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Semi-$\hbar$-Open Sets and Semi-$\hbar$-Continuity in Topological Spaces

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Abstract

In this study, we generalized two known open sets, called: $semi$-open and $\hbar$-open sets to find a new type called $semi$-$\hbar$-open set. In topological space, we have presented the relationship of several famous open sets to this set. We have also studied the $semi$-$\hbar$-open continuity in topological space.

Keywords: $semi$-open, $\hbar$-open, $semi$-$\hbar$-open, $semi$-$\hbar$-continuity.

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1. Introduction

In [5], Levin gave the term semi-open for the set $\mathcal{A}$, in the topological space $(\mathcal{X},\tau)$ and studied its topological properties; it is a term given to the set if $\mathcal{A}$ fulfills the condition $\mathcal{A} \subseteq cl(int(\mathcal{A}))$. Since then, it has been widely explored in several literary works [4, 6, and 10]. Njastad [9] introduced a set called α-open; if $\mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{A} \subseteq int(cl(int(\mathcal{A})))$, since then, this notion has been studied by many authors see [3, 5 and 8]. The concept of the $\hbar$-open of the set $\mathcal{A}$ was introduced by [1]; if $\mathcal{A} \subseteq \mathcal{X}$ and if for every $\mathcal{G} \neq \emptyset, \mathcal{X}$, and $\mathcal{G} \in \tau$ such that $\mathcal{A} \subseteq int(\mathcal{A} \cup \mathcal{G})$. Abdullah and etc. [2], introduced and defined the $\hbar\alpha$-open set and studied their properties.

2. Semi-$\hbar$-Open Sets

Definition (2.1): For the set $\mathcal{A} \subseteq \mathcal{X}$ in $(\mathcal{X},\tau)$ is $semi$-$\hbar$-open, if there exists $\mathcal{H}$ set which is $\hbar$-open such that $\mathcal{H} \subseteq \mathcal{A} \subseteq cl(\mathcal{H})$.

Example (2.2): Let $\mathcal{X} = \{1,2,3\}$, and $\tau = \{\emptyset, \mathcal{X}, \{2\}, \{1,2\}\}$ Then
$\tau^{\hbar} = \{\emptyset, \mathcal{X}, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}$ represent $\hbar$-open sets
$\tau^{s\hbar} = \{\emptyset, \mathcal{X}, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ represent $semi$-$\hbar$-open sets.
In this example, the $\tau^{\hbar} \neq \tau^{s\hbar}$ but $\tau^{s\hbar}$ isn't equal to the power set of $\mathcal{X}$. Clearly that $\tau^{s\hbar}$ of this example isn't T.S. because $\{1,3\} \cap \{2,3\} = \{3\}$ isn't $semi$-$\hbar$-open set.

Proposition (2.3): A subset $\mathcal{A}$ in $(\mathcal{X},\tau)$ is $semi$-$\hbar$-open if and only if $\mathcal{A} \subseteq cl\big(int_{\hbar}(\mathcal{A})\big)$.

Proof: Let $\mathcal{A} \subseteq cl\big(int_{\hbar}(\mathcal{A})\big)$. Then take $\mathcal{H} = int_{\hbar}(\mathcal{A})$. We have $\mathcal{H} \subseteq \mathcal{A} \subseteq cl(\mathcal{H})$. Conversely, let the set $\mathcal{A}$ be an $semi$-$\hbar$-open, then there exists $\mathcal{H}$ which is $\hbar$-open, where $\mathcal{H} \subseteq \mathcal{A} \subseteq cl(\mathcal{H})$ we have $\mathcal{H} \subseteq int_{\hbar}(\mathcal{A})$ therefore $cl(\mathcal{H}) \subseteq cl\big(int_{\hbar}(\mathcal{A})\big)$, so $\mathcal{A} \subseteq cl\big(int_{\hbar}(\mathcal{A})\big)$.

Proposition (2.4): Every open set is $semi$-$\hbar$-open.

Proof: Let $\mathcal{G}$ be an open subset of $(\mathcal{X},\tau)$, by [1, Theorem 2.1], $\mathcal{G}$ is an $\hbar$-open set. So $\mathcal{G} \subseteq \mathcal{G} \subseteq cl(\mathcal{G})$.

Reverse of the aforementioned proposition is false as given in Example (2.2), {1}, {1,3} and {2,3} are $semi$-$\hbar$-open sets. But it is not open sets.

Proposition (2.5): for all $\hbar$-open set in any T.S. is $semi$-$\hbar$-open.

Proof: Let $\mathcal{H}$ an $\hbar$-open subset of $(\mathcal{X},\tau)$, since $\mathcal{H} \subseteq \mathcal{H} \subseteq cl(\mathcal{H})$. So $\mathcal{H}$ is $semi$-$\hbar$-open.

Reverse of the aforementioned proposition is incorrect as given in Example (2.2), {2,3} is $semi$-$\hbar$-open but not $\hbar$-open.

Theorem (2.6): The union of any family of $semi$-$\hbar$-open sets is $semi$-$\hbar$-open.

Proof: Let's consider a collection $\{\mathcal{A}_i\}_{i\in I}$ of $semi$-$\hbar$-open sets, where each $\mathcal{A}_i$ is $semi$-$\hbar$-open. For each $\mathcal{A}_i$, there is an $\hbar$-open set $\mathcal{H}_i$ so that means $\mathcal{H}_i \subseteq \mathcal{A}_i \subseteq cl(\mathcal{H}_i)$. Now, consider the family of $\hbar$-open sets $\{\mathcal{H}_i\}_{i\in I}$. Let $\mathcal{H} = \bigcup_{i\in I} \mathcal{H}_i$. $\mathcal{H}$ is $\hbar$-open. Since $= \bigcup_{i\in I} \mathcal{A}_i$, for any point $z \in \mathcal{U}$, there exists $j \in I$ such that $z \in \mathcal{A}_j$. Since $\mathcal{A}_j$ is $semi$-$\hbar$-open, we have $\mathcal{A}_j \subseteq cl(\mathcal{H}_j)$, where $\mathcal{H}_j$ is $\hbar$-open. Therefore, $z \in cl(\mathcal{H}_j)$, and consequently, $\mathcal{U} \subseteq \bigcup_{i\in I} cl(\mathcal{H}_i) = cl(\mathcal{H})$. We have shown that $\mathcal{U} = \bigcup_{i\in I} \mathcal{A}_i$ is $semi$-$\hbar$-open, as there is an $\hbar$-open set $\mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{U} \subseteq cl(\mathcal{H})$.

Remark (2.7): The intersection of $semi$-$\hbar$-open sets need not be $semi$-$\hbar$-open as shown in Example (2.2) $\{1,3\} \cap \{2,3\} = \{3\}$ and {3} is not $semi$-$\hbar$-open set.

Proposition (2.8): Every $semi$-open sets are $semi$-$\hbar$-open.

Proof: Let $\mathcal{S}$ be a $semi$-open subset in $(\mathcal{X},\tau)$, then there exist $\mathcal{G} \in \tau$, such that $\mathcal{G} \subseteq \mathcal{S} \subseteq cl(\mathcal{G})$. Since $\mathcal{G}$ is also $\hbar$-open set by [1. Theorem 2.1] we get that $\mathcal{S}$ is a $semi$-$\hbar$-open set.

Reverse of the aforementioned proposition is incorrect, as in example 2.9.

Example (2.9): Let $\mathcal{X} = \{1,2,3,4\}$, and

$\tau = \{\emptyset, \mathcal{X}, \{4\}, \{3,4\}\}$

$\tau^{s} = \{\emptyset, \mathcal{X}, \{3,4\}, \{1,4\}, \{2,3,4\}, \{1,2,4\}, \{2,4\}, \{1,3,4\}, \{4\}\}$

$\tau^{\hbar} = \{\emptyset, \mathcal{X}, \{3,4\}, \{1,2,3\}, \{3\}, \{4\}\}$

$\tau^{s\hbar} = \{\emptyset, \mathcal{X}, \{3,4\}, \{1,4\}, \{2,3,4\}, \{1,2,4\}, \{3\}, \{2,4\}, \{1,2,3\}, \{1,3\}, \{1,3,4\}, \{4\}\}$

We see that the $semi$-$\hbar$-open sets {3},{1,2,3},{1,2} are not $semi$-open.

Theorem (2.10): Let $\mathcal{A}$ be a $semi$-$\hbar$-open set in $(\mathcal{X},\tau)$ if there exists $\mathcal{B} \subseteq \mathcal{X}$, with $\mathcal{A} \subseteq \mathcal{B} \subseteq cl(\mathcal{A})$ we get $\mathcal{B}$ is $semi$-$\hbar$-open.

Proof: Let $\mathcal{A} \subseteq \mathcal{B}$ be a subset in $(\mathcal{X},\tau)$, $\mathcal{A}$ is $semi$-$\hbar$-open set, then there exists $\mathcal{H} \subseteq \mathcal{X}$, $\mathcal{H}$ is $\hbar$-open set with $\mathcal{H} \subseteq \mathcal{A} \subseteq cl(\mathcal{H})$, now by $\mathcal{H} \subseteq \mathcal{A} \subseteq \mathcal{B}$ so $\mathcal{H} \subseteq \mathcal{B}$, also $\mathcal{A} \subseteq cl(\mathcal{H})$ implies $cl(\mathcal{A}) \subseteq cl\big(cl(\mathcal{H})\big) = cl(\mathcal{H})$ since $\mathcal{B} \subseteq cl(\mathcal{A})$ so $\mathcal{B} \subseteq cl(\mathcal{H})$, then $\mathcal{H} \subseteq \mathcal{B} \subseteq cl(\mathcal{H})$, so $\mathcal{B}$ is $semi$-$\hbar$-open.

Example (2.11): Let $\mathcal{X} = \{1,2,3,4\}$,

$\tau = \{\emptyset, \mathcal{X}, \{2\}, \{2,3,4\}, \{2,3\}, \{1,2,4\}, \{3\}, \{2,4\}\}$

$\tau^{c} = \{\{1,4\}, \{1,2,4\}, \{3\}, \{1\}, \emptyset, \{1,3\}, \{1,3,4\}\}$

$\tau^{\hbar} = \{\{2\}, \{2,3,4\}, \{2,3\}, \{1,2,4\}, \{3\}, \{2,4\}, \mathcal{X}, \emptyset\}$

$\tau^{s\hbar} = \{\{2\}, \{2,3,4\}, \{2,3\}, \{1,2,4\}, \{1,2\}, \{3\}, \{2,4\}, \emptyset, \mathcal{X}, \{1,2,3\}\}$

We see that {2} is $semi$-$\hbar$-open set $cl(\{2\}) = \{1,2,4\}$ and $\{2\} \subseteq \{2,4\} \subseteq cl(\{2\})$, $\{2\} \subseteq \{1,2\} \subseteq cl(\{2\})$. So {2,4},{1,2} is $semi$-$\hbar$-open.

Proposition (2.12): Every α-open set in $(\mathcal{X},\tau)$ is $semi$-$\hbar$-open.

Proof: Let $\mathcal{G}$ be an α-open subset in $(\mathcal{X},\tau)$, so $\mathcal{G} \subseteq int\Big(cl\big(int(\mathcal{G})\big)\Big) \subseteq cl\big(int(\mathcal{G})\big)$ so $int(\mathcal{G}) \subseteq \mathcal{G} \subseteq cl(int(\mathcal{G}))$ since $int(\mathcal{G})$ is $\hbar$-open set by Proposition (2.4), so $\mathcal{G}$ is $semi$-$\hbar$-open.

Reverse of the aforementioned theorem is false, as given in the next example

Example (2.13): Let $= \{3,2,1\}$, $\tau = \{\emptyset, \mathcal{X}, \{3\}\}$

$\tau^{\alpha} = \{\emptyset, \mathcal{X}, \{3\}, \{2,3\}, \{1,3\}\}$,

$\tau^{\hbar} = \{\emptyset, \mathcal{X}, \{3\}, \{1,2\}\}$

$\tau^{s\hbar} = \{\emptyset, \mathcal{X}, \{2,3\}, \{1,2\}, \{3\}, \{1,3\}\}$

We see that {1,2} $semi$-$\hbar$-open set but not α-open, also {1,2} is $\hbar$-open set but not α-open.

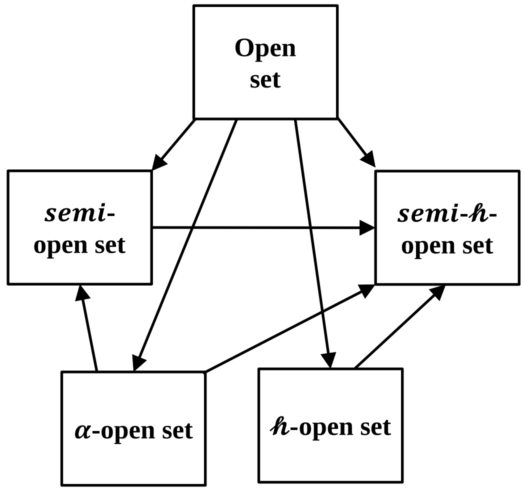

Fig. 1. The relationship of the $semi$-h-open set with some other types of open sets.

The following example show that the converse of all directions isn't true except between α-open and $semi$-open set.

Example (2.14): Let $\mathcal{X} = \{4,2,1,3\}$ and
$\tau = \{\emptyset, \mathcal{X}, \{4\}, \{1,2,4\}, \{1,4\}, \{2,4\}, \{1,3,4\}\}$
$\tau^{s} = \tau^{\alpha} = \{\emptyset, \mathcal{X}, \{3,4\}, \{2,4\}, \{2,3,4\}, \{1,2,4\}, \{2,4\}, \{1,3,4\}, \{4\}\}$
$\tau^{h} = \{\emptyset, \mathcal{X}, \{2,4\}, \{1,4\}, \{1,2,4\}, \{1,2\}, \{3\}, \{1\}, \{1,2,3\}, \{1,3\}, \{1,3,4\}, \{4\}\}$
$\tau^{sh} = \{\emptyset, \mathcal{X}, \{3,4\}, \{1,4\}, \{2,3,4\}, \{1,2,4\}, \{1,2\}, \{3\}, \{2,4\}, \{1\}, \{1,2,3\}, \{1,3\}, \{1,3,4\}, \{4\}\}$

3. Applications of *Semi-h*-Open Sets

Definition (3.1): For $\mathcal{A} \subseteq \mathcal{X}$ in $(\mathcal{X}, \tau)$. A $semi$-h-limit point $x \in \mathcal{X}$ of $\mathcal{A}$ is defined by: for every $semi$-h-$open\ \mathcal{G} \ni x$; $(\mathcal{G} \cap \mathcal{A}) \setminus \{x\} \neq \emptyset$. The semi-h-derived set of $\mathcal{A}$ is the set containing all of 's semi-h-limit points. ($\mathcal{D}_{sh}(\mathcal{A})$).

Theorem (3.2): Let $(\mathcal{X}, \tau_1)$ and $(\mathcal{X}, \tau_2)$ be T.S. such that $\tau_1^{sh} \subseteq \tau_2^{sh}$. $\mathcal{A} \subseteq \mathcal{X}$ and $x \in \mathcal{D}_{sh_2}(\mathcal{A})$, then $x \in \mathcal{D}_{sh_1}(\mathcal{A})$.

Proof: Let $x \in \mathcal{D}_{sh_2}(\mathcal{A})$. Then $(\mathcal{G} \cap \mathcal{A}) \setminus \{x\} \neq \emptyset$ for all $\mathcal{G} \in \tau_2^{sh}$ such that $x \in \mathcal{G}$. But $\tau_1^{sh} \subseteq \tau_2^{sh}$, so, in particular, $(\mathcal{G} \cap \mathcal{A}) \setminus \{x\} \neq \emptyset$, $\forall\, \mathcal{G} \in \tau_1^{sh}$ s.t. $x \in \mathcal{G}$. Hence, $x \in \mathcal{D}_{sh_1}(\mathcal{A})$.

Reverse of the aforementioned theorem is false, as in example 3.3.

Example (3.3): Let $\mathcal{X} = \{3,2,1\}$ with
$\tau_1 = \{\{1,2\}, \emptyset, \{3\}, \mathcal{X}\}$
$\tau_2 = \{\mathcal{X}, \emptyset, \{3\}\}$. Then
$\tau_1^{sh} = \{\emptyset, \{3\}, \{1,2\}, \mathcal{X}\}$ and
$\tau_2^{sh} = \{\emptyset, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \mathcal{X}\}$. Hence $\tau_1^{sh} \subseteq \tau_2^{sh}$. If $\mathcal{A} = \{1,2\}$, $1 \in \mathcal{D}_{sh_1}(\mathcal{A})$ and $1 \notin \mathcal{D}_{sh_2}(\mathcal{A})$.

Theorem (3.4): Let $F \subseteq \mathcal{B}$ be subsets of $(\mathcal{X}, \tau)$ then

(i) $\mathcal{D}_{sh}(F) \subset \mathcal{D}_{sh}(\mathcal{B})$.
(ii) $\mathcal{D}_{sh}(F) \subseteq \mathcal{D}(F)$
(iii) $\mathcal{D}_{sh}(F) \subseteq \mathcal{D}_{s}(F)$
(iv) $\mathcal{D}_{sh}(F) \subseteq \mathcal{D}_{\alpha}(F)$
(v) $\mathcal{D}_{sh}(F \cup \mathcal{B}) = \mathcal{D}_{sh}(F) \cup \mathcal{D}_{sh}(\mathcal{B})$.

Proof: The proof is easy for (i)-(iv).

(v) Since
$$\left.\begin{matrix} F \subset F \cup \mathcal{B} \\ \mathcal{B} \subset F \cup \mathcal{B} \end{matrix}\right\} \text{ imply } \left.\begin{matrix} \mathcal{D}_{sh}(F) \subset \mathcal{D}_{sh}(F \cup \mathcal{B}) \\ \mathcal{D}_{sh}(\mathcal{B}) \subset \mathcal{D}_{sh}(F \cup \mathcal{B}) \end{matrix}\right\}$$
Hence, $\mathcal{D}_{sh}(F) \cup \mathcal{D}_{sh}(\mathcal{B}) \subset \mathcal{D}_{sh}(F \cup \mathcal{B})$ ... (1)

Now, Let $x \in \mathcal{D}_{sh}(F \cup \mathcal{B})$. This means that for every $semi$-h-open set $\mathcal{G}$ containing x, $\mathcal{G} \cap (F \cup \mathcal{B}) \setminus \{x\} \neq \emptyset$. This leads to that $(\mathcal{G} \cap F) \cup (\mathcal{G} \cap \mathcal{B}) \setminus \{x\} \neq \emptyset$.

Now, consider two cases:
a. If $(\mathcal{G} \cap F) \setminus \{x\} \neq \emptyset$, then $x \in \mathcal{D}_{sh}(F)$.
b. If $(\mathcal{G} \cap \mathcal{B}) \setminus \{x\} \neq \emptyset$, then $x \in \mathcal{D}_{sh}(\mathcal{B})$.
In either case, x belongs to $\mathcal{D}_{sh}(F)$ or $\mathcal{D}_{sh}(\mathcal{B})$, so $x \in \mathcal{D}_{sh}(F) \cup \mathcal{D}_{sh}(\mathcal{B})$. Hence $\mathcal{D}_{sh}(F \cup \mathcal{B}) \subset \mathcal{D}_{sh}(F) \cup \mathcal{D}_{sh}(\mathcal{B})$...(2). From (1) and (2), we have $\mathcal{D}_{sh}(F \cup \mathcal{B}) = \mathcal{D}_{sh}(F) \cup \mathcal{D}_{sh}(\mathcal{B})$

Definition (3.5): Let $W \subseteq \mathcal{X}$ in $(\mathcal{X}, \tau)$. We say that W is $semi$-h-closed if $\mathcal{D}_{sh}(W) \subset W$.

Example (3.6): Let $\mathcal{X} = \{1,2,3\}$ and
$\tau = \{\emptyset, \mathcal{X}, \{2\}, \{2,3\}, \{3\}, \{1,3\}\}$. Then
$\tau^{sh} = \{\emptyset, \{3\}, \{2,3\}, \mathcal{X}, \{1,3\}, \{2\}\}$
If $\mathcal{E} = \{1,3\} \subset \mathcal{X}$, then $\mathcal{D}_{sh}(\mathcal{E}) = \{1\} \subseteq \mathcal{E}$. Hence, $\mathcal{E}$ is $semi$-h-closed sets.

Corollary (3.7): A subset $\mathcal{E} \subset (\mathcal{X}, \tau)$ is $semi$-h-closed if and only if $\mathcal{E}^c$ is $semi$-h-open.

Proof: Suppose that $\mathcal{E}$ is $semi$-h-closed. Let $x \in \mathcal{E}^c$ that is $x \notin \mathcal{E}$, but $\mathcal{E}$ is $semi$-h-closed, then there exists $semi$-h-$open\ \mathcal{G}_x$; $x \in \mathcal{G}_x \subset \mathcal{E}^c$, implies that for all $x \in \mathcal{E}^c$, there exists $semi$-h-$open\ \mathcal{G}_x$; $x \in \mathcal{G}_x \subset \mathcal{E}^c$. Hence $\mathcal{E}^c = \bigcup_{x \in \mathcal{E}^c} \{\mathcal{G}_x : x \in \mathcal{G}_x\}$ $semi$-h-open. Hence $\mathcal{E}^c$ is $semi$-h-open

Conversely, Assume that $\mathcal{E}^c$ is $semi\text{-}\hbar$-open. We need to explain that $\mathcal{E}$ is $semi\text{-}\hbar$-closed
for $x \in \mathcal{D}_{s\hbar}(\mathcal{E})$ with $x \notin \mathcal{E}$, so $x \in \mathcal{D}_{s\hbar}(\mathcal{E})$, $x \in \mathcal{E}^c$. But $\mathcal{E}^c$ is $semi\text{-}\hbar$-open and $\mathcal{E} \cap \mathcal{E}^c = \emptyset$. This implies that for some $semi\text{-}\hbar\text{-}open\ \mathcal{G} = \mathcal{E}^c \ni x; (\mathcal{E} \cap \mathcal{E}^c) - \{x\} = \emptyset$. Hence $x \notin \mathcal{D}_{s\hbar}(\mathcal{E})$, a contradiction. So for every $x \in \mathcal{D}_{s\hbar}(\mathcal{E}), x \in \mathcal{E}$
This imply that $\mathcal{D}_{s\hbar}(\mathcal{E}) \subset \mathcal{E}$ by Def.(3.5) $\mathcal{E}$ is $semi\text{-}\hbar$-closed.

Definition (3.8): Let $\mathcal{E} \subseteq \mathcal{X}$ in $(\mathcal{X}, \tau)$, we define the $semi\text{-}\hbar$-closure of $\mathcal{E}$, denoted by $cl_{s\hbar}(\mathcal{E})$ as: $cl_{s\hbar}(\mathcal{E}) = \bigcap\{\mathcal{F}: \mathcal{F}$ is $semi\text{-}\hbar\text{-}closed\ and\ \mathcal{E} \subseteq \mathcal{F}\}$.

Remark (3.9):

(1) Since $\mathcal{E} \subset \mathcal{F}$, for every $semi\text{-}\hbar\text{-}closed\ \mathcal{F}$ containing $\mathcal{E}$. Thus, $cl_{s\hbar}(\mathcal{E})$ is the smallest closed set containing $\mathcal{E}$.

(2) If $\mathcal{E}$ is itself closed, then $cl_{s\hbar}(\mathcal{E}) = \mathcal{E}$.

Theorem (3.10): Let $W \subseteq \mathcal{X}$ in $(\mathcal{X}, \tau)$. Then $cl_{s\hbar}(W) = W \cup \mathcal{D}_{s\hbar}(\mathcal{E})$

Proof: We have $W \subset cl_{s\hbar}(W)$ and $\mathcal{D}_{s\hbar}(W) \subset cl_{s\hbar}(W)$ implies that $\cup \mathcal{D}_{s\hbar}(W) \subset cl_{s\hbar}(W)$ ..(1)
Now, Let $x \in cl_{s\hbar}(W)$. There are two cases: If $x \in W$, so $x \in cl_{s\hbar}(W)$, since $W \subseteq cl_{s\hbar}(W)$. Otherwise if x not in $\mathcal{E}$, then x is a limit point of W (because $cl_{s\hbar}(W)$ includes all limit points of W). Hence, $x \in \mathcal{D}_{s\hbar}(W)$, and $x \in W \cup \mathcal{D}_{s\hbar}(W)$. This proves $cl_{s\hbar}(W) \subseteq W \cup \mathcal{D}_{s\hbar}(W)$…(2). From (1) and (2) we get $cl_{s\hbar}(W) = W \cup \mathcal{D}_{s\hbar}(W)$.

Proposition (3.11): Let $\mathcal{A} \subseteq \mathcal{B}$ be sets in $(\mathcal{X}, \tau)$, then

(i) $\mathcal{A} \subset cl_{s\hbar}(\mathcal{A})$.
(ii) $cl_{s\hbar}(\mathcal{A}) \subset cl_{s\hbar}(\mathcal{B})$.
(iii) $cl_{s\hbar}(\mathcal{A} \cup \mathcal{B}) = cl_{s\hbar}(\mathcal{A}) \cup cl_{s\hbar}(\mathcal{B})$.
(iv) $cl_{s\hbar}(\mathcal{A}) \subseteq cl(\mathcal{A})$
(v) $cl_{s\hbar}(\mathcal{A}) \subseteq cl_{s}(\mathcal{A})$
(vi) $cl_{s\hbar}(\mathcal{A}) \subseteq cl_{\hbar}(\mathcal{A})$
(vii) $cl_{s\hbar}(\mathcal{A}) \subseteq cl_{\alpha}(\mathcal{A})$

Proof: (i)-(ii) The proofs are easy.
(iv) Let $\mathcal{A}$ be a $semi\text{-}\hbar$-open set and let $x \in cl_{s\hbar}(\mathcal{A})$. By the definition of $semi\text{-}\hbar$-closure, x is in every $semi\text{-}\hbar$-closed set containing $\mathcal{A}$. Now, let consider the closure set $cl(\mathcal{A})$. By definition of $cl(\mathcal{A})$ and since every $semi\text{-}\hbar$-closed set is also closed, it follow that $cl_{s\hbar}(\mathcal{A})$ is a closed set containing $\mathcal{A}$, so x is also in $cl(\mathcal{A})$. Thus, we have shown that $cl_{s\hbar}(\mathcal{A}) \subseteq cl(\mathcal{A})$, as desired.
(v)-(vii) By similar to the above way.

The converse of (iv)-(vii) in the Reverse of the aforementioned proposition is not valid as in next example

Example (3.12): Let $\mathcal{X} = \{1, 2, 3\}$ and
$\tau = \{\emptyset, \mathcal{X}, \{2\}, \{1,2\}\}$
$\tau^{s} = \tau^{\alpha} = \{\emptyset, \mathcal{X}, \{2\}, \{2,3\}, \{1,2\}\}$
$\tau^{\hbar} = \{\emptyset, \mathcal{X}, \{1,2\}, \{2\}, \{1\}, \{1,3\}\}$
$\tau^{s\hbar} = \{\emptyset, \mathcal{X}, \{1,3\}, \{1,2\}, \{2,3\}, \{2\}, \{1\}\}$

Let $\mathcal{A} = \{2,3\}$, then $cl(\mathcal{A}) = cl_{s}(\mathcal{A}) = cl_{\alpha}(\mathcal{A}) = \mathcal{X}$ and $cl_{s\hbar}(\mathcal{A}) = \{2,3\}$. Clearly $cl(\mathcal{A}) = cl_{s}(\mathcal{A}) = cl_{\hbar}(\mathcal{A}) \nsubseteq cl_{sl}(\mathcal{A})$. Also let $\mathcal{B} = \{1\}$, then $cl_{\hbar}(\mathcal{B}) = \{1,3\}$ and $cl_{s\hbar}(\mathcal{B}) = \{1\}$. Clearly $cl_{\hbar}(\mathcal{B}) \nsubseteq cl_{s\hbar}(\mathcal{B})$.

Definition (3.13): Let $\mathcal{E} \subseteq \mathcal{X}$ in $(\mathcal{X}, \tau)$, we define the $semi\text{-}\hbar$-interior of a set $\mathcal{E}$, denoted by $int_{s\hbar}(\mathcal{E})$ as $int_{s\hbar}(\mathcal{E}) = \bigcup\{\mathcal{G}: \mathcal{G}\ is\ semi\text{-}\hbar$-open and $\mathcal{G} \subseteq \mathcal{E}\}$.

Proposition (3.14): Let $\mathcal{A} \subseteq \mathcal{X}$ in $(\mathcal{X}, \tau)$, then

(i) $int_{s\hbar}(\mathcal{A})$ is the largest $semi\text{-}\hbar$-open set in $\mathcal{A}$
(ii) $int_{s\hbar}(\mathcal{A}) \subset \mathcal{A}$
(iii) $int(\mathcal{A}) \subseteq int_{s\hbar}(\mathcal{A})$
(iv) $int_{s}(\mathcal{A}) \subseteq int_{s\hbar}(\mathcal{A})$
(v) $int_{\hbar}(\mathcal{A}) \subseteq int_{s\hbar}(\mathcal{A})$
(vi) $int_{\alpha}(\mathcal{A}) \subseteq int_{s\hbar}(\mathcal{A})$

Proof: By definition (3.13) we get the proofs of (i) and (ii).
(iii) Let $\mathcal{A}$ be a $semi\text{-}\hbar$-open set and let $x \in int(\mathcal{A})$, meaning there exists an open set $\mathcal{U}$ such that $x \in \mathcal{U} \subseteq \mathcal{A}$. Now, consider the semi-$\hbar$-interior set $int_{s\hbar}(\mathcal{A})$, which is the union of for all $semi\text{-}\hbar$-open sets in $\mathcal{A}$. Since $\mathcal{U}$ is an open set containing x, it is also an $semi\text{-}\hbar$-open set. Therefore, $\mathcal{U}$ is one of the $semi\text{-}\hbar$-open sets contained in $\mathcal{A}$, and $x \in int_{s\hbar}(\mathcal{A})$. Since x was an arbitrary point in $int(\mathcal{A})$. Thus, we have proved that $int(\mathcal{A}) \subseteq int_{s\hbar}(\mathcal{A})$.
(iv)-(vi) By similar to the above way.

Reverse of the Proposition (3.14) (ii)-(vi) need not be valid as in Example 3.12, Let $\mathcal{A} = \{2,3\}$, then $int(\mathcal{A}) = int_{\hbar}(\mathcal{A}) = \{2\}$ and $int_{s\hbar}(\mathcal{A}) = \{2,3\}$. Clearly, $int_{s\hbar}(\mathcal{A}) \nsubseteq int(\mathcal{A}) = int_{\hbar}(\mathcal{A})$. Also let

$\mathcal{B} = \{1,3\}$, then $int_\alpha(\mathcal{B}) = int_s(\mathcal{B}) = \emptyset$ and $int_{sh}(\mathcal{B}) = \{1,3\}$. Clearly, $int_{sh}(\mathcal{A}) \nsubseteq int_\alpha(\mathcal{B}) = int_s(\mathcal{B})$.

Theorem (3.15): Let $\mathcal{A} \subseteq \mathcal{B}$ be sets in $(\mathcal{X}, \tau)$. Then

(i) $int_{sh}(\mathcal{A}) \subset int_{sh}(\mathcal{B})$

(ii) $int_{sh}(int_{sh}(\mathcal{A})) = int_{sh}(\mathcal{A})$

Proof: The proofs are easy.

4. *Semi-ℏ*-Continuous Functions and *Semi-ℏ*-Homeomorphism

Definition (4.1): For $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is a function, if $\mathcal{S}^{-1}(\mathcal{G}) \in \tau^{sh}$ for every $\mathcal{G} \in \sigma$, then $\mathcal{S}$ is called a *semi-ℏ*-continuous function.

Example (4.2): Let $\mathcal{X} = \{3,2,1\} = \mathcal{Y}$, and
$\tau = \{\emptyset, \mathcal{X}, \{1\}, \{3\}, \{1,2\}, \{1,3\}\}$,
$\tau^{sh} = \{\emptyset, \mathcal{X}, \{1\}, \{3\}, \{1,2\}, \{1,3\}\}$ and
$\sigma = \{\emptyset, \mathcal{Y}, \{1, 3\}\}$. Clearly, if $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is the identity function, then $\mathcal{S}$ is a *semi-ℏ*-continuous.

Theorem (4.3): Any continuous function is *semi-ℏ*-continuous.

Proof. Let a continuous function be $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ and $\mathcal{G} \in \sigma$. Since, $\mathcal{S}$ is continuous, then $\mathcal{S}^{-1}(\mathcal{G}) \in \tau$. By Proposition (2.4) every open set is *semi-ℏ*-open, therefore $\mathcal{S}^{-1}(\mathcal{G})$ is *semi-ℏ*-open set in $\mathcal{X}$. So, $\mathcal{S}$ is *semi-ℏ*-continuous.
Reverse of the aforementioned theorem is not valid as seen in example 4.4.

Example (4.4): Let $\mathcal{X} = \{c, b, a\}$ and
$\mathcal{Y} = \{1,2,3\}$, $\tau = \{\{b\}, \mathcal{X}, \emptyset\}$,
$\tau^{sh} = \{\{a, c\}, \{b\}, \{b, c\}, \mathcal{X}\}$,
$\sigma = \{\emptyset, \{1\}, \mathcal{Y}, \{1,2\}\}$.
The function $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ define as $\mathcal{S}(a) = 2, \mathcal{S}(b) = 1, \mathcal{S}(c) = 3$. $\mathcal{S}$ is *semi-ℏ*-continuous, but $\mathcal{S}$ isn't continuous.

Theorem (4.5): Every *ℏ*-continuous function is *semi-ℏ*-continuous.

Proof. The proof is easy.
Reverse of the aforementioned theorem is not valid as seen in example 4.6.

Example (4.6): Let $\mathcal{X} = \{c, b, a\}$ and
$\mathcal{Y} = \{1,2,3\}$, $\tau = \{\mathcal{X}, \emptyset, \{b\}\}$,
$\tau^{h} = \{\{b\}, \{a, c\}, \emptyset, \mathcal{X}\}$ and
$\tau^{sh} = \{\{a, c\}, \{a, b\}, \{b\}, \mathcal{X}, \emptyset, \{b, c\}\}$,
$\sigma = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}\}$. We define a function $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ as $\mathcal{S}(a) = 2, \mathcal{S}(b) = 1, \mathcal{S}(c) = 3$. $\mathcal{S}$ is *semi-ℏ*-continuous, but $\mathcal{S}$ isn't *ℏ*-continuous.

Theorem (4.7): Every *semi*-continuous function is *semi-ℏ*-continuous.

Proof. The proof is easy.
Reverse of the aforementioned theorem need not be valid as given in example 4.8.

Example (4.8): Let $\mathcal{X} = \{c, b, a\}$ and
$\mathcal{Y} = \{1,2,3\}$, $\tau = \{\{a, c\}, \mathcal{X}, \emptyset\}$,
$\tau^{s} = \{\{a, c\}, \mathcal{X}, \emptyset, \}$ and
$\tau^{sh} = \{\{a, b\}, \emptyset, \mathcal{X}, \{b\}, \{b, c\}, \{a\}, \{c\}, \{a, c\}\}$,
$\sigma = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}\}$. We define a function $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ as $\mathcal{S}(a) = 1, \mathcal{S}(b) = 2, \mathcal{S}(c) = 3$. $\mathcal{S}$ is *semi-ℏ*-continuous, but $\mathcal{S}$ isn't *semi*-continuous.

Theorem (4.9): Every α-continuous function is *semi-ℏ*-continuous.

Proof. The proof is easy.
Reverse of the aforementioned theorem not valid as in example 4.10.

Example (4.10): Let $\mathcal{X} = \{c, b, a\}$ and
$\mathcal{Y} = \{1,2,3\}$, $\tau = \{\mathcal{X}, \{b, c\}, \emptyset\}$,
$\tau^{\alpha} = \{\emptyset, \mathcal{X}, \{b, c\}\}$ and
$\tau^{sh} = \{\{b\}, \{a\}, \emptyset, \{a, c\}, \{c\}, \{a, b\}, \mathcal{X}, \{b, c\}\}$,
$\sigma = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}\}$. We define a function $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ as $\mathcal{S}(a) = 1, \mathcal{S}(b) = 2, s(c) = 3$. $\mathcal{S}$ is *semi-ℏ*-continuous, but $\mathcal{S}$ isn't α-continuous.

Theorem (4.11): If $g: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is *semi-ℏ*-continuous and $\mathcal{S}: (\mathcal{Y}, \sigma) \to (\mathcal{Z}, \vartheta)$ is continuous, then $\mathcal{S} \circ g: (\mathcal{X}, \tau) \to (\mathcal{Z}, \vartheta)$ is *semi-ℏ*-continuous.

Proof: For $g: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is *semi-ℏ*-continuous and $f: (\mathcal{Y}, \sigma) \to (\mathcal{Z}, \vartheta)$ be continuous. Let $\mathcal{G} \in \vartheta$. Since, $\mathcal{S}$ is continuous, then $\mathcal{S}^{-1}(\mathcal{G}) \in \sigma$. Since, g is *semi-ℏ*-continuous, so $g^{-1}((\mathcal{S}^{-1}(\mathcal{G})) = (\mathcal{S} \circ g)^{-1}(\mathcal{G})$ is *semi-ℏ*-open set in $\mathcal{X}$. We get, $\mathcal{S} \circ g: (\mathcal{X}, \tau) \to (\mathcal{Z}, \vartheta)$ is *semi-ℏ*-continuous.

Definition (4.12): Let $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ be mapping if $\mathcal{S}(\mathcal{G}) \in \sigma^{s\hbar}$ for every $\mathcal{G} \in \tau$, then $\mathcal{S}$ is called $semi$-$\hbar$-open function.

Example (4.13): Let $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ be an identity function, where $\mathcal{X} = \mathcal{Y} = \{1, 2, 3\}$, and $\tau = \{\emptyset, \mathcal{X}, \{2, 3\}\}$, $\sigma = \{\emptyset, \mathcal{Y}, \{1\}\}$, $\sigma^{s\hbar} = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Clearly, the function $\mathcal{S}$ is $semi$-$\hbar$-open.

Theorem (4.14): Any open function is also $semi$-$\hbar$-open.
Proof. The proof is clear.

Example (4.15): In Example (4.13), the identity function $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is $semi$-$\hbar$-open but not open.

Theorem (4.16): Every $\hbar$-open function is $semi$-$\hbar$-open.
Proof. The proof is easy.
Reverse of the aforementioned theorem is not valid, as seen in the example 4.17.

Example (4.17): Let $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ be the identity function, where $\mathcal{X} = \{3, 2, 1\} = \mathcal{Y}$, and $\tau = \{\emptyset, \mathcal{X}, \{1,3\}\}$, $\sigma = \{\emptyset, \mathcal{Y}, \{1\}\}$,
$\sigma^{s\hbar} = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}\}$,
$\sigma^{\hbar} = \{\emptyset, \mathcal{Y}, \{1\}, \{2,3\}\}$.
Clearly, $\mathcal{S}$ is $semi$-$\hbar$-open but not $\hbar$-open.

Theorem (4.18): Every $semi$-open function is also $semi$-$\hbar$-open.
Proof. The proof is easy.
Reverse of the aforementioned theorem need not be valid as in example 4.19.

Example (4.19): Let $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ be the identity function, where $\mathcal{X} = \mathcal{Y} = \{1, 2, 3\}$, and $\tau = \{\emptyset, \mathcal{X}, \{2, 3\}\}$, $\sigma = \{\emptyset, \mathcal{Y}, \{1\}\}$,
$\sigma^{s\hbar} = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}\}$,
$\sigma^{s} = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}, \{1,3\}\}$.
Clearly, the function $\mathcal{S}$ is $semi$-$\hbar$-open but not $semi$-open.

Theorem (4.20): Every α-open function is $semi$-$\hbar$-open.
Proof. The proof is easy.
Reverse of the aforementioned theorem is incorrect as in example 4.21.

Example (4.21): Let $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ be an identity function, where $\mathcal{X} = \{3, 2, 1\} = \mathcal{Y}$, $\tau = \{\emptyset, \mathcal{X}, \{2, 3\}\}$, $\sigma = \{\emptyset, \mathcal{Y}, \{1\}\}$, $\sigma^{s\hbar} = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ and $\sigma^{\alpha} = \{\emptyset, \mathcal{Y}, \{1\}, \{1,2\}, \{1,3\}\}$. Clearly, the function $\mathcal{S}$ is $semi$-$\hbar$-open however, it is not. α-open.

Theorem (4.22): If $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is open and $g: (\mathcal{Y}, \sigma) \to (\mathcal{Z}, \vartheta)$ is $semi$-$\hbar$-open, then $g \circ \mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Z}, \vartheta)$ is $semi$-$\hbar$-open.
Proof. Clear

Definition (4.23): Let bijective function $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ A function S is considered a semi-h-homeomorphism when it is both semi-h-continuous and semi-h-open.

Theorem (4.24): For $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is homeomorphism, we get $\mathcal{S}$ is $semi$-$\hbar$-homeomorphism.
Proof: By Theorem (4.3), we get that every continuous mapping is semi-h-continuous. Furthermore, it can be deduced from Theorem (4.14) that every open mapping is also semi-h-open. Also, $\mathcal{S}$ is bijective then, $\mathcal{S}$ is $semi$-$\hbar$-homeomorphism.
The converse of the above theorem is incorrect as in example (4.25).

Example (4.25): Let $\mathcal{S}: (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ be the identity function, with $\mathcal{X} = \{1, 2, 3\} = \mathcal{Y}$, for $\tau = \{\mathcal{X}, \emptyset, \{1, 3\}\}$, $\tau^{s\hbar} = \{\emptyset, \mathcal{X}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2,3\}\}$, $\sigma = \{\emptyset, \mathcal{Y}, \{2,3\}\}$ and $\sigma^{s\hbar} = \{\emptyset, \mathcal{Y}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2,3\}$.
Clearly, $\mathcal{S}$ is $semi$-$\hbar$-homeomorphism, but it isn't homeomorphism.

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