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Solving Newell-Whitehead-Segel Equation By using Elzaki Transform and its inverse with The Homotopy Perturbation Method

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Abstract

This research is a combination of the homotopy perturbation method with the Elzaki transform method and Elzaki inverse to solve some nonlinear partial differential equations. The method used is proved to be an effective and easy way to solve the nonlinear from kind Newell-Whitehead-Segel partial differential equations, which are classified and belong to the category of homogeneous partial differential equations of the second degree. The Elzaki transform method and Homotopy perturbation were shown to be very potent and successful integral transform methods for solving some non-linear equations when this method was compared to other well-known methods that handle the problems under consideration.

Keywords:

(NWS) Equation; Elzaki Transform; Elzaki inverse; Homotopy Perturbation Method; Linear and Non-linear partial differential equations.

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1. Introduction

It is known that some of the phenomena formed and arised in the fields of mathematics, engineering applications, applied and mathematical physics and many other fields of science can be described by (PDEs). In physics there are some examples of the phenomenon of wave propagation, the phenomenon of heat flow and other phenomena and applications in the environment that can be described by (PDEs) [1]. Moreover, some population models are dominated by PDEs [2]. The dispersion of reactive chemicals can also be expressed by equations (PDEs). Fluid dynamics as a physical phenomenon, electricity, plasma physics phenomenon, shallow water wave propagation phenomenon, and many other phenomena and models can be expressed by (PDEs). A very useful way to describe these phenomenas in several different fields of engineering and different sciences, and their contemporary applications. Therefore, it has become necessary to have sufficient understanding and knowledge of most of the traditional approaches and methods, the stages in which they are developed, the recent development of solving (PDEs) equations, and how the methods are implemented. However, this study will be useful through our work on solving equations (PDEs) with the given conditions, which are often called initial conditions or boundary conditions for the dependent variable to arrive at solutions to equations (PDEs) [3]. There are many methods applied to many linear and nonlinear (PDEs), including the HPM method which are used lately and regarded as effective tools. In addition to many other methods used recently, such as the (ADM) method and the (HPM) method [4], the linear and non-linear (PDEs) play an important role in many engineering applications and fields in various sciences. We mention among these methods used the Laplace method and the Sumodo method, etc. which were used to solve (PDEs) [5]. Moreover, many researchers put their interest in finding solutions to nonlinear equations (PDEs) by using and applying different methods. One of these methods used by

researchers is (ADM) [6]. Moreover, the HPM method was used to solve the problem of the non-linear part [7]. It is noted that the Elzaki transformation with its inverse is completely unable to deal with and solve the problem of the nonlinear part because of the problems and difficulties caused by the nonlinear equations [8]. Other methods were used to solve the problem of the non-linear part with the method of Elzaki transform, and among them is (HPM) method and the (ADM) method, which is the method used in this research.
The HPM method was used to solve the problem of the nonlinear part of an equation Newell-Whitehead-Segel. It was found at the solution that the results obtained from the direction of convergence are distinctive and effective, and it gave very accurate results when comparing the approximate solution (U-approximate) with the exact solution (u-exact) and programming the solution using (maple) software [9]. Nevertheless, the Newell-Whitehead-Segel (NWS) equation is written in its general form as follows[10]:

$$u_t = ku_{xx} + au - bu^m$$

If $x \epsilon R$ and $t \geq 0$, and $u(x,t)$ is a function of the spatial variable x and the temporal variable. The function $u(x,t)$ can be compared to the nonlinear temperature distribution in an indefinitely long, thin rod or the fluid flow velocity in an infinitely long, narrow pipe, where a, b, and k are real values with $k > 0$ and m is a positive integer.

2. Elzaki transform with Homotopy Perturbation Method

We take an encompassing nonlinear partial differential equation using the initial conditions of the type [11, 12] to illustrate the basic idea of (HPTM).
Let's look at the starting initial condition in the Newell-Whitehead-Segel equation, which is expressed as follows:

$$u_t(x,t) = ku_{xx}(x,t) + au(x,t) - bu^m(x,t) \quad (1)$$

The Elzaki transform is applied to the two sides of Equation (1) and substituting $m = 2$, as follows:

$$E[u_t] = kE[u_{xx}] + aE[u] - bE[u^2] \quad (2)$$

By utilizing a distinctive characteristic of Elzaki transform [13], we can express Equation (2) as follows:

$$\frac{1}{r}E[u(x,t)] - ru(x,0) = kE[u_{xx}] + aE[u] - bE[u^2] \quad (3)$$

Arrange the above equation as follows:

$$E[u(x,t)] = r^2u(x,0) + rkE[u_{xx}] + raE[u] - rbE[u^2] \quad (4)$$

Equation (4) is solved by using the Elzaki inverse for the two sides:

$$E^{-1}\left[E[u(x,t)]\right] = E^{-1}[r^2\, u(x,0)] + E^{-1}\left[rkE[u_{xx}]\right] + E^{-1}\left[raE[u]\right] - E^{-1}\left[rbE[u^2]\right] \quad (5)$$

After simplifying a above equation using Elzaki's inverse [14, 15]:

$$u(x,t) = u(x,0) + E^{-1}\left[rkE[u_{xx}]\right] + E^{-1}\left[raE[u]\right] - E^{-1}\left[rbE[u^2]\right] \quad (6)$$

By taking the Homotopy Perturbation formula for the linear and non-linear parts [16], respectively:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) = u_0 + pu_1 + p^2u_2 + \cdots \quad (7)$$

$$N[u(x,t)] = \sum_{n=0}^{\infty} p^n H_n(u) \quad (8)$$

By applying each of formulas (7) and (8) to Eq. (6), as follows:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + pE^{-1}\left[rk\, E[\sum_{n=0}^{\infty} p^n H_n(u)]\right] + E^{-1}\left[ra\, E[\sum_{n=0}^{\infty} p^n u_n(x,t)]\right] - pE^{-1}\left[rb\, E[\sum_{n=0}^{\infty} p^n u_n^2]\right] \quad (9)$$

By comparing the coefficients of similar powers for p, it was obtained:

$$p^0 : u_0(x,t) = u(x,0)$$

$$p^1 : u_1(x,t) = E^{-1}\left[r\, E\left[k\frac{\partial^2 u_0}{\partial x^2} + au_0 - bH_0(u)\right]\right],$$

$$p^2 : u_2(x,t) = E^{-1}\left[r\, E\left[kp\frac{\partial^2 u_1}{\partial x^2} + au_1 - bH_1(u)\right]\right],$$

$$p^3: u_3(x,t) = E^{-1}\left[r\, E\left[k\frac{\partial^2 u_2}{\partial x^2} + a u_2 - b\, H_2(u)\right]\right], \quad (10)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$p^n: u_{n+1}(x,t) = E^{-1}\left[r\, E\left[k\frac{\partial^2 u_n}{\partial x^2} + a u_n - b\, H_n(u)\right]\right]$$

Where H_n is the polynomial that represents the non-linear terms, which represent:

$$H_n(u) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i\, u_i\right)\right]_{p=0}, n = 0, 1, 2, \dots \quad (11)$$

$$H_0(u) = u_0^2$$
$$H_1(u) = 2u_0u_1 \quad (12)$$
$$H_2(u) = 2u_0u_2 + u_1^2$$
$$H_3(u) = 2u_0u_3 + 2u_1u_2$$

3. Application

By taking the following first initial condition:

$$u(x,0) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2} \quad (13)$$

The values of $H_n(u)$ are found with $u_n\,(x,t)$, where $n = 0,1,2,3,\dots,k, a, b = 1$, as follows:

$$u_0(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2}$$

$$H_0(u) = u_0^2 = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4},$$

$$u_1(x,t) = \frac{5}{3}\,\frac{t\, e^{\frac{x}{\sqrt{6}}}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3},$$

$$H_1(u) = 2u_0u_1 = \frac{10}{3}\,\frac{t\, e^{\frac{x}{\sqrt{6}}}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^5}, \quad (14)$$

$$u_2(x,t) = \frac{25}{36}\,\frac{t^2\, e^{\frac{x}{\sqrt{6}}}\left(2\, e^{\frac{x}{\sqrt{6}}}-1\right)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4},$$

$$H_2(u) = 2u_0u_2 + u_1^2 = \frac{25}{18}\,\frac{t^2\, e^{\frac{x}{\sqrt{6}}}\left(4\, e^{\frac{x}{\sqrt{6}}}-1\right)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^6},$$

$$u_3(x,t) = -\frac{125}{648}\,\frac{t^3\, e^{\frac{x}{\sqrt{6}}}\left(-4\, e^{\frac{\sqrt{6}\,x}{3}}+7e^{\frac{x}{\sqrt{6}}}-1\right)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^5},$$

$$\vdots \qquad \vdots$$

Substituting $p = 1$, we get the approximate solution for $u(x,t)$:

$$u(x,t) = lim_{p=1}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots)$$

$$u(x,t) = \frac{1}{648}\,\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^5}\Big(648 + 1944\, e^{\frac{x}{\sqrt{6}}} + 1944\, e^{\frac{\sqrt{6}\,x}{3}} + 648\, e^{\frac{\sqrt{6}\,x}{2}} + 1080\, t\; e^{\frac{x}{\sqrt{6}}} + 2160\, t\; e^{\frac{\sqrt{6}\,x}{3}} + 1080\, t\; e^{\frac{\sqrt{6}\,x}{2}} + 900\, t^2\; e^{\frac{\sqrt{6}\,x}{2}} + 450\, t^2\; e^{\frac{\sqrt{6}\,x}{3}} - 450\, t^2\; e^{\frac{x}{\sqrt{6}}} + 500\, t^3\; e^{\frac{\sqrt{6}\,x}{2}} - 875\, t^3\; e^{\frac{\sqrt{6}\,x}{3}} + 125\, t^3\; e^{\frac{x}{\sqrt{6}}}\Big) \quad (15)$$

We can discover the exact solution within the series form:

$$u(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}-\frac{5\,t}{6}}\right)^2} \quad (16)$$

To find the exact solution, we substitute the exact solution into the equation Newell-Whitehead-Segel:

$$u_t = u_{xx} + u - u^2$$

After substituting the exact solution into the equation and simplifying it, we get the value of the exact solution as follows:

$$\frac{\partial\, u_{exact}}{\partial t} - \frac{\partial\, u_{exact}}{\partial x^2} - u_{exact} + u^2{}_{exact} = 0 \quad (17)$$

Table (1):

Numerical results for a exact solution $u(x,t)$ value of the (NWS) equation using (ET - HPM),
for $k = 1, a = 1, b = 1 \; and \; m = 2$,
$x = \{-6, -4, -2, 0, 2, 4, 6\}$,
$t = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.

x/t	0	0.1	0.2	0.3
-6	0.84736 4671490	0.858237 423468	0.86842 6740717	0.877962 493700
-4	0.69986 4857380	0.718519 101372	0.73634 8517953	0.753345 099239
-2	0.48093 1370198	0.505399 495554	0.52959 8845450	0.553431 981277
0	0.25000 0000000	0.271254 811297	0.29329 8588292	0.316042 418149
2	0.093947 0585384	0.105300 872201	0.11766 8117390	0.131077 025806
4	0.02670 6338435 3	0.030672 4100580	0.035156 8540636	0.040211 7655480
6	0.006316 4141214 1	0.007359 94354982	0.008566 5340526 7	0.0099593 0543468

x/t	0.4	0.5	0.6
-6	0.88687 5219615	0.89519 5743593	0.902954 854057
-4	0.76950 7901779	0.78484 2271947	0.799359 062983
-2	0.57680 9640615	0.59965 1587054	0.621887 181535
0	0.33938 8045459	0.36322 9616614	0.387455 619002
2	0.14554 6407699	0.16108 4395565	0.177687 367565
4	0.04589 1182175 1	0.052250 4462807	0.059345 4162921
6	0.01156 4075759 4	0.013409 5366124	0.015527 4046759

Figure 1:

$u(x,t)$ exact (u_{exact}), of the (NWS) equation using (ET-HPM),
for $k = 1, a = 1, b = 1 \; and \; m = 2$,
$x = \{-6, -4, -2, 0, 2, 4, 6\}$,
$t = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.

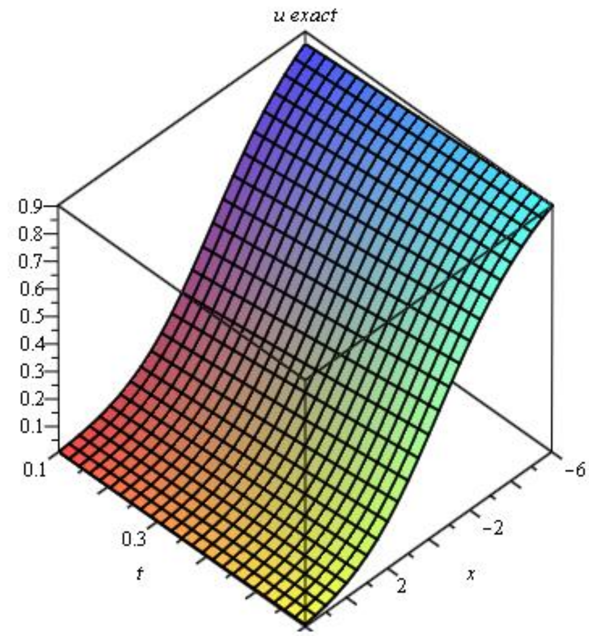

Fig. 1. (u_{exact}) of the (NWS) equation using (ET-HPM).

Table (2):

Numerical solutions for the (NWS) equation's approximate solution value utilizing (ET-HPM),
for $k = 1, a = 1, b = 1 \; and \; m = 2$,
$x = \{-6, -4, -2, 0, 2, 4, 6\}$,
$t = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.

x/t	0	0.1	0.2	0.3
-6	0.84736 4671490	0.85823 7361624	0.868425 817170	0.877958 138638
-4	0.69986 4857380	0.71851 8752587	0.736343 028822	0.753317 788647
-2	0.48093 1370198	0.50539 9246068	0.529594 666756	0.553409 940621
0	0.25000 0000000	0.27125 5304784	0.293306 327161	0.316080 729167
2	0.093947 0585384	0.10530 1159845	0.117672 912095	0.131102 261863
4	0.026706 3384353	0.030672 2953052	0.035155 0658299	0.040202 9842978
6	0.006316 41412141	0.007359 84557517	0.008564 94006464	0.0099511 0358499

x/t	0.4	0.5	0.6
-6	0.88686 2426536	0.895166 781374	0.902899 303662
-4	0.76942 3134620	0.784639 169296	0.798945 995238
-2	0.57673	0.599469	0.621497

	7376019	281307	964841
0	0.339506172839	0.363510320216	0.388020833333
2	0.145629155721	0.161293540245	0.178135362008
4	0.0458643849973	0.0521876022166	0.0592209702442
6	0.0115377421312	0.0133442616986	0.0153900682821

Figure 2:
$u(x,t)$ approximate $(U_{approximate})$
of the (NWS) equation using (ET-HPM),
for $k = 1, a = 1, b = 1$ and $m = 2$,
$x = \{-6,-4,-2,0,2,4,6\}$,
$t = \{0,0.1,0.2,0.3,0.4,0.5,0.6\}$.

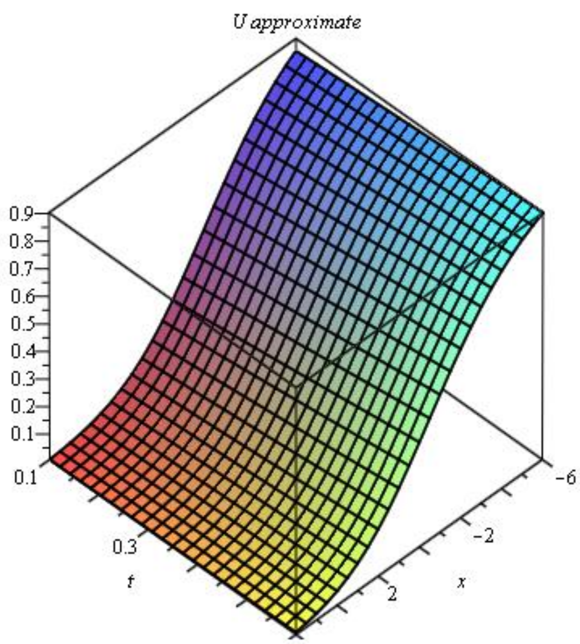

Fig. 2. $u(x,t)$ approximate of the (NWS) equation using (ET-HPM).

Table (3):

The absolute error of u_{exact} and $u_{approximate}$, of the (NWS) equation using (ET-HPM),
for $k = 1, a = 1, b = 1$ and $m = 2$,

$x = \{-6,-4,-2,0,2,4,6\}$,
$t = \{0,0.1,0.2,0.3,0.4,0.5,0.6\}$.

x/t	0	0.1	0.2	0.3
-6	0.	$6.184383473 \times 10^{-8}$	$9.235471260 \times 10^{-7}$	0.000004355063080
-4	0.	$3.4878574634 \times 10^{-7}$	0.0000054891304966	0.0000273105920754
-2	0.	$2.494864896 \times 10^{-7}$	0.000004178693725	0.000022040655861
0	0.	$4.934870175 \times 10^{-7}$	0.0000077388684262	0.000038311017667
2	0.	$2.8764437459 \times 10^{-7}$	0.0000047947057086	0.000025236056498
4	0.	$1.1475281142 \times 10^{-7}$	0.0000017882337016	0.000008781250575
6	0.	$9.797465690 \times 10^{-8}$	0.0000015939880258	0.0000082018496877

x/t	0.4	0.5	0.6
-6	0.0000127930799 38	0.000028962219212	0.00005555039418
-4	0.0000847671602 82	0.000203102650846	0.000413067745684
-2	0.00007226459605	0.00018230574694	0.00038921669417
0	0.000118127380571	0.00028070360234	0.00056521433133
2	0.000082748021991	0.000209144679354	0.00044799444118
4	0.000026797178237	0.00006284406461	0.00012444604847
6	0.000026333628123	0.000065274913767	0.000137336393754

Figure 3:
$(Urr) = |u_{exact} - u_{approximate}|$
Value of the (NWS) equation using (ET-HPM),

when $k = 1, a = 1, b = 1 \; and \; m = 2$,
$x = \{-6, -4, -2, 0, 2, 4, 6\}$,
$t = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.

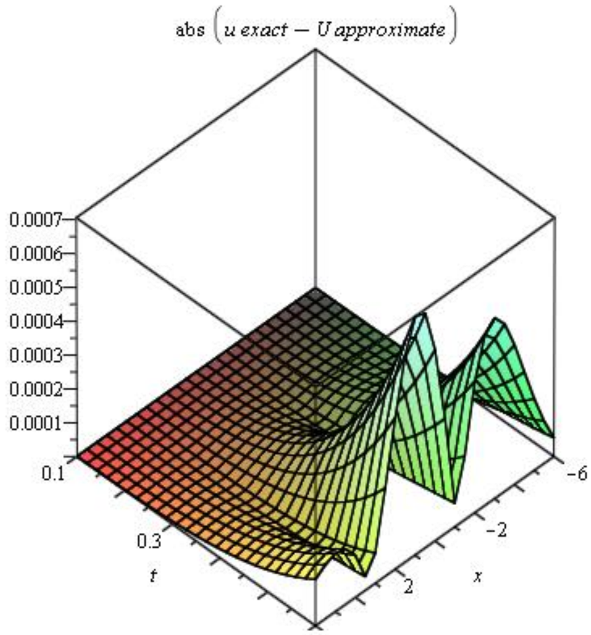

Fig. 3. $(Urr) = |u_{exact} - u_{approximate}|$ Value of the (NWS) equation using (ET-HPM).

4. Conclusions

To solve Newell-Whitehead-Segel (NWS) nonlinear partial differential equations, in this study, we combined the Elzaki transform and its inverse with the homotopy perturbation approach. The Elzaki with the (HPM) proved to be a highly effective tool for tackling nonlinear equations. We effectively employed this method to solve the Newell-Whitehead-Segel equation, and the results show that the suggested method is exceptionally efficient, straightforward, and applicable to both linear and non-linear problems.

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