$, $(r(a^n) = 0)$. So, $1 = ba^n + x$for some $b \in R$ and $x \in X_a^n$, $a^n = a^n b a^n + a^n x$, $(a^n - a^n ba^n) = a^n x \in Ra^n \cap X_a^n = 0$, $a^n(1 - ba^n) = 0$, $1 - ba^n \in r(a^n) = 0$. Thus $1 = ba^n$. This shows that $R = Ra^n$.

(2) Let $x \in Y(R)$, $a \in R$, then $r(1-ax) = 0$, implies that $b(1-ax) = 1$, for some $b \in R$ by (1). Therefore, $x \in J(R)$.

(3) Let $P = cR$, $c \in R$, be a non zero reduced principal right ideal. Since $c^2 \in N_2(R)$ and R is almost weakly np-injective, then there exists a positive integer n such that $Rc^n \oplus X_c^n = lr(c^{2n}) = Rc^{2n} \oplus X_c^{2n}$. $X_c^{2n} \leq R$ [P is reduced $r(c^n) = r(c^{2n})$]. So there exists $r \in R$, $x \in X_c^{2n}$ such that $c^n = rc^{2n} + x$, $c^{2n} = c^n rc^{2n} + c^n x$, $c^n x = (1 - c^n r) c^{2n} \in R$, $c^{2n} \cap X_c^{2n} = 0$, $c^{2n} = c^n rc^{2n}$, $1 - c^n r \in l(c^{2n}) = l(c)$, $c = c^n rc = cc^{n-1} rc = cbc$, where $b = c^{n-1} r$, whence P is generated by the idempotent $e = bc$. ■

The following lemma is given in [5]

Lemma 2.4:
If an R is a left (right) WPSI ring, then $J \subseteq Z$ [$J \subseteq Y$].

From Theorem (2.3) and Lemma (2.4) we get

Corollary 2.5:
If an R is a right WPSI and almost weakly np-injective ring. Then, $Y(R) = J(R)$.

Proof :

Since R is right WPSI ring $J(R) \subseteq Y(R)$ and by Theorem (2.3), $Y(R) = J(R)$. ■

Following [6], a ring R is said to be W – regular if $a \in aRa$ for all $a \in N_2(R)$. According to [5], a ring R is called n – regular if, for every $a \in N(R)$, $a=aba$ for some $b \in R$. Clearly, R is regular iff R is W – regular and n – regular.

Following [4], a ring R is called right SXM if, for each $0 \neq a \in R$, $r(a^n) = r(a)$ for all positive integer n satisfying $a^n \neq 0$.

Theorem 2.6:
Let R be a right SXM ring. Then, R is W-regular iff R is almost weakly np-injective and right weakly PP ring .

Proof :

Assume that for any $a \in N_2(R)$, then there exists a positive integer n such that $l_R r_R(a^n) = Ra^n \oplus X_a^n$, $X_a^n \leq R$. Since, R is right weakly PP, then $r(a^n) = (1-e)R$, $e^2 = e \in R$ by the assumption $lr(a^n) = Re = Ra^n \oplus X_a^n$, thus there exists $r \in R$, $x \in X_a^n$ such that $e = ra^n + x$,so $a^n = a^n e = a^n ra^n + a^n x$, $(1 - a^n r) a^n = a^n x \in Ra^n \cap X_a^n = 0$ and $a^n = a^n ra^n$, this implies that $(1 - ra^n) \in r(a^n) = r(a)$ (R is SXM), and $a = ara^n = ara^{n-1} a = aba$, $b = ra^{n-1} \in R$. Hence, R is W-regular.Conversely, it is clear. ■

Recall that R is a right quasi duo (MERT), [7], if every maximal (resp. maximal essential) ideal of R is a two sided ideal.

Theorem 2.7 :
If R is a right quasi duo ring. The following conditions are equivalent:

1- Every right R- module is almost weakly np-injective.
2- Every cyclic right R- module is almost weakly np-injective.

3- Every simple right R- module is almost weakly np-injective.
4- Every element of $N_2(R)$ is strongly regular.
5- R is W-regular.
Proof:

Obviously $1 \to 2 \to 3$ and $4 \to 5$, $5 \to 1$ is easy by [6,Theorem4.3]

(3) $\to$ (4). For any $0 \neq a \in N_2(R)$. We will show that aR + r(a) = R. Suppose not. Then, there exists a maximal right ideal M of R containing aR+ r(a). Since, R/M is almost weakly np-injective, then there exists a positive integer n such that $l_{R/M}\, r_R(a^n) = Ra^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$. Let $f: a^nR \to R/M$ be defined by $f(a^n r) = r + M$. Since, $aR + r(a) \subseteq M$, f is a well defined R- homomorphism. Thus, by Lemma (2.2), $f(a^n) = ca^n + M + x$, $c \in R$, $x \in X_{a^n}$ and $f(a^n) = 1 + M$ and so $1 - ca^n + M = x \in (R/M) \cap X_{a^n} = 0$, Thus, $1 \in M$ (R is quasi duo), which is a contradiction. Hence, aR+r(a)=R and az +b= 1, $z \in R$, $b \in r(a)$, so $a^2z = a$. Thus, a is strongly regular element. ■

A ring R is called biregular [9], if for any a R, RaR is generated by a central idempotent.

Theorem 2.8:

Let R be a reduced ring whose every simple singular right R-module is almost weakly np- injective. Then, R is a biregular ring.
Proof:

For any $0 \neq a \in R$, $l(RaR) = r(RaR) = l(a) = r(a)$. If $RaR \oplus r(a) \neq R$, then there exists a maximal right ideal M of R containing $RaR \oplus r(a)$. If M is not essential in R, then $M = r(e)$, $e^2 = e \in R$. Therefore, ea = 0 . Since, R is a belianae = 0. Hence, $e \in r(a) \subseteq r(e)$. Which is a contradiction . So, M is essential in R .By hypothesis, R/M is almost weakly np-injective. Since, R is reduced N(R)=0.Hence, there exists a positive integer n and $l_{R/M} r_R(a^n) = (R/M)\, a^n \oplus X_a{}^n$. Let $f: a^nR \to R/M$ be defined by $f(a^n r) = r + M$. f is a well-defined (R is reduced). Thus, by Lemma(2.2), $f(a^n) = ca^n + M + x$, $c \in R$, $x \in R$ and $f(a^n) = 1 + M$, and so $1 - ca^n + M = x \in (R/M) \cap X_a{}^n = 0$, $1 - ca^n \in M$, Since $ca^n \in RaR \subseteq M$, $1 \in M$, which is a contradiction. Hence,
$RaR \oplus r(a) = R$. R is a biregular. ■

The following definition is given in [8]

Definition 2.9:

R is called a right CAM – ring if, for any essential maximal right ideal M of R, (if it exists), for any right subideal I of M which is either a complement right subideal of M or a right annihilator ideal in R, I is an ideal of M.

Right CAM – rings generalize semisimpleartinian rings. In [8,Proposition 4], it is shown that semiprime right CAM – ring R is either semisimpleartinian or reduced.

Definition 2.10 [2]

Let I be a right (left) ideal of R. Then, R/I is a right (left) Π-flat R-modules if and only if for each $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$ ($a^n = a^n b$). The ring R is called right (left) simple Π-flat if every simple right (left) R-module is Π-flat.

The following theorem which extends [6, Theorem 5.2]

Theorem 2.11: The following conditions are equivalent for a ring R
1- R is either semi simple artinian or strongly regular.
2- R is a semi prime right CAM-ring whose singular simple right R modules are π-flat.
3- R is a semi prime right CAM-ring almost weakly np-injective.

4- R is a semi prime right MERT,CAM-ring whose simple singular right R-modules are GP-injective .
5- R is a semi prime right MERT,CAM-ring whose simple singular right R-modules are np-injective .
6-R is a semi prime right MERT,CAM-ring whose simple singular right R-modules are weakly np-injective .
7- R is a semi prime right MERT,CAM-ring whose simple singular right R-modules are almost weakly np- injective .

Proof :

(1) implies (2) and (1)$\Leftrightarrow$(3) are evident. (2)$\rightarrow$(1). If an R is not a semisimpleartinian ring, then R is reduced. Let $0 \neq a \in R$. If $aR \oplus r(a) \neq R$, then $aR \oplus r(a) \subseteq M$ for some maximal right ideal M of R. If M is not essential right ideal of R, then M =eR, where $e^2 = e \in R$. Because R is reduced, ae = ea = 0 and $e \in r(a) \subseteq M = r(e)$, a contradiction. Hence, M is an essential right ideal of R and so R/M is a simple singular right R-module. By (2), R/M is π-flat, then there exists a positive integer n and $a^n \neq 0$ such that $a^n = ba^n$ for some $b \in M$,this impliesthat $(1-b) \in l(a^n) = r(a) \subseteq M$ (R is reduced) and so $1 \in M$, a contradiction. Hence, $aR \oplus r(a) = R$ and then R is strongly regular ring.

(3)$\rightarrow$(1) : If R is not a semi simple artinian ring, then R is reduced. For any $0 \neq a \in R$, $a \in N_2(R)$ (R is reduced). Since, R is almost weakly np-injective , $lr(a^n) = Ra^n \oplus X_a^n$, where X_a^n is a left ideal of R and for some positive integer n. Hence $a^n \in lr(a^n) = lr(a^{2n}) = Ra^n \oplus X_a^n$ then $a^n = ra^{2n} + x$ for some $x \in X_a^n$, $r \in R$. Therefore, $a^{2n} = a^n ra^{2n} + a^n x$ implies that $a^{2n} - a^n ra^{2n} = a^n x \in Ra^{2n} \cap X_a^n = 0$. $a^{2n} = a^n ra^{2n}$ implies $(1- a^n r) \in l(a^{2n}) = l(a^n)$. Thus, $a = a^n ra$ which implies a=aba, where $b = a^{n-1} r \in R$. Therefore, R is strongly regular (R is reduced).

(1) $\rightarrow$(4)$\rightarrow$(5)$\rightarrow$(6)$\rightarrow$(7) are clear.

(4)$\rightarrow$(1), We can assume directly that R is reduced. Let $0 \neq a \in R$. If $aR \oplus r(a) \neq R$, then there exists a maximal right ideal L of R containing $aR \oplus r(a)$. With the similar discussion to the proof of (2$\rightarrow$1), we get L is an essential right ideal of R. By assumption, R/L is a right GP-injective. Thus, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from $a^n R \rightarrow R/L$ extends to one from R into R/L. Since, R is reduced, we can define a right R- homomorphis f: $a^n R \rightarrow R/L$ by $f(a^n r) = r+L$ for all $r \in R$. Then, there exists $c \in R$ such that $1-ca^n \in L$. Since, R is MERT, L is an ideal of R and $RaR \subseteq L$, which implies that $1 \in L$.This is a contradiction.Therefore, there exists $b \in R$, $d \in r(a)$ such that 1 = ab + d, and $a = a^2 b$. So R is strongly regularity.

(5)$\rightarrow$(1) and (6)$\rightarrow$(1) [6, Theorem (5.2)].

(7)$\rightarrow$(1), We can assume directly that R is reduced. So, N(R)= 0. Let $0 \neq a \in R$. If $aR \oplus r(a) \neq R$, Then, $aR \oplus r(a) \subseteq M$ for some maximal right ideal of R. With the similar discussion to the proof of (2$\rightarrow$1), we get M is anessential right ideal of R. Hence, R/M is a simple singular right R-module. By hypothesis, R/M is right almost weakly np-injective.Then, there exists a positive integer n and $l_{R/M} r_R (a^n) = (R/M) a^n \oplus X_a^n$, $X_a^n \leq R/M$.

Let f: $a^n R \rightarrow R/M$ be defined by $f(a^n r) = r+M$.Since R is reduced, f is a well defined R-homomorphism. Thus, by Lemma(2.2), $f(a^n) = ca^n + M + x$, $c \in R$, $x \in X_a^n$and $f(a^n) = 1+M$ and so $1- ca^n + M = x \in (R/M) \cap X_a^n = 0$, $1- ca^n \in M$.But, then $1 \in M$, because R is a MERT ring and M is an ideal. It is a contradiction. Hence, $aR \oplus r(a) = R$ and then R is strongly regular ring. ■

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