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On Types of Invo-Clean Rings: Review

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Abstract

In this paper, three types of rings were reviewed: invo-clean, invo-t-clean and invo-k-clean, the ring invo-clean is invo-t-clean and invo-k-clean. Since invo-t-clean ring is invo-k-clean when $k = 3$. Various examples representing elements of the three rings were presented, as were other rings in Z_n that satisfy the three types, especially for the invo-k-clean ring, where different examples of rings were taken at different values of k. There are properties that all rings share and others that differ among them, the most important of which is characteristic, where the invo-clean ring is 24, and the invo-t-clean ring is 120 while invo-k-clean is difficult to prove.

Keywords:
invo-clean, invo-t-clean, invo-k-clean, involution, tripotent.

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1. Introduction

Throughout this review, R is associated ring with identity. $Idm(R) = \{e \in R: e^2 = e\}$, $Invo(R) = \{v \in R: v^2 = 1\}$, $Tri(R) = \{t \in R: t^3 = t\}$, $k - potent(R) = \{p \in R: p^k = p, k \in \mathbb{N}\}$, Z_n the set of integer number modulo n.

The definition of the invo-clean in 2017 appeared to the researcher Danchev in [1], where this definition is a particular case of the definition of the clean ring. In contrast, the last definition appeared in 1977 in [2], by which he adopted the sum of the idempotent with the unit element as the clean element. Danchev replaced the unit element with the involution element to get the invo-clean element. While each the invo-clean ring is also clean, the inverse does not hold true.

Several studies by researcher Danchev dealt with the definition of invo-clean, as the corners of the invo-clean rings were studied in his research [3]. He got that if the ring R is invo-clean, then the ring eRe is invo-clean for the idempotent element e in the ring R. In addition, he also showed that if the ring of square matrices of dimension n, where $n \in \mathbb{N}$ defined on the ring R is invo-clean, the ring R will also be the invo-clean. His research [4] proved that every weakly tripotent ring would be strongly invo-clean and vice versa. Many studies dealt with this definition; see [5], [6], [7].

Two definitions later emerged: invo-t-clean and invo-k-clean therefore, where the definition appeared as invo-t-clean at the end of October 2022 [8]. This is generalized to invo-clean rings, where researchers have replaced the idempotent element with the tripotent element.

Now, let's go back to the definition of invo-k-clean, which appeared at the end of December 2022 for the Rashedi in [9], where this researcher replaced the idempotent element of the invo-clean element with the k-potent element. This definition is a representation of the two previous definitions when $k = 2$, the invo-k-clean ring is also invo-clean, and when $k = 3$, the invo-k-clean ring is also invo-t-clean, and when $k \geq 4$ is not studied from the others and its relation with the previous two rings. This definition is, therefore, a good representation of prior definitions.

2. Types Of Invo-Clean Rings

We will present the main definitions and take up in this research and review them according to their historical sequence, which are invo-clean, involution-t-clean, and invo-k-clean rings respectively.

Definition 2.1 invo-clean ring [1]

An element r in the ring R is invo-clean if $r = v + e$ where $v \in Invo(R)$ and $e \in Idm(R)$. $invo_c(R)$ denoted the set of invo-clean elements in R. If $invo_c(R) = R$, then we call R invo-clean ring. The element r is said to be strongly invo-clean if $ve = ve$. when every element in the ring R is strongly invo-clean, we call R strongly invo-clean ring.

Definition 2.2 involution-t-clean ring [8]

An involution-t-clean element (in short, invo-t-clean) is the element r in the ring R such that $r = v + t$ where $v \in Invo(R)$ and $t \in Trip(R)$. $invo_{tc}(R)$ denoted the set of every invo-t-clean elements in the ring R.$invo_{tc}(R) = R$, then we called R invo-t-clean ring. The element r in the ring R is called strongly invo-t-clean if $vt = vt$. when every element in R is strongly invo-t-clean, we call R strongly invo-t-clean ring.

Definition 2.3 invo-k-clean ring [9]

An element r in the ring R is called invo-k-clean if it will be written as $r = v + p$ where $v \in Invo(R)$ and $p \in k - potent(R)$. The set of all invo-k-clean elements in R denoted by $invo_{kc}(R)$. If $invo_{kc}(R) = R$, then we called R invo-k-clean ring. An element r in the ring R is said to be strongly invo-k-clean if $vp = vp$. If every element in R is strongly invo-k-clean, we call R strongly invo-k-clean ring.

We will provide examples of rings containing elements of the three types, but to calculate these species, we must calculate elements of involution, impotent, and tripotent. Also, we calculate the k-potent element for different values of k.

Examples 2.4:

The examples presented here are characterized by different elements at each Idempotent, tripotent plus k-potent values that k equals 4,5,6,7,8.

1. For $Z_{61,}$ we have the elements for each type shown below.

$Idm(Z_{61}) = \{0,1\}$

$Tri(Z_{61}) = \{0, 1, 60\}$,

$4 - potent(Z_{61}) = \{0, 1, 13, 47\}$

$5 - potent(Z_{61}) = \{0, 1, 11, 50, 60\}$

$6 - potent(Z_{61}) = \{0, 1, 9, 20, 34, 58\}$

$7 - potent(Z_{61}) = \{0, 1, 13, 14, 47, 48, 60\}$

$8 - potent(Z_{61}) = \{0, 1\}$

$Invo(Z_{61}) = \{1, 60\}$

$invo_c(Z_{61}) = \{0, 1, 2, 60\}$

$invo_{tc}(Z_{61}) = \{0, 1, 2, 59, 60\}$

$invo_{4c}(Z_{61}) = \{0, 1, 2, 12, 14, 46, 48, 60\}$

$invo_{5c}(Z_{61}) = \{0, 1, 2, 10, 12, 49, 51, 59, 60\}$

$invo_{6c}(Z_{61}) = \{0, 1, 2, 35, 33, 8, 10, 19, 21, 57, 59, 60\}$

$invo_{7c}(Z_{61}) = \{0, 1, 2, 12, 13, 14, 15, 48, 49, 46, 47, 59, 60\}$

$invo_{8c}(Z_{61}) = \{0, 1, 2, 60\}$

We note from this ring that the invo-clean elements are present in all other species, while the invo-t-clean elements are present in some species, such as $invo_{5c}(Z_{61})$, a$invo_{6c}(Z_{61})$ and $invo_{7c}(Z_{61})$ are not present in $invo_c(Z_{61})$, $invo_{4c}(Z_{61})$ and $invo_{8c}(Z_{61})$, in this example we note that $invo_{8c}(Z_{61}) = invo_c(Z_{61})$

2. For $Z_{175,}$ we have the set of elements for each type shown below.

$Idm(Z_{175}) = \{0, 1, 50, 126\}$

$Tri(Z_{175}) = \{0, 1, 49, 50, 76, 99, 125, 126, 174\}$,

$4 - potent(Z_{175}) = \{0, 1, 25, 50, 51, 100, 126, 151\}$

$$5 - potent(Z_{175}) = \left\{\begin{matrix} 0, 1, 7, 43, 49, 50, 57, 76, 99, 118, \\ 125, 126, 132, 168, 174 \end{matrix}\right\}$$

$$6 - potent(Z_{175}) = \left\{\begin{matrix} 0, 1, 21, 36, 50, 56, 71, 91, 106, \\ 126, 141, 161 \end{matrix}\right\}$$

$$7 - potent(Z_{175}) = \left\{\begin{matrix} 0, 1, 24, 25, 26, 49, 50, 51, 74, 75, \\ 76, 99, 100, 101, 124, 125, 126, \\ 149, 150, 151, 174 \end{matrix}\right\}$$

$8 - potent(Z_{175}) = \{0, 1, 50, 126\}$

$Invo(Z_{175}) = \{1, 76, 99, 174\}$

$$invo_c(Z_{175}) = \left\{\begin{matrix} 0, 1, 2, 99, 100, 76, 77, 174, 49, \\ 50, 51, 149, 27, 125, 126, 127 \end{matrix}\right\}$$

$$invo_{tc}(Z_{175}) = \left\{\begin{matrix} 0, 1, 2, 148, 149, 23, 152, 26, 27, 173, 174, \\ 48, 49, 50, 51, 75, 76, 77, 98, 99, 100, 124, \\ 125, 126, 127 \end{matrix}\right\}$$

$$invo_{4c}(Z_{175}) = \left\{\begin{matrix} 0, 1, 2, 149, 150, 152, 24, 26, 27, 174, \\ 49, 50, 51, 52, 75, 76, 77, 99, \\ 100, 101, 124, 125, 126, 127 \end{matrix}\right\}$$

$$invo_{5c}(Z_{175}) = \left\{\begin{matrix} 0,1,2,131,133,6,8,142,19,148, \\ 149,23,152,26,27,156,33,167,169, \\ 42,44,173,174,48,49,50,51,56, \\ 58,69,75,76,77,83,92,98,99, \\ 100,106,117,119,124,125,126,127 \end{matrix}\right\}$$

$$invo_{6c}(Z_{175}) = \left\{\begin{matrix} 0,1,2,132,7,135,140,142,15,147, \\ 20,149,22,27,155,30,160,162,35, \\ 37,167,42,170,174,49,50,51, \\ 55,57,62,65,70,72,76,77,85, \\ 90,92,97,99,100,105,107,112, \\ 120,125,126,127 \end{matrix}\right\}$$

$$invo_{7c}(Z_{175}) = \left\{\begin{matrix} 0,1,2,148,149,150,151,152,25, \\ 26,27,23,24,173,174,48,49, \\ 50,51,52,73,74,75,76,77,98, \\ 99,100,101,102,123,124, \\ 125,126,127 \end{matrix}\right\}$$

$$invo_{8c}(Z_{175}) = \left\{\begin{matrix} 0,1,2,99,100,76,77,174,49, \\ 50,51,149,27,125,126,127 \end{matrix}\right\}$$

We note from this ring that the invo-clean elements are present in all other species, while the invo-t-clean elements are present in some species, such as $invo_{5c}(Z_{175})$, and $invo_{7c}(Z_{175})$ are not present in $invo_{c}(Z_{175})$, $invo_{4c}(Z_{175})$, $invo_{6c}(Z_{175})$ and $invo_{8c}(Z_{175})$, in this example we note that also $invo_{8c}(Z_{175}) = invo_{c}(Z_{175})$

The reason that elements of type invo-t-clean are present in elements of type $invo_{5c}(R)$ and $invo_{7c}(R)$ is that the tripotent element in invo-t-clean is $t^3 = t$, so $t^5 = t^3 t^2 = t$ in similar way $t^7 = t$ and this is true for any $t^{2n+1} = t$ and that for any $n \in \mathbb{N}$.

The following examples present rings of the three types in Z_n where n is mentioned, all of which satisfy the type required.

Examples 2.5:

The following examples of the rings in Z_n and we're going to mention at what value n is going to be invo-clean, invo-t-clean and invo-k-clean at different values of k. $n = \{2,3,4,6,8,12,24\}$ are invo-clean.

$n = \{2,3,4,5,6,8,10,12,15,20,24,30,40,60,120\}$ are invo-t-clean.

$n = \{2,3,4,6,8,12,24\}$ are invo-4-clean.

$n = \{2,3,4,5,6,8,10,12,15,20,24,30,40,60,120\}$ are invo-5-clean.

$n = \{2,3,4,6,8,12,24\}$ are invo-6-clean.

$$n = \left\{\begin{matrix} 2,3,4,5,6,7,8,9,10,12,14,15,18,20,21,24,28, \\ 30,35,36,40,42,45,56,60,63,70,72,84,90, \\ 105,120,126,140,168,180,210,252,280 \end{matrix}\right\}$$

are invo-7-clean ring which is less than order 300.

$n = \{2,3,4,6,8,12,24\}$ are invo-8-clean.

As in the example 2.4, we note that the invo-clean rings it will be invo-t-clean and invo-k-clean, while the invo-t-clean rings It will be invo-k-clean rings when $k = 5,7$. Generally, the invo-t-clean ring for Z_n is the invo-k-clean ring when k is an odd number.

Now, we'll give an example of an invo-k-clean ring, but it's not an invo-clean or an invo-t-clean.

Examples 2.6:

The ring Z_9 is invo-7-clean, but not invo-clean or invo-t-clean, since

$Idm(Z_9) = \{0,1\}$

$Tri(Z_9) = \{0,1,8\}$,

$4 - potent(Z_9) = \{0,1,4,7\}$

$5 - potent(Z_9) = \{0,1,8\}$

$6 - potent(Z_9) = \{0,1\}$

$7 - potent(Z_9) = \{0,1,2,4,5,7,8\}$

$8 - potent(Z_9) = \{0,1\}$

$Invo(Z_9) = \{1,8\}$

$invo_c(Z_9) = \{8,1,2,0\}$

$invo_{tc}(Z_9) = \{0,1,2,7,8\}$

$invo_{4c}(Z_9) = \{0,1,2,3,5,6,8\}$

$invo_{5c}(Z_9) = \{0,1,2,7,8\}$

$invo_{6c}(Z_9) = \{8,1,2,0\}$

$invo_{7c}(Z_9) = \{0,1,2,3,4,5,6,7,8\}$

$invo_{8c}(Z_9) = \{8,1,2,0\}$

Remark 2.7:

1. Danchev, in [1], said that every idempotent element is invo-clean since for $e \in Idm(R)$, $e = 2e - 1 + (1 - e)$, $(2e-1)^2 = 1$, $(1-e)^2 = 1 - e$. Since $(1-e)^3 = 1 - e$, Thus, each idempotent element is invo-t-clean. Also, in [9], Rashedi stated that each idempotent element it will be invo-k-clean, because $(1-e)^k = 1 - e$.
2. Rashedi, in [9], shows that the tripotent element in proposition 2.4 is also invo-k-clean. since if $t^3 = t$, then $1 - t^2 \in Idm(R)$ and $t^2 + t - 1 \in invo(R)$ so $t = (t^2 + t - 1) + (1 - t^2)$. The tripotent element is both invo-clean and invo-k-clean, as well as invo-t-clean.

We can ask when the k-potent element is invo-clean. The answer is true when $k = 2,3$ as shown above, but the k-potent element is invo-clean in the case for every element in the ring R is k-potent is shown Rashedi in Lemma 2.2 in [9], but it has a printing error she writes "$a = 1 + (1 - a)$," and the write is $a = 1 + (-1 + a)$. This error is repeated in Lemma 2.3.

We discuss three results in [9], which we present below

"Let R be a ring in which every element is k-potent. Then R is invo-k-clean." Lemma 2.2.

"Let $2 \leq k \in \mathbb{N}$ and R be a ring in which every element is involution. Then R is invo-k-clean" Lemma 2.3.

"Let $2 \leq k \in \mathbb{N}$. Then every tripotent ring is invo-k-clean" Proposition 2.4.

We see that Lemma 2.3 and Proposition 2.4 are special cases from Lemma 2.2. since every involution element is tripotent and the last one is 3-potent. So we can reduce the three results by the first one. In the second and third results, the research gets the invo-k-clean ring, but it can get better than the result, which is invo-clean, since every tripotent element is invo-clean.

3. Characteristics of rings

For studying the characteristics of the rings on the three types of rings. We present the result below.

- The first result in [1], Lemma 2.2. "If R is an invo-clean ring, then 24=0 in particular, $6 \in Nil(R)$." $Nil(R)$ is represent the set of nilpotent elements in the ring R. This is true when R is the invo-clean ring, but in the proof, one element is used, which is 3, and is the invo-clean, so the proof can be limited to this element instead of the entire ring to get it is 24=0, and that $6 \in Nil(R)$.

Example 2.8

So let's have the 2×2 upper triangle matrix ring defined on Z_4 where this example is mentioned in [8]. The researcher explained that this ring is not an invo-t-clean, so this ring needs not to be invo-clean, but the element $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is a tripotent element, so this element is the invo-clean. It is clear that $4 \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $24 \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ which is nilpotent.

- The second result in [8], Proposition 2.4. "Let 3 be an invo-t-clean element in a ring R, then $15t = 0$ and 120=0". The researchers in [8], in proposition 2.4, referred to element 3 in the ring R, when he was invo-t-clean and got the results without relying entirely on all elements in the ring to get 120=0 since it's $3 = v + t$ where $v \in invo(R)$ and $t \in Tri(R)$) so we get that $15t = 0$. Whoever mentioned that any element is nilpotent, but we'll clarify that element 30 will be nilpotent. $(30)^3 = 120 \cdot 225 = 0$.

- The third result in [8] is Proposition 2.7. "Let $2 \leq k \in \mathbb{N}$ and R be an invo-k-clean ring. Then $30 \in Nil(R)$". The researcher in [8], in proposition 2.7, This episode is confirmed when R is the invo-clean ring, but in the proof, one element is used, which is 3, which is the invo-k-clean, so the proof can be limited to this element instead of the entire ring to get $30 \in Nil(R)$. The proof of the proposition is true when $k = 2$, but when $3 \leq k$, the proof needs not be true because if the proof takes $(3 - e)^2 = 1$, then the researcher gets that $8 = 5e$ where $e \in k - potent$. If we take $k = 3$, then $e^3 = e$, then it is not necessary that $e^2 = e$ we have that $1 = (3 - e)^2 = 3 - 6e + e^2$. The following example shows a ring that invo-k-clean but $8 \neq 5e$.

Example 2.9

we see that ring Z_{30} is invo-3-clean, but it is not invo-clean since

$$Idm(Z_{30}) = \{0, 1, 6, 10, 15, 16, 21, 25\}$$

$$Tri(Z_{30}) = \begin{Bmatrix} 0, 1, 4, 5, 6, 9, 10, 11, 14, 15, 16, 19, \\ 20, 21, 24, 25, 26, 29 \end{Bmatrix}$$

$$Invo(Z_{30}) = \{1, 11, 19, 29\}$$

$$invo_c(Z_{30}) = \begin{Bmatrix} 0, 1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 14, \\ 15, 16, 17, 19, 20, 21, 22, \\ 24, 25, 26, 27, 29 \end{Bmatrix}$$

$$invo_{tc}(Z_{30}) = \begin{Bmatrix} 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \\ 13, 14, 15, 16, 17, 18, 19, 20, 21, \\ 22, 23, 24, 25, 26, 27, 28, 29 \end{Bmatrix}$$

We see that element 3 is not an invo-clean element but it is invo-t-clean so it is invo-3-clean and the ring Z_{30} is invo-3-clean $3 = 29 + 4$, so $(3 - e)^2 = 1$ where $e^3 = e$, so $(3 - 4)^2 = 1$ but the result which the research get is $8 = 5e$, $8 = 5 \cdot 4 = 20$ which is a contradiction since $8 \neq 20$ in Z_{30}. The result of $30 \in Nil(R)$ is still true for this ring.

4. Conclusion

There are relationships between the three rings invo-clean, invo-t-clean and invo-k-clean, so that the invo-clean rings lead to the invo-t-clean and invo-k-clean, while the invo-t-clean rings lead to the invo-k-clean when k is an odd integer number.

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