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Strongly Invo. T- Clean Rings

Rand K. Zaki [1, *], Nazar H. Shuker [2]

[1,2] *Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq*
Emails: randkanaan92@gmail.com, nazarhamdoon7@gmail.com

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Abstract

In this paper, we present the idea of a strongly invo. T-clean rings, which we define as rings with every a in R having the formula $a = t + v$, where t is a tripotent and v is an order two unit that commute. In such rings fundamental properties are given, and its connection with other related rings is obtained. We consider a strongly Tri nil clean ring as a ring R with every element a in $\boldsymbol{R}$ can be expressed as a sum of a tripotent and a nilpotent that commute, and we explore the Jacobson radical over strongly Invo. T-clean ring.

Keywords:
Clean ring, Tri nil clean ring, idempotent, Tripotent

Correspondence:
Author : Rand K. Zaki
Email:randkanaan92@gmail.com

1. Introduction

All rings are taken in this paper to be associative and have an identity element. The Jacobson radical, the set of units of R, idempotents, and nilpotents are represented by the letters J(R), U(R), Id(R), and N(R) respectively. We also denote Z_n as the ring of integers modulo n and Tr(R) as the set of tripotents. In 1917 Danchev in [3] introduced the concept of invo- clean rings as a ring R with every a in R, there exists a unit v of order 2 and an idempotent η such that $a = \eta + v$. If further $\eta v = v\eta$, R is referred to be a strongly invo- clean ring.
These rings were studied by [3], [7] and [4].

2. Preliminaries

Definition 2.1 [1]. If $\eta = \eta^2$, then η is called an idempotent element of R. Clearly if η is an idempotent, then (1-η) is also idempotent, $since\ (1-\eta)^2 = 1 - 2\eta + \eta^2 = 1 - \eta$.

Example 2.2. In the ring Z_6, the idempotent elements are $\{0, 1, 3, 4\}$.

Definition 2.3 [11]. A ring element n is referred to as nilpotent if there is positive integer s. such that $a^s = 0$.

Example 2.4. In the ring Z_{16}. then the nilpotent elements of Z_{16} are $\{0, 2, 4, 6, 8, 10, 12, 14\}$.

Definition 2.5 [3]. If each element a in R, with $a = \eta + v$, where η is an idempotent element and v is a unit element of order two, R is said to be invo clean ring. If further $\eta v = v\eta$, R is called a strongly invo clean ring.

Example 2.6. The rings Z_2, Z_3, Z_4, Z_6, Z_8, are all invo –clean rings. But Z_5 and Z_7 are not invo-clean rings.

Example 2.7. Let R=$\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in Z_2 \right\}$ .The elements of R are:

$\{\delta_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \delta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \delta_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$
$\delta_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \delta_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \delta_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \delta_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$
$\delta_7 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\}, U(R) = \{\delta_5, \delta_7\}, and\ Id\ (R) = \{\delta_0, \delta_1, \delta_3, \delta_4, \delta_5, \delta_6\}$. Clearly every element in R is the form U+ Id (R). So R is invo- clean ring.

Definition 2.8 [2]. An element t is said to be a tripotent if $t = t^3$. R is called tripotent ring if all its elements are tripotent [10].

Example 2.9. The ring Z_6 is tripotent ring. Since $0^3=0$; $1^3=1$, $2^3=2$, $3^3=3$, $4^3=4$, $5^3=5$.

Lemma 2.10 [6]. If n is a nilpotent and v is a unit and $vn = nv$, then

1- 1+n is a unit.

2- v+n is a unit.

Lemma 2. 11. If t is a tripotent element, then

1- t^2 and $1-t^2$ are idempotent.

2- t^2+t-1 is a unit of order 2.

Definition 2.12 [8]. A ring R is regarded as Tri nil clean (TNC for short) ring. If each element $a \in R$ may be expressed by the formula $a = t + n$, for $t \in Tr(R)$ and $n \in N(R)$.

3. Strongly Invo. T- Clean Rings

In this part, the notion of a strongly invo.T-clean rings is introduced, along with some of its fundamental properties and its connection to other related rings.

Definition 3.1. R is considered to be strongly invo. T-clean ring (SITC for short) if every element $a \in R$ then $a = v + t$, where v is a unit element of order two , t is a tripotent element of R, and vt=tv.

Example 3.2. The rings Z_2, Z_3, Z_4, Z_5, Z_6, Z_8 are all SITC rings. But the ring of integers Z_7 is not SITC ring.

Lemma 3.3. Let R be a SITC ring. Then 360 = 0, and $30 \in Nil(R)$.

proof. Let $a = t + v$, where $t \in Tr(R)$, $v^2 = 1$ and $tv = vt$, then $3 = t + v$, $3 - v = t$, since $t^3 = t$. Then we get $(3-v)^3 = 3 - v$. $27 - 27v + 9v^2 - v^3 = 3 - v$.

$27 - 3 - 27v + v + 9v^2 - v^3 = 0.$

$24 - 26v + 9v^2 - v^3 = 0.$

Since $v^2 = 1$. $24 - 26v + 9 - v = 0$. $33 - 27v = 0$, $33 = 27v$. By squaring both sides of the equality. $1089 = 729\,v^2$, since $v^2 = 1$, $108 = 729$, so $1089 - 729 = 0$, thus $360 = 0$. Note that $(30)^3 = 0$, hence $30 \in Nil(R)$.

Theorem 3.4. Suppose R is a SITC ring, with $2 \in U(R)$ and $Tr(R) = \{0, 1, -1\}$. Then $R \cong Z_5$.

proof. Let b in R. Then b may be written as either $b = v + 1$, $b = v$ or $b = v - 1$, where v is a unit of order 2. Note that :

$$\left(\frac{v-1}{2}\right)^3 = \frac{v^3 - 3v^2 + 3v - 1}{8}$$
$$= \frac{v - 3 + 3v - 1}{8}$$
$$= \frac{4v-4}{8} = \frac{v-1}{2}.$$

Therefore $\frac{v-1}{2}$ is a tripotent, but the tripotent elements in R are 0, 1 or -1.Hence $\frac{v-1}{2} = 0, \frac{v-1}{2} = 1$ or $\frac{v-1}{2} = -1$. If $\frac{v-1}{2} = 0$, gives $v = 1$, if $\frac{v-1}{2} = 1$, gives $v = 3$, and if $\frac{v-1}{2} = -1$, gives $v = -1$. So all the elements of R are $\{0, 1, -1, 2, 3, 4\}$. But it must that $4 = -1$, since the only solution of $4(-1) = 1$ or $4.4 = 1$ is possible, so $5 = 0$ and $R = \{0, 1, 2, 3, 4\}$. Therefore $R \cong Z_5$.

Proposition 3.5. If R is a SITC ring and if $2, 5 \in U(R)$. Then R is a tripotent ring.

Proof. From Lemma (3.3), $360 = 0$. So $360 = 2^3.5^3.3 = 0$. Since $2, 5 \in U(R)$ by assumption, then $3 = 0$. For any $a \in R$, $a = t + v$, where $t = t^3$, $v^2 = 1$, and $tv = vt$. Thus

$$a^3 = t^3 + 3t^2v + 3tv^2 + v^3$$
$$= t + v = a.$$

Therefore R is a tripotent ring.

Proposition 3.6. If R is a SITC ring, and if 2 is a unit , then $J(R) = 0$.

Proof. Let $a \in J(R)$, then $a = v + t$, for $t = t^3, v^2 = 1$ and $tv = vt$. So $v^{-1}a - 1$ is a unit, say v_1 , but $a - v = t \in U(R)$. Since $t = t^3$, then $t(1 - t^2) = 0$, gives $t^2 = 1$. Further more $(a - v)^2 = t^2$ gives $a^2 - 2av + v^2 = t^2$, implies $a^2 - 2av + 1 = 1$. So $a^2 - 2av = 0$, implies $a(a - 2v) = 0$. Since $a \in J(R)$, then $a - 2v \in U(R)$, this gives $a = 0$. Therfore $J(R) = 0$.

Proposition 3.7. If 2 is a unit in a ring R. Then R is a tripotent if and only if R is a strongly invo – clean ring.

Proof. Assume R is a strongly invo- clean ring, and $a \in R$, then $a = \eta + v$, $\eta v = v\eta$ and $v^2 = 1$. Let $3 = \eta + v$, so $3 - v = \eta$, hence $(3 - v)^2 = 3 - v$, this gives $9 - 6v + v^2 = 3 - v$, implies $9 - 6v + 1 + 3 + v = 0$, so $7 = 5v$, by squaring both sides of the equality we get $49 = 25$, then 24=0, since $2 \in U(R)$. Then $3 = 0$. Now

$$a^3 = (\eta + v)^3 = \eta^3 + 3\eta^2 v + 3\eta\, v^2 + v^3$$
$$= \eta + 3\eta v + 3\eta + v$$
$$= \eta + v = a.$$

Hence $a^3 = a$. Thus R is a tripotent ring.

Conversely, suppose that $t \in Tr(R)$, then $t = t^3$. If we set $t = 1 - t^2 + (t^2 + t - 1)$. Since $1 - t^2$ is idempotent and since $t^2 + t - 1$ is a unit of order two. R is therefore a strongly invo- clean ring.

4. Strongly Tnc-Rings

In this section, we give the definition of strongly TNC-rings, and their connection with a SITC rings and other related rings.

Definition 4.1. A ring R is considered to be a strongly TNC-ring if for each a in R, $a = t + n$. where $t \in Tr(R)$, $n \in N(R)$ and $nt = tn$.

Example 4.2. In the ring Z_{12}, then $N(Z_{12}) = \{0,6\}$, while $(Z_{12}) = \{1,4,5,7,9\}$. It is easy to find that Z_{12} is a strongly TNC-ring.

proposition 4.3. Assume R is a strongly TNC ring, and let $n^2 = 2n$, for any nilpotent element n, then R is a SITC ring.

proof. Let $a \in R$. Then $1 + a = t + n$, where $t \in Tr(R)$, $n \in N(R)$ and $nt = tn$. Hence $a = t + n + 1$. Since $n + 1 \in U(R)$, say w.

So $a = t + w$. Note that $w^2 = (n+1)^2 = n^2 + 2n + 1 = 1$. R is therefore SITC ring.

proposition 4.4. If R is a SITC ring and if $2 \in N(R)$, then R is a strongly TNC ring.

proof. For any $a \in R$. Then $a - 1 = t + \beta$, where $t \in Tr(R)$, $\beta^2 = 1$ and $\beta t = t\beta$. Then $a = t + \beta + 1$. Since $2 \in N(R)$, then $2^k = 0$ for some positive integer k. Now consider $(\beta + 1)^2 = \beta^2 + 2\beta + 1 = 1 + 2\beta + 1 = 2(1 + \beta)$.

$(\beta + 1)^3 = (\beta + 1)^2 (\beta + 1) = 2(\beta + 1)(\beta + 1) = 2^2 (\beta + 1)$. Repeat this k times, we get $(\beta + 1)^k = 2^k(\beta + 1) = 0.$ *Hence* $\beta + 1 \in N(R)$ say, n. Therefore $a = t + n$.

Proposition 4.5. If R is a ring, with $3 \in N(R)$, and if for every a in R, $a = t + \beta,\ t = t^3,\ \beta^3 = 1,\ t\beta = \beta t$. Then R is strongly TNC ring.

Proof. Let a in R, then $a + 1 = t + \beta,\ t \in Tr(R),\ \beta^3 = 1$ and $t\beta = \beta t$. So $a = t + \beta - 1$.

Consider $(\beta - 1)^3 = \beta^3 - 3\beta^2 + 3\beta - 1$

$$= 1 - 3\beta^2 + 3\beta - 1$$

$$= 3\beta(1 - \beta).$$

Since $3 \in N(R)$ by assumption, then we have $(\beta - 1)^3 \in N(R)$. Thus $\beta - 1 \in N(R)$. Therefore $a = t + n$, where $n = \beta - 1 \in N(R)$.

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