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The Modified Integral Transform Method to Solve Heat Equation in a Cylindrical Coordinate

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ABSTRACT

This paper investigated a modified integral transform method used to solve heat equation in cylindrical coordinate, this modification method has been obtained based on L_2 integral transform (x-coordinate), we expand L_2 integral transform (x-coordinate) to L_{24} integral transform (x,y,z,t-coordiantes) and convert it to cylindrical coordinate denoted by L_{24c} integral transform (r,θ,z,t-coordinates). Finally we used L_{24c} integral transform to solve heat equation in cylindrical coordinate.

Key Words: Heat Equation, Integral Transform Method , Cylindrical Coordinate.

1. Introduction:

Heat transfer topic have great importance in several problems of industrial and environmental. In advance, in energy production and transformation applications, In this field there is no single application that does not include the effects of heat transfer In a way or another. It have a wide participation of power from conventional fossil fuels, sources of nuclear or the use of geothermal energy sources [9]. Many science fields and engineering confrontation linear or non-linear partial differential equations describing the physical phenomena. Many of methods (for instance, exact and approximate methods) can be used to solve differential equations. Often ,of difficulty to solve these equations analytically. Such equations can be solved by integral transforms like Fourier and Laplace transforms and the importance of Fourier and Laplace transforms lies in their ability to get algebraic equations from differential equations [3]. In general several researchers used integral transform method to solve heat equation such as:

Kamel Al-Khaled solve heat equation by using finite Fourier Transform [4]. Xiao-Jun Yang found the solution of one dimensional heat-diffusion equation in cartesian coordinate by depending on a new integral transform operator [11]. Also Xiao-Jun Yang used new integral transform for solve a steady heat transfer problem [12]. Ranjit R.Dhunde and G.L.Waghmare used double Laplace Transform to solve one dimensional heat equation in cartesian coordinate [7]. Hamood Ur Rehman, Muzammal Iftikhar, Shoaib Saleem, Muhammad Younis and Abdul Mueed used Quadruple Laplace Transform to solve heat equation in cartesian coordinate [3]. V. S. Kulkarni, K. C. Deshmukh and P. H.Munjankar used finite hankel transform to solve steady state temperature of the cylinder satisfies the heat conduction equation(r,z,t coordinate) [10]. Also the solution of steady state heat equation(r,θ,t coordinate) was found by using mellin transform [1]. Also the solution of one dimension heat equation in cylindrical coordinate got by Laplace Transform [6].

2. Modified Integral Transform Method

L_2 integral transform method given by [5]:

$$F(p) = L_2\big(f(x)\big) = \int_0^\infty x e^{-p^2x^2} f(x)dx \quad , p \ is \ parameter$$

the above transform extended to L_{22} transform:

$$F(p,q) = L_{22}\big(f(x,y)\big) = \int_0^\infty \int_0^\infty xy\, e^{-p^2x^2-q^2y^2} f(x,y)dx\, dy \ , \ \ p,q \ are \ parameters$$

put L_{22} transform in the fourth dimension:

$$F(p,q,v,s) = L_{24}\big(f(x,y,z,t)\big)$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty xyzt\, e^{-p^2x^2-q^2y^2-v^2z^2-s^2t^2} f(x,y,z,t)dx\, dy\, dz\, dt \ , \ \ p,q,v,s \ are \ parameters$$

by use $x = rcos\theta \ , y = rsin\theta \ , z = z$ convert the above integral transform to cylindrical coordinate [2]:

$$F(p,q,v,s) = L_{24c}\big(f(r,\theta,z,t)\big)$$
$$= \int_0^\infty \int_0^\infty \int_0^{\pi/2} \int_0^\infty (r^2 sin\theta\, cos\theta\, z\, t) e^{-r^2(p^2cos^2\theta+q^2sin^2\theta)} e^{-v^2z^2-s^2t^2} f(r,\theta,z,t)\, rdr\, d\theta\, dz\, dt$$
$$, \ \ p,q,v,s \ are \ parameters$$

whereas L_{24c}denoted to cylindrical coordinate of L_{24} transform .In the next step, we use L_{24c}which is modified integral transform method to solve heat equation in cylindrical coordinate. Also L_{24c} inverse is denoted by L_{24c}^{-1} and defined by:

$$L_{24c}^{-1}F(p,q,v,s) = f(r,\theta,z,t)$$
$$= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\beta-i\infty}^{\beta+i\infty} \int_{\gamma+i\infty}^{\gamma+i\infty} \int_{\delta-i\infty}^{\delta+i\infty} (r^3 sin\theta\, cos\theta\, z\, t) e^{r^2(p^2cos^2\theta+q^2sin^2\theta)} e^{v^2z^2+s^2t^2} F(p,q,v,s)\, ds\, dv\, dq\, dp$$

3. Apply L_{24c}integral transform to solve heat equation

The expression of heat equation in cylindrical coordinate is [8]:

$$\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial T}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = \left(\frac{\rho c_p}{k}\right)\frac{\partial T}{\partial t} \qquad ...(1)$$

ρ : fluid density

c_p : fluid specific heat

k : fluid thermal conductivity

with boundary and initial conditions:

$$T(a,\theta,z,t) = 0 \ , T(r,0,z,t) = 0 \ , T\left(r,\frac{\pi}{2},z,t\right) = T_1 \ , T(r,\theta,0,t) = 0 \ ,$$
$$T(r,\theta,b,t) = 0 \ , T(r,\theta,z,0) = T_a$$
$$0 < r \le a \ , 0 \le \theta \le \frac{\pi}{2} \ , 0 \le z \le b \ , t \ge 0$$

We take L_{24c} to equation (1):

$$L_{24c}\left[\frac{\partial^2 T}{\partial r^2}\right] + L_{24c}\left[\left(\frac{1}{r}\right)\frac{\partial T}{\partial r}\right] + L_{24c}\left[\left(\frac{1}{r^2}\right)\frac{\partial^2 T}{\partial \theta^2}\right] + L_{24c}\left[\frac{\partial^2 T}{\partial z^2}\right] = L_{24c}\left[\left(\frac{\rho c_p}{k}\right)\frac{\partial T}{\partial t}\right] \ ...(2)$$

Now we must find:

$$L_{24c}\left[\frac{\partial^2 \mathrm{T}}{\partial \mathrm{r}^2}\right]$$

$$=\int_0^\infty\int_0^\infty\int_0^{\pi/2}\int_0^\infty (r^2 sin\theta\ cos\theta\ z\ t)e^{-r^2(p^2cos^2\theta+q^2sin^2\theta)}e^{-v^2z^2-s^2t^2}(\frac{\partial^2 \mathrm{T}}{\partial \mathrm{r}^2})\ rdr\ d\theta\ dz\ dt$$

by integrate twice with respect to r we get:

$$\mathrm{L}_{24c}\left[\frac{\partial^2 \mathrm{T}}{\partial \mathrm{r}^2}\right]=\mathrm{L}_{24c}[4\mathrm{r}^2((\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)^2\ \mathrm{T})]-\mathrm{L}_{24c}[8(\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)\ \mathrm{T}]$$
$$-\mathrm{L}_{24c}[6(\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)\ \mathrm{T}]+\mathrm{L}_{24c}[6\left(\frac{1}{\mathrm{r}^2}\right)\ \mathrm{T}]\ ...\ (3)$$

similarly:

$$L_{24c}\left[\left(\frac{1}{\mathrm{r}}\right)\frac{\partial \mathrm{T}}{\partial \mathrm{r}}\right]=L_{24c}[2(p^2cos^2\theta+q^2sin^2\theta)\ T]-L_{24c}[\left(\frac{2}{r^2}\right)\ T]\ ...\ (4)$$

$$L_{24c}\left[\left(\frac{1}{\mathrm{r}^2}\right)\frac{\partial^2 \mathrm{T}}{\partial \theta^2}\right]=L_{24c}[(4sin^2\theta\ cos^2\theta(p^2-q^2)^2)\ T]-L_{24c}[6(p^2-q^2)sin^2\theta\ T]$$
$$+L_{24c}[6(p^2-q^2)cos^2\theta\ T]+\left(\frac{1}{2q^2}\right)\left(\frac{1}{2v^2}\right)\left(\frac{1}{2s^2}\right)T\left(\frac{\pi}{2}\right)-L_{24c}[4\left(\frac{1}{r^2}\right)\ T]\ \ ...\ (5)$$

$$L_{24c}\left[\frac{\partial^2 \mathrm{T}}{\partial \mathrm{z}^2}\right]=L_{24c}[4z^2v^4\ T]-L_{24c}[4v^2\ T]-L_{24c}[2v^2\ T]\ ...\ (6)$$

$$L_{24c}\left[(\frac{\rho c_p}{k})\frac{\partial \mathrm{T}}{\partial \mathrm{t}}\right]=L_{24c}\left[2s^2(\frac{\rho c_p}{k})\ t\ T\right]-L_{24c}[\left(\frac{1}{t}\right)(\frac{\rho c_p}{k})\ T]\ ...\ (7)$$

substituting (3), (4), (5), (6) and (7) in (2) we get:

$$\mathrm{L}_{24c}[4\mathrm{r}^2((\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)^2\ \mathrm{T})]-\mathrm{L}_{24c}[8(\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)\ \mathrm{T}]$$
$$-\mathrm{L}_{24c}[6(\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)\ \mathrm{T}]+\mathrm{L}_{24c}\left[6\left(\frac{1}{\mathrm{r}^2}\right)\ \mathrm{T}\right]+L_{24c}[2(p^2cos^2\theta+q^2sin^2\theta)\ T]$$
$$-L_{24c}[\left(\frac{2}{r^2}\right)\ T]+L_{24c}[(4sin^2\theta\ cos^2\theta(p^2-q^2)^2)\ T]-L_{24c}[6(p^2-q^2)sin^2\theta\ T]$$
$$+L_{24c}[6(p^2-q^2)cos^2\theta\ T]+\left(\frac{1}{2q^2}\right)\left(\frac{1}{2v^2}\right)\left(\frac{1}{2s^2}\right)T\left(\frac{\pi}{2}\right)-L_{24c}\left[4\left(\frac{1}{r^2}\right)\ T\right]$$
$$+L_{24c}[4z^2v^4\ T]-L_{24c}[4v^2\ T]-L_{24c}[2v^2\ T]=L_{24c}\left[2s^2\left(\frac{\rho c_p}{k}\right)t\ T\right]$$
$$-L_{24c}[\left(\frac{1}{t}\right)(\frac{\rho c_p}{k})\ T]$$

arrange the above equation:

$$\mathrm{L}_{24c}[4\mathrm{r}^2((\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)^2\ \mathrm{T})]-\mathrm{L}_{24c}[8(\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)\ \mathrm{T}]$$
$$-\mathrm{L}_{24c}[6(\mathrm{p}^2\cos^2\theta+\mathrm{q}^2\sin^2\theta)\ \mathrm{T}]+\mathrm{L}_{24c}\left[6\left(\frac{1}{\mathrm{r}^2}\right)\ \mathrm{T}\right]+L_{24c}[2(p^2cos^2\theta+q^2sin^2\theta)\ T]$$
$$-L_{24c}[\left(\frac{2}{r^2}\right)\ T]+L_{24c}[(4sin^2\theta\ cos^2\theta(p^2-q^2)^2)\ T]-L_{24c}[6(p^2-q^2)sin^2\theta\ T]$$
$$+L_{24c}[6(p^2-q^2)cos^2\theta\ T]-L_{24c}\left[4\left(\frac{1}{r^2}\right)\ T\right]+L_{24c}[4z^2v^4\ T]-L_{24c}[4v^2\ T]$$

$$-L_{24c}[2v^2\,T] - L_{24c}\left[2s^2\left(\frac{\rho c_p}{k}\right) t\,T\right] + L_{24c}\left[\left(\frac{1}{t}\right)\left(\frac{\rho c_p}{k}\right) T\right] = -\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)}\right)$$

multiply both sides of the above equation by $(\frac{1}{2p^2})$:

$$\mathrm{L}_{24c}\left[2r^2(\frac{1}{p^2})\,((p^2\cos^2\theta + q^2\sin^2\theta)^2\,T)\right] - \mathrm{L}_{24c}\left[4(\frac{1}{p^2})\,(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right]$$
$$-\mathrm{L}_{24c}\left[3\left(\frac{1}{p^2}\right)(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right] + \mathrm{L}_{24c}\left[3\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)\,T\right]$$
$$+L_{24c}\left[(\frac{1}{p^2})\,(p^2cos^2\theta + q^2sin^2\theta)\,T\right] - L_{24c}\left[\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)\,T\right]$$
$$+L_{24c}\left[(\frac{1}{p^2})\,(2sin^2\theta\,cos^2\theta(p^2-q^2)^2)\,T\right] - L_{24c}\left[3(\frac{1}{p^2})\,(p^2-q^2)sin^2\theta\,T\right]$$
$$+L_{24c}\left[3\left(\frac{1}{p^2}\right)(p^2-q^2)cos^2\theta\,T\right] - L_{24c}\left[2\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)\,T\right] + L_{24c}\left[2\left(\frac{1}{p^2}\right)z^2v^4\,T\right]$$
$$-L_{24c}\left[2(\frac{1}{p^2})\,v^2\,T\right] - L_{24c}\left[(\frac{1}{p^2})\,v^2\,T\right] - L_{24c}\left[s^2\left(\frac{\rho c_p}{k}\right)(\frac{1}{p^2})\,t\,T\right]$$
$$+L_{24c}\left[\left(\frac{1}{t}\right)(\frac{1}{2p^2})\,\left(\frac{\rho c_p}{k}\right)T\right] = -\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)(2p^2)}\right)$$

take L_{24c}^{-1} to the above equation:

$$\left[2r^2(\frac{1}{p^2})\,((p^2\cos^2\theta + q^2\sin^2\theta)^2\,T)\right] - \left[4(\frac{1}{p^2})\,(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right]$$
$$-\left[3\left(\frac{1}{p^2}\right)(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right] + \left[3\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)\,T\right]$$
$$+\left[(\frac{1}{p^2})\,(p^2cos^2\theta + q^2sin^2\theta)\,T\right] - \left[\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)\,T\right]$$
$$+\left[(\frac{1}{p^2})\,(2sin^2\theta\,cos^2\theta(p^2-q^2)^2)\,T\right] - \left[3(\frac{1}{p^2})\,(p^2-q^2)sin^2\theta\,T\right]$$
$$+\left[3\left(\frac{1}{p^2}\right)(p^2-q^2)cos^2\theta\,T\right] - \left[2\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)\,T\right] + \left[2\left(\frac{1}{p^2}\right)z^2v^4\,T\right]$$
$$-\left[2(\frac{1}{p^2})\,v^2\,T\right] - \left[(\frac{1}{p^2})\,v^2\,T\right] - \left[s^2\left(\frac{\rho c_p}{k}\right)(\frac{1}{p^2})\,t\,T\right]$$
$$+\left[\left(\frac{1}{t}\right)(\frac{1}{2p^2})\,\left(\frac{\rho c_p}{k}\right)T\right] = -L_{24c}^{-1}\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)(2p^2)}\right)$$

the above equation reduce to:

$$T[(\frac{2r^2}{p^2})(p^2cos^2\theta + q^2sin^2\theta) - \frac{6}{p^2}(p^2cos^2\theta + q^2sin^2\theta) + \frac{2}{p^2}sin^2\theta cos^2\theta(p^2-q^2)^2 + \frac{3}{p^2}(p^2-q^2)cos2\theta + \frac{2}{p^2}z^2v^4 - 3(\frac{1}{p^2})v^2 - \frac{s^2}{p^2}\left(\frac{\rho c_p}{k}\right)t + \left(\frac{\rho c_p}{k}\right)\frac{1}{2p^2t}] = -L_{24c}^{-1}\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)(2p^2)}\right) \ldots (8)$$

it is clear that:

$L_{24c}(k) = \frac{k}{(2s^2)(2v^2)(2q^2)(2p^2)}$, so $k = L_{24c}^{-1}(\frac{k}{(2s^2)(2v^2)(2q^2)(2p^2)})$ where k is any constant.

rewrite the equation (8) gives:

$$T[(\frac{2r^2}{p^2})(p^2cos^2\theta + q^2sin^2\theta) - \frac{6}{p^2}(p^2cos^2\theta + q^2sin^2\theta) + \frac{2}{p^2}sin^2\theta cos^2\theta(p^2 - q^2)^2 + \frac{3}{p^2}(p^2 - q^2)cos2\theta + \frac{2}{p^2}z^2v^4 - 3(\frac{1}{p^2})v^2 - \frac{s^2}{p^2}\left(\frac{\rho c_p}{k}\right)t + \left(\frac{\rho c_p}{k}\right)\frac{1}{2p^2t}] = -T(\frac{\pi}{2}) \quad \dots (9)$$

from boundary condition:

$$T\left(\frac{\pi}{2}\right) = T\left(r, \frac{\pi}{2}, z, t\right) = T_1$$

after substituting in (9) and simplicity we get:

$$T = T(r,\theta,z,t) = -T_1/[(\frac{2r^2}{p^2})(p^2cos^2\theta + q^2sin^2\theta) - \frac{6}{p^2}(p^2cos^2\theta + q^2sin^2\theta) + \frac{2}{p^2}sin^2\theta cos^2\theta(p^2 - q^2)^2 + \frac{3}{p^2}(p^2 - q^2)cos2\theta + \frac{2}{p^2}z^2v^4 - 3(\frac{1}{p^2})v^2 - \frac{s^2}{p^2}\left(\frac{\rho c_p}{k}\right)t + \left(\frac{\rho c_p}{k}\right)\frac{1}{2p^2t}]$$

Which is the solution of the heat equation in cylindrical coordinate.

4. Conclusion

In this paper, we extend L_2transform to L_{24} transform which is converted to the cylindrical coordinate denoted by L_{24c} transform. L_{24c} transform has an advantage compare with hankel transform, millen transform and quadruple Laplace transform because of L_{24c} transform solve heat equation in cylindrical coordinate (r,θ,z,t-coordinates) which is partial differential equation with variable coefficients and hankel transform solve heat equation in cylindrical coordinate (r,z,t-coordinates) [10], millen transform solve heat equation in cylindrical coordinate (r,θ,t-coordinates) [1], quadruple Laplace transform solve heat equation in cartesian coordinate (x,y,z,t-coordinates) which is partial differential equation with constant coefficients [3].

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