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New Games via Grill-Generalized Open Sets

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ABSTRACT

This paper presents some games via 𝔾-g-open sets by using the concept of grill topological space which is Ǵ(Ŧi, 𝔾), where i={0, 1, 2}. By many figures and proposition, the relationships between these types of games have been studied with explaining some examples.

Keywords. 𝔾-g-open set, 𝔾-g-closed set, Ǵ (Ŧ0, 𝔾), Ǵ(Ŧ1, 𝔾), Ǵ(Ŧ, 𝔾).

1. Introduction

A nonempty collection 𝔾 of nonempty subsets of a topological space Ӿ is named a grill if

i. Ạ ∈ 𝔾 and Ạ ⊆ $\mathcal{B}$ ⊆ Ӿ, then $\mathcal{B}$ ∈ 𝔾.

ii. Ạ, $\mathcal{B}$ ⊆ Ӿ and Ạ ∪ $\mathcal{B}$ ∈ 𝔾, then Ạ ∈ 𝔾 or $\mathcal{B}$ ∈ 𝔾 [1]. Let Ӿ be a nonempty set. Then the following families are grills on Ӿ. [1], [8], [9]

- {∅} and P(Ӿ)\{∅} are trivial examples of grills on Ӿ.
- $𝔾_{\infty}$, the grill of all infinite subsets of Ӿ.
- $𝔾_{co}$ the grill of all uncountable subsets of Ӿ.
- $𝔾_p$ ={ Λ: Λ ∈ P(Ӿ), p ⊆ Λ } is a specific point grills on Ӿ.
- $𝔾_{Ạ}$ ={ $\mathcal{B}$: $\mathcal{B}$ ∈ P(Ӿ), $\mathcal{B}$ ∩ $Ạ^c$≠ ∅}, and

If (Ӿ,ŧ) is a topological space, then the family of all non-nowhere dense subsets called $𝔾_{\varsigma}$ ={Ạ: $\text{int}_ŧ$ $\text{cl}_ŧ$(Ạ) ≠ ∅}, is the one of kinds of grill on Ӿ.

Let 𝔾 be a grill on a topological space(Ӿ,ŧ). The operator φ: P(Ӿ)→P(Ӿ) was defined by φ (Ạ)={ӿ∈ Ӿ \ Ư∩ Ạ∈𝔾, for all Ư∈ ŧ(ӿ)}, ŧ(ӿ) denotes the neighborhood of ӿ . A mapping Ψ: P(Ӿ)→P(Ӿ) is defined as Ψ (Ạ) = Ạ ∪ φ (Ạ) for all Ạ ∈ P(Ӿ) [3], [10]. The map Ψ satisfies Kuratowski closure axioms: [3], [9]

(i) Ψ (∅) = ∅,

(ii) If Ạ ⊆$\mathcal{B}$, then Ψ (Ạ) ⊆ Ψ ($\mathcal{B}$),

(iii) If Ạ ⊆ Ӿ, then Ψ (Ψ (Ạ)) = Ψ (Ạ),

(iv) If Ạ, $\mathcal{B}$ ⊆ Ӿ, then Ψ (Ạ ∪ $\mathcal{B}$) = Ψ (Ạ) ∪ Ψ ($\mathcal{B}$).

Let Ǵ be a game between two players $?_1$ and $?_2$. The set of choices $Ĩ_1$, $Ĩ_2$, $Ĩ_3$, …., $Ĩ_?$, for each player. These choice are called moves or options [4,5]. Alternating game which is, one of players $?_1$ chooses one of the options $Ĩ_1$, $Ĩ_2$, $Ĩ_3$, …., $Ĩ_?$. Next player $?_2$ choices one of these moves where knowing the chooses of $?_1$. In alternating games must determine the player who who starts the game [6,7].

In this research provides the sorts of game through a given set. The winning and losing strategy for any player Ꝑ in the game Ǵ, if Ꝑ has a winning strategy in Ǵ shortly by (Ꝑ ↪ Ǵ) and if Ꝑ does not have a winning strategy shortly by (Ꝑ ↬ Ǵ), if Ꝑ has a losing strategy shortly by (Ꝑ ↩ Ǵ) and if Ꝑ does not have a losing strategy shortly by (Ꝑ ↫ Ǵ).

2. Preliminaries.

The following results are given in [2]

Definition 2.1: Let (Ӿ, ƀ) be a topological space, define a game Ǵ($Ŧ_0$, Ӿ) as follows: The two players $□_1$ and $□_2$ are play an inning for each natural numbers, in the □- th inning, the first round, $□_1$ will choose $ӿ_□$ ≠ $ş_□$, whenever $ӿ_□$, $ş_□$ ∈ Ӿ.
Next, $□_2$ choose $Ư_□$ ∈ ƀ such that $ӿ_□$ ∈ $Ư_□$ and $ş_□$ ∉ $Ư_□$, $□_2$ wins in the game, whenever $\mathcal{B}$ = { $Ư_1$, $Ư_2$, $Ư_3$, ... , $Ư_□$, ... } satisfies that for all $ӿ_□$ ≠ $ş_□$ in Ӿ there exist $Ư_□$ ∈ $\mathcal{B}$ such that $ӿ_□$ ∈ $Ư_□$ and $ş_□$ ∉ $Ư_□$. Otherwise $□_1$ wins.

Definition 2.2: Let (Ӿ, ƀ) be a topological space, define a game Ǵ($Ŧ_1$, Ӿ) as follows: The two players $□_1$ and $□_2$ are play an inning for each natural numbers, in the □- th inning, the first round, $□_1$ will choose $ӿ_□$ ≠ $ş_□$, such that $ӿ_□$, $ş_□$ ∈ Ӿ.
Next, $□_2$ choose $Ư_□$, $Ꞷ_□$ ∈ ƀ such that $ӿ_□$ ∈ ($Ư_□$-$Ꞷ_□$) and $ş_□$ ∈ ($Ꞷ_□$-$Ư_□$), $□_2$ wins in the game, whenever $\mathcal{B}$={{$Ư_1$, $Ꞷ_1$}, {$Ư_2$, $Ꞷ_2$}, …., {$Ư_□$, $Ꞷ_□$}, …. } satisfies that for all $ӿ_□$ ≠ $ş_□$ in Ӿ there exists {$Ư_□$, $Ꞷ_□$ } ∈ $\mathcal{B}$ such that $ӿ_□$ ∈ ($Ư_□$- $Ꞷ_□$) and $ş_□$ ∈ ($Ꞷ_□$- $Ư_□$). Other hand $□_1$ wins.

Definition 2.3: Let (Ӿ, ƀ) be a topological space, define a game Ǵ($Ŧ_2$, Ӿ) as follows: The two players $□_1$ and $□_2$ are play an inning for each natural numbers, in the □- th inning, the first round, $□_1$ will choose $ӿ_□$ ≠ $ş_□$, whenever $ӿ_□$, $ş_□$ ∈ Ӿ.
Next, $□_2$ choose $Ư_□$, $Ꞷ_□$ are disjoint, $Ư_□$, $Ꞷ_□$ ∈ ƀ such that $ӿ_□$ ∈ $Ư_□$ and $ş_□$ ∈ $Ꞷ_□$, $□_2$ wins in the game, whenever $\mathcal{B}$ = {{$Ư_1$, $Ꞷ_1$ }, {$Ư_2$, $Ꞷ_2$ } ,....,{$Ư_□$, $Ꞷ_□$ },....} satisfies that for all $ӿ_□$ ≠ $ş_□$ in Ӿ there exists {$Ư_□$, $Ꞷ_□$ } ∈ $\mathcal{B}$ such that $ӿ_□$ ∈ $Ư_□$ and $ş_□$ ∈ $Ꞷ_□$. Other hand $□_1$ wins.

3. 𝔾-g-Openness On Game

Definition 3.1: In space (Ӿ, ŧ, 𝔾), let 𝔻 ⊆ Ӿ. 𝔻 is named to be grill-g-closed set denoted by "𝔾-g-closed", if (𝔻 - Ư) ∉ 𝔾 then, (cl(𝔻) - Ư) ∉ 𝔾 where, Ư ⊆ Ӿ and Ư ∈ ŧ. Now, D^c is a grill-g-open set denoted by "𝔾-g-open" . The family of all "𝔾-g-closed" "𝔾-g-open" sets denoted the 𝔾**g**C(Ӿ), 𝔾**g**O(Ӿ).

Example 3.2: Consider the space (Ӿ, ƀ, 𝔾), where Ӿ = { $ƫ_1$, $ƫ_2$, $ƫ_3$}, ƀ = {Ӿ, ∅, {$ƫ_1$}, {$ƫ_2$}, {$ƫ_1$, $ƫ_2$}}, and 𝔾 ={Ӿ, {$ƫ_1$}, {$ƫ_1$, $ƫ_2$}, {$ƫ_1$, $ƫ_3$}}. Then, 𝔾gC(Ӿ) = { Ӿ, ∅, {$ƫ_1$}, {$ƫ_2$}, {$ƫ_3$}, {$ƫ_1$, $ƫ_2$}, {$ƫ_1$, $ƫ_3$}, {$ƫ_2$, $ƫ_3$}}, 𝔾gO(Ӿ) = { Ӿ, ∅, { $ƫ_1$}, { $ƫ_2$}, { $ƫ_3$}, { $ƫ_1$, $ƫ_2$}, { $ƫ_1$, $ƫ_3$}, { $ƫ_2$, $ƫ_3$}}.

Remark 3.3: For any space (Ӿ, ŧ, 𝔾) then
i. Every closed set is a 𝔾**g**C(Ӿ).
ii. Every open set is a 𝔾**g**O(Ӿ).
The Example 3.2 shows the opposite of Remark 3. 3 is not true.
i. {$ƫ_1$} is a 𝔾**g**C(Ӿ), but {$ƫ_1$} is not closed set.
ii. {$ƫ_1$, $ƫ_3$} is a 𝔾**g**O(Ӿ), but {$ƫ_1$, $ƫ_3$} ∉ ŧ.

Definition 3.4: The space (Ӿ, ŧ, 𝔾) is a 𝔾-g-$Ŧ_0$-space shortly "𝔾g-$Ŧ_0$-space" if for each ɱ ≠ ȵ and ɱ, ȵ ∈ Ӿ, then there exist Ư ∈ 𝔾g-O(Ӿ) whenever, ɱ ∈ Ư and ȵ ∉ Ư or ɱ ∉ Ư and ȵ ∈ Ư.

Definition 3.5: The space (Ӿ, ŧ, 𝔾) is a 𝔾-g-$Ŧ_1$-space shortly "𝔾g-$Ŧ_1$-space" if for each ɱ, ɲ ∈ Ӿ and ɱ ≠ ɲ. Then there exists $Ư_1, Ư_2$ ∈ 𝔾g-O(Ӿ) whenever ɱ ∈ $Ư_1$, ɲ ∉ $Ư_1$ and ɲ ∈ $Ư_2$, ɱ ∉ $Ư_2$.

Definition 3.6: The space (Ӿ, ŧ, 𝔾) is a 𝔾-g-$Ŧ_2$-space shortly "𝔾g-$Ŧ_2$-space" if for each ɱ ≠ ɲ. Then there exists $Ư_1$, $Ư_2$ ∈ 𝔾g-O(Ӿ) whenever ɱ ∈ $Ư_1$, ɲ ∈ $Ư_2$ and $Ư_1 \cap Ư_2 = \emptyset$.

Remark 3.7: The space (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_{i+1}$-space then it is a 𝔾g-$Ŧ_i$-space (for every $i \in \{0, 1\}$).

Definition 3.8: Let (Ӿ, ŧ, 𝔾) be a grill topological space, define a game $Ǵ(Ŧ_0, 𝔾)$ as follows: The two players $□_1$and $□_2$ are play an inning for each natural numbers, in the □- th inning, the first round, $□_1$ will choose $x_□ \neq ş_□$, whenever $x_□$, $ş_□$ ∈ Ӿ. Next, $□_2$ choose $Ư_□$ ∈ 𝔾g-O(Ӿ) such that $x_□ \in Ư_□$ and $ş_□ \notin Ư_□$, $□_2$ wins in the game, whenever $\mathcal{B}$={$Ư_1$, $Ư_2$, $Ư_3$,…., $Ư_□$,….} satisfies that for all $x_□ \neq ş_□$ in Ӿ there exist $Ư_□ \in \mathcal{B}$ such that $x_□ \in Ư_□$ and $ş_□ \notin Ư_□$. Other hand $□_1$ wins.

Example 3.9: Let $Ǵ(Ŧ_0, 𝔾)$ be a game Ӿ ={ x, ş, ɼ}, and ŧ = {Ӿ, ∅, {x }, {ş}, {x, ş}, 𝔾 = P(Ӿ)\{∅}, then 𝔾g-C(Ӿ) ={ Ӿ , ∅, {ɼ }, {x, ɼ}, {ş, ɼ}}, 𝔾g-O(Ӿ) ={ Ӿ, ∅, {x}, {ş}, {x, ş}}, then in the first round $□_1$ will choose x ≠ ş, whenever x, ş ∈ Ӿ. Next, $□_2$ choose {x } ∈ 𝔾g- O(Ӿ) such that x ∈ {x} and ş ∉ {x}, in the second round $□_1$ will choose x ≠ ɼ, whenever x, ɼ ∈ Ӿ. Next , $□_2$ choose {x} ∈ 𝔾g- O(Ӿ) such that x ∈ {x} and ɼ ∉ {x}, in the third round $□_1$ will choose ş ≠ ɼ, whenever ş, ɼ ∈ Ӿ. Next, $□_2$ choose {ş} ∈ 𝔾g- O(Ӿ) such that ş ∈ {ş} and ɼ ∉ {ş}, $□_2$ wins in the game, whenever $\mathcal{B}$ = {{x}, {ş}} satisfies that for all $x_□ \neq ş_□$in Ӿ there exist $Ư_□ \in \mathcal{B}$ such that $x_□ \in Ư_□$and $ş_□ \notin Ư_□$whenever ,$Ư_□$ ∈ 𝔾g- O(Ӿ). So $□_2$ is the winner of the game.

Remark 3.10: For any grill topological space (Ӿ, ŧ, 𝔾):

- **i.** if $□_2 \hookrightarrow Ǵ(Ŧ_0, Ӿ)$, then $□_2 \hookrightarrow Ǵ(Ŧ_0, 𝔾)$.
- **ii.** if $□_2 \hookleftarrow Ǵ(Ŧ_0, Ӿ)$, then $□_2 \hookleftarrow Ǵ(Ŧ_0, 𝔾)$.
- **iii.** if $□_1 \hookrightarrow Ǵ(Ŧ_0, 𝔾)$, then $□_1 \hookrightarrow Ǵ(Ŧ_0, Ӿ)$.

Proof: Is clear by Remark 3. 3: (ii)

Theorem 3.11: Let (Ӿ, ŧ, 𝔾) is a 𝔾g- $Ŧ_0$- space if and only if $□_2 \hookrightarrow Ǵ(Ŧ_0, 𝔾)$.
Proof: Since (Ӿ, ŧ, 𝔾) is a 𝔾g- $Ŧ_0$- space then in the □- th inning any choice for the first player $□_1$ $x_□ \neq ş_□$ whenever $x_□$, $ş_□$ ∈ Ӿ. The second player $□_2$ can be found $Ư_□$ ∈ 𝔾gO(Ӿ). Thus $\mathcal{B}$ = { $Ư_1$, $Ư_2$, $Ư_3$, … . , $Ư_□$, … . } is the winning strategy for $□_2$.
Conversely Clear.

Theorem 3.12: The grill topological space (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_0$-space if and only if for each elements ɱ ≠ ɲ there is a 𝔾g-closed set containing only one of them.
Proof: Let ɱ and ɲ are two distinct elements in Ӿ. Since Ӿ is a 𝔾g-$Ŧ_0$-space, then there is a 𝔾g-open set Ư containing only one of them, then (Ӿ- Ư) is a 𝔾g-closed set containing the other one.
Conversely Let ɱ and ɲ are two distinct elements in Ӿ and there is a 𝔾g-closed set Ѷ containing only one of them. Then (Ӿ- Ѷ) is a 𝔾g-open set containing the other one.
By Theorem (3. 11) and (3. 12) we get.

Corollary 3.13: For a space (Ӿ, ŧ, 𝔾), $□_2 \hookrightarrow Ǵ(Ŧ_0, 𝔾)$ if and only if, for every $x_1 \neq x_2$ in Ӿ, there exist Ѷ ∈ 𝔾gC(Ӿ) such that x_1 ∈ Ѷ and x_2 ∉ Ѷ.

Corollary 3.14: Let (Ӿ, ŧ, 𝔾) 𝔾g-$Ŧ_0$-space if and only if $□_1 \not\hookrightarrow Ǵ(Ŧ_0, 𝔾)$.

Proof: By Theorem 3. 11, the proof is over.

Theorem 3.15: Let (Ӿ, ƫ, 𝔾) is not 𝔾g- $Ƭ_0$- space if and only if $□_1$ ↪ Ǵ($Ƭ_0$, 𝔾) .
Proof: In the □-th inning $□_1$ in Ǵ($Ƭ_0$, 𝔾) choose x□ ≠ ȿ□, whenever x□, ȿ□ ∈ Ӿ, $□_2$ in Ǵ($Ƭ_0$, 𝔾), cannot be found Ư□ is a 𝔾g-open set containing only one elements of them, because (Ӿ, ƫ, 𝔾) is not 𝔾g- $Ƭ_0$- space, hence $□_1$ ↪ Ǵ($Ƭ_0$, 𝔾).
Conversely clear.

Corollary 3.16: Let (Ӿ, ƫ, 𝔾) is not 𝔾g-$Ƭ_0$-space if and only if $□_2$ ↬ Ǵ($Ƭ_0$, 𝔾).
Proof: It is clear.

Definition 3.17: Let (Ӿ, ƫ, 𝔾) be a grill topological space, define a game Ǵ($Ƭ_1$, 𝔾) as follows: The two players $□_1$and $□_2$ are play an inning for each natural numbers, in the □- th inning, the first round, $□_1$ will choose x□ ≠ ȿ□, whenever x□, ȿ□ ∈ Ӿ. Next,$□_2$ choose Ư□, Ɯ□ ∈ 𝔾g-O(Ӿ) such that x□∈ (Ư□-Ɯ□) and ȿ□,∈ (Ɯ□-Ư□), $□_2$ wins in the game, whenever 𝓑={{Ư$_1$, Ɯ$_1$}, {Ư$_2$, Ɯ$_2$},….,{Ư□, Ɯ□},….} satisfies that for all x□ ≠ ȿ□ in Ӿ there exists {Ư□, Ɯ□} ∈ 𝓑 such that x□∈ (Ư□-Ɯ□) and ȿ□∈ (Ɯ□-Ư□). Other hand $□_1$wins.

Example 3.18: Let Ǵ($Ƭ_1$, 𝔾) be a game Ӿ = { x, ȿ, ʈ}, and ƫ = {Ӿ , ∅, {x}, {ȿ}, {ʈ}, {x, ȿ}, {x, ʈ}, {ȿ, ʈ}}, 𝔾 = {Ӿ, ∅, {x }, {x, ȿ}, {x, ʈ}}, 𝔾g- C(Ӿ) =P(Ӿ) = 𝔾g- O(Ӿ) then in the first round, $□_1$will choose x ≠ ȿ whenever x, ȿ ∈ Ӿ. Next, $□_2$ choose {x}, {ȿ} ∈ 𝔾g-O(Ӿ) such that x ∈ ({x} - {ȿ}) and ȿ ∈ ({ȿ} - {x}), in the second round, $□_1$will choose x ≠ ʈ, whenever x, ʈ ∈ Ӿ. Next, $□_2$ choose {x}, {ʈ} ∈ 𝔾g-O(Ӿ) such that x ∈ ({x} - {ʈ}) and ʈ ∈ ({ʈ}-{x}) in the third round, $□_1$will choose ȿ ≠ ʈ, whenever ȿ, ʈ ∈ Ӿ. Next, $□_2$ choose {ȿ}, {ʈ} ∈ 𝔾g-O(Ӿ) such that ȿ ∈ ({ȿ} - {ʈ}) and ʈ ∈ ({ʈ} - {ȿ}), so $□_2$ wins in the game whenever 𝓑 ={{{x}, {ȿ}}, {{x}, {ʈ}}, {{ȿ}, {ʈ}}} satisfies that for all x□ ≠ ȿ□ in Ӿ there exists {{x□},{ȿ□}} ∈ 𝓑 such that x□ ∈ ({x□} - {ȿ□}) and ȿ□ ∈ ({ȿ□} - {x□}), $□_2$ is the winner in the game.

Example 3.19: Let (Ӿ, ƫ, 𝔾) be a game Ӿ = {ʄ$_1$, ʄ$_2$, ʄ$_3$}, ƫ = {Ӿ, ∅, {ʄ$_1$}, {ʄ$_2$}, {ʄ$_1$, ʄ$_2$}}, and 𝔾 = {Ӿ, {ʄ$_3$}, {ʄ$_1$, ʄ$_3$}, {ʄ$_2$, ʄ$_3$}}. 𝔾gC(Ӿ) = { Ӿ, ∅, {ʄ$_3$}, {ʄ$_1$, ʄ$_3$}, {ʄ$_2$, ʄ$_3$}}, 𝔾gO(Ӿ) = { Ӿ, ∅, {ʄ$_1$}, {ʄ$_2$}, {ʄ$_1$, ʄ$_2$}}, then in the first round $□_1$will choose ʄ$_1$≠ ʄ$_3$ whenever ʄ$_1$, ʄ$_3$ ∈ Ӿ. Next, $□_2$ cannot be found Ư□, Ɯ□ ∈ 𝔾g-O(Ӿ) such that ʄ$_1$∈ (Ư□- Ɯ□) and ʄ$_3$ ∈ (Ɯ□ - Ư□), so $□_1$ wins in the game.

Remark 3.20: For any grill topological space (Ӿ, ƫ, 𝔾):
i. If $□_2$ ↪ Ǵ($Ƭ_1$, Ӿ), then $□_2$ ↪ Ǵ($Ƭ_1$, 𝔾).
ii. If $□_2$ ↩ Ǵ($Ƭ_1$, Ӿ), then $□_2$ ↩ Ǵ($Ƭ_1$, 𝔾).
iii. If $□_1$ ↪ Ǵ($Ƭ_1$, 𝔾), then $□_1$ ↪ Ǵ($Ƭ_1$, Ӿ).
Proof: Is clear by Remark 3.3: (ii)

Theorem 3.21: Let (Ӿ, ƫ, 𝔾) is a 𝔾g- $Ƭ_1$- space if and only if $□_2$ ↪ Ǵ($Ƭ_1$, 𝔾).
Proof: Let (Ӿ, ƫ, 𝔾) be a grill topological space, in the first round $□_1$ will choose x$_1$ ≠ ȿ$_1$, whenever x$_1$, ȿ$_1$ ∈ Ӿ. Next, since (Ӿ, ƫ, 𝔾) is a 𝔾g- $Ƭ_1$- space $□_2$ can be found Ư$_1$, Ɯ$_1$ ∈ 𝔾g- O(Ӿ) such that x$_1$∈ (Ư$_1$- Ɯ$_1$) and ȿ$_1$∈ (Ɯ$_1$- Ư$_1$) in the second round $□_1$ will choose x$_2$ ≠ ȿ$_2$, whenever x$_2$, ȿ$_2$ ∈ Ӿ. Next, $□_2$ can be found Ư$_2$, Ɯ$_2$ ∈ 𝔾g- O(Ӿ) such that x$_2$∈ (Ư$_2$ - Ɯ$_2$) and ȿ$_2$ ∈ (Ɯ$_2$ - Ư$_2$), in the □- th round, $□_1$ will choose x□ ≠ ȿ□, whenever x□, ȿ□ ∈ Ӿ.
Next, $□_2$ can be found Ư□, Ɯ□ ∈ 𝔾g- O(Ӿ) such that x□ ∈ (Ư□ - Ɯ□) and ȿ□ ∈ (Ɯ□ - Ư□). Thus 𝓑 = {{Ư$_1$, Ɯ$_1$}, {Ư$_2$, Ɯ$_2$}, … , {Ư□, Ɯ□}, … } is the winning strategy for $□_2$.
Conversely Clear.

Theorem 3.22: The grill topological space (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space if and only if for each elements ɱ ≠ ɒ there exists two 𝔾g-closed sets $\ddot{V}_1$ and $\ddot{V}_2$ such that ɱ ∈ ($\ddot{V}_1$ - $\ddot{V}_2$) and ɒ ∈ ($\ddot{V}_2$ - $\ddot{V}_1$).

***Proof*:** Let ɱ and ɒ are two distinct elements in Ӿ. Since Ӿ is a 𝔾g-$Ŧ_1$-space, then there exists $Ư_1$ and $Ư_2$ ∈ 𝔾g-open such that ɱ ∈ ($Ư_1$ - $Ư_2$) and ɒ ∈ ($Ư_2$ - $Ư_1$). Then there exists 𝔾g-closed sets (Ӿ- $Ư_1$) and (Ӿ- $Ư_2$) such that ɱ ∈ ((Ӿ- $Ư_2$) - (Ӿ- $Ư_1$)), ɒ ∈ ((Ӿ- $Ư_1$) - (Ӿ- $Ư_2$)) whenever (Ӿ- $Ư_2$) = $\ddot{V}_1$ and (Ӿ- $Ư_1$) = $\ddot{V}_2$. then there exists two 𝔾g-closed sets $\ddot{V}_1$ and $\ddot{V}_2$ satisfy ɱ ∈ ($\ddot{V}_1 \cap \ddot{V}_2{}^c$) and ɒ ∈ ($\ddot{V}_2 \cap \ddot{V}_1{}^c$) there for ɱ ∈ ($\ddot{V}_1$ - $\ddot{V}_2$) and ɒ ∈ ($\ddot{V}_2$ - $\ddot{V}_1$).

Conversely Let ɱ and ɒ are two distinct elements in Ӿ and there exists two 𝔾g-closed sets $\ddot{V}_1$ and $\ddot{V}_2$ satisfy ɱ ∈ ($\ddot{V}_1 \cap \ddot{V}_2{}^c$) and ɒ ∈ ($\ddot{V}_2 \cap \ddot{V}_1{}^c$) then there exists 𝔾g-open set (Ӿ - $\ddot{V}_1$) and (Ӿ - $\ddot{V}_2$) whenever ɱ ∈ ((Ӿ - $\ddot{V}_2$) - (Ӿ - $\ddot{V}_1$)), ɒ ∈ ((Ӿ - $\ddot{V}_1$) - (Ӿ - $\ddot{V}_2$)) whenever (Ӿ - $\ddot{V}_2$) = $Ư_1$ and (Ӿ - $\ddot{V}_1$) = $Ư_2$.

Corollary 3.23: For a space (Ӿ, ŧ, 𝔾), $□_2$ ↪ Ǵ($Ŧ_1$, 𝔾) if and only if, for every $ҳ_1 \neq ҳ_2$ in Ӿ, there exists $\ddot{V}_1, \ddot{V}_2$ ∈ 𝔾gC(Ӿ) such that $ҳ_1$ ∈ ($\ddot{V}_1$ - $\ddot{V}_2$) and $ҳ_2$ ∈ ($\ddot{V}_2$ - $\ddot{V}_1$).

Proof: Let $ҳ_1 \neq ҳ_2$ whenever $ҳ_1$, $ҳ_2$ ∈ Ӿ, since $□_2$ ↪ Ǵ($Ŧ_1$, 𝔾) then by Theorem 3. 21, the space (Ӿ, ŧ, 𝔾) is a 𝔾g- $Ŧ_1$- space. Then Theorem 3. 22 is applicable.

Conversely By Theorem 3. 22 the grill topological space (Ӿ, ŧ, 𝔾) is a 𝔾g-$Ŧ_1$-space Then Theorem 3. 21 is applicable.

Corollary 3.24: Let (Ӿ, ŧ, 𝔾) is a 𝔾g- $Ŧ_1$- space if and only if $□_1$ ↬ Ǵ($Ŧ_1$, 𝔾).

Proof: By Theorem 3. 21, the proof is over.

Proposition 3.25: Let (Ӿ, ŧ, 𝔾) is not 𝔾g- $Ŧ_1$- space if and only if $□_1$ ↪ Ǵ($Ŧ_1$, 𝔾)

Proof: In the □-th inning $□_1$ in Ǵ($Ŧ_1$, 𝔾) choose $ҳ_□ \neq ş_□$, whenever $ҳ_□, ş_□$ ∈ Ӿ, $□_2$ in Ǵ($Ŧ_1$, 𝔾), cannot be found $Ư_□$, $Ɯ_□$ are two 𝔾g-open sets such that $ҳ_□$ ∈ ($Ư_□$ - $Ɯ_□$) and $ş_□$ ∈ ($Ɯ_□$ - $Ư_□$) because (Ӿ, ŧ, 𝔾) is not 𝔾g-$Ŧ_1$-space hence $□_1$ ↪ Ǵ($Ŧ_1$, 𝔾).

Conversely Clear.

Corollary 3.26: Let (Ӿ, ŧ, 𝔾) is not 𝔾g-$Ŧ_1$-space if and only if $□_2$ ↬ Ǵ($Ŧ_1$, 𝔾).

Proof: By theorem 3. 25. The proof is over.

Definition 3.27: Let (Ӿ, ŧ, 𝔾) be a grill topological space, define a game Ǵ($Ŧ_2$, 𝔾) as follows: The two players $□_1$ and $□_2$ are play an inning for each natural numbers, in the □- th inning, the first round, $□_1$ will choose $ҳ_□ \neq ş_□$, whenever $ҳ_□$, $ş_□$ ∈ Ӿ. Next, $□_2$ choose $Ư_□$, $Ɯ_□$ are disjoint, $Ư_□$, $Ɯ_□$ ∈ 𝔾g- O(Ӿ) such that $ҳ_□$ ∈ $Ư_□$ and $ş_□$ ∈ $Ɯ_□$. $□_2$ wins in the game, whenever $\mathcal{B}$={{$Ư_1$, $Ɯ_1$}, {$Ư_2$, $Ɯ_2$},….,{$Ư_□$, $Ɯ_□$},….} satisfies that for all $ҳ_□ \neq ş_□$ in Ӿ there exists {$Ư_□$, $Ɯ_□$} ∈ $\mathcal{B}$ such that $ҳ_□$ ∈ $Ư_□$ and $ş_□$ ∈ $Ɯ_□$. Other hand $□_1$ wins.

By the same way of Example 3. 18 we can be explained that $□_2$ wins in the game Ǵ($Ŧ_2$, 𝔾), whenever Ư, Ɯ are two disjoint, 𝔾g-open sets and $\mathcal{B}$ be a collection of all disjoint 𝔾g-open sets in Ӿ other hand $□_1$ wins.

Example 3.28: Let Ǵ($Ŧ_2$, 𝔾) be a game Ӿ={ā̈, ɓ, ɕ} and ŧ={Ӿ, ∅}, 𝔾={Ӿ, {ā̈}, {ā̈, ɓ}, {ā̈, ɕ}}, 𝔾gC(Ӿ)={Ӿ, ∅, {ā̈}, {ā̈, ɓ}, {ā̈, ɕ}}, 𝔾gO(Ӿ)={Ӿ, ∅, {ɓ}, {ɕ}, {ɓ, ɕ}} then in the first round $□_1$ will choose ā̈ ≠ ɓ, whenever ā̈, ɓ ∈ Ӿ. Next $□_2$ cannot be found $Ư_ɱ$, $Ɯ_ɱ$ ∈ 𝔾gO(Ӿ) such that ā̈ ∈ $Ư_ɱ$ and ɓ ∈ $Ɯ_ɱ$, $Ư_ɱ \cap Ɯ_ɱ$ = ∅ thus $□_1$ wins in the game.

Remark 3.29: For any (Ӿ, ŧ, 𝔾):

i. If $\square_2$ ↪ Ǵ(Ŧ$_2$, Ӿ), then $\square_2$ ↪ Ǵ(Ŧ$_2$, 𝔾).

ii. If $\square_2$ ↩ Ǵ (Ŧ$_2$, Ӿ), then $\square_2$ ↩ Ǵ(Ŧ$_2$, 𝔾).

iii. If $\square_1$ ↪ Ǵ(Ŧ$_2$, 𝔾), then $\square_1$ ↪ Ǵ(Ŧ$_2$, Ӿ).

Proof: Is clear by Remark 3. 3: (ii)

Theorem 3.30: A space (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space if and only if $\square_2$ ↪ Ǵ(Ŧ$_2$, 𝔾).

Proof: Let (Ӿ, ŧ, 𝔾) be a grill topological space in the first round $\square_1$ will choose ӿ$_1$ ≠ ş$_1$, whenever ӿ$_1$, ş$_1$ ∈ Ӿ. Next since (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space $\square_2$ can be found Ư$_1$ and Ɯ$_1$∈ 𝔾g-O(Ӿ) such that ӿ$_1$ ∈ Ư$_1$ *and* ş$_1$ ∈ Ɯ$_1$, Ư$_1$ ∩ Ɯ$_1$ = ∅ in the second round $\square_1$ will choose ӿ$_2$ ≠ş$_2$. whenever ӿ$_2$, ş$_2$ ∈ Ӿ. Next $\square_2$ choose Ư$_2$ and Ɯ$_2$ ∈ 𝔾gO(Ӿ) such that ӿ$_2$ ∈ Ư$_2$ and ş$_2$ ∈ Ɯ$_2$, Ư$_2$ ∩ Ɯ$_2$ = ∅ in the ɱ–th round $\square_1$ will choose ӿ$_ɱ$≠ ş$_ɱ$. whenever ӿ$_ɱ$, ş$_ɱ$ ∈ Ӿ. Next $\square_2$ choose Ư$_ɱ$ and Ɯ$_ɱ$ ∈ 𝔾gO(Ӿ) such that ӿ$_ɱ$ ∈ Ư$_ɱ$ and ş$_ɱ$ ∈ Ɯ$_ɱ$, Ư$_ɱ$ ∩ Ɯ$_ɱ$ = ∅. Thus $\mathcal{B}$ = {{Ư$_1$, Ɯ$_1$}, {Ư$_2$, Ɯ$_2$}, …,{Ư$_ɱ$, Ɯ$_ɱ$}…} is the winning strategy for $\square_2$.

Conversely Clear.

From Theorem (3. 30) we get

Corollary 3.31: A space (Ӿ, ŧ, 𝔾) is a 𝔾g-Ŧ$_2$-space if and only if $\square_1$↬ Ǵ(Ŧ$_2$, 𝔾).

Theorem 3.32: A space (Ӿ, ŧ, 𝔾) is not 𝔾g-Ŧ$_2$-space if and only if $\square_1$ ↪Ǵ(Ŧ$_2$, 𝔾).

Proof: By corollary 3. 31 the proof is over.

Corollary 3.33: A space (Ӿ, ŧ, 𝔾) is not 𝔾g-Ŧ$_2$-space if and only if $\square_2$↬Ǵ(Ŧ$_2$, 𝔾).

Proof: By theorem 3. 32 the proof is over.

Remark 3.34: For any space (Ӿ, ŧ, 𝔾):

i. If $\square_2$ ↪Ǵ(Ŧ$_{i+1}$, 𝔾), then $\square_2$ ↪Ǵ(Ŧ$_i$, 𝔾), whenever i={0, 1}.

ii. If $\square_2$ ↪Ǵ(Ŧ$_i$, Ӿ), then $\square_2$ ↪Ǵ(Ŧ$_i$, 𝔾), whenever i={0, 1, 2}.

The following Diagram 3. 1 clarifies the relationships given in the Remark 3. 34.

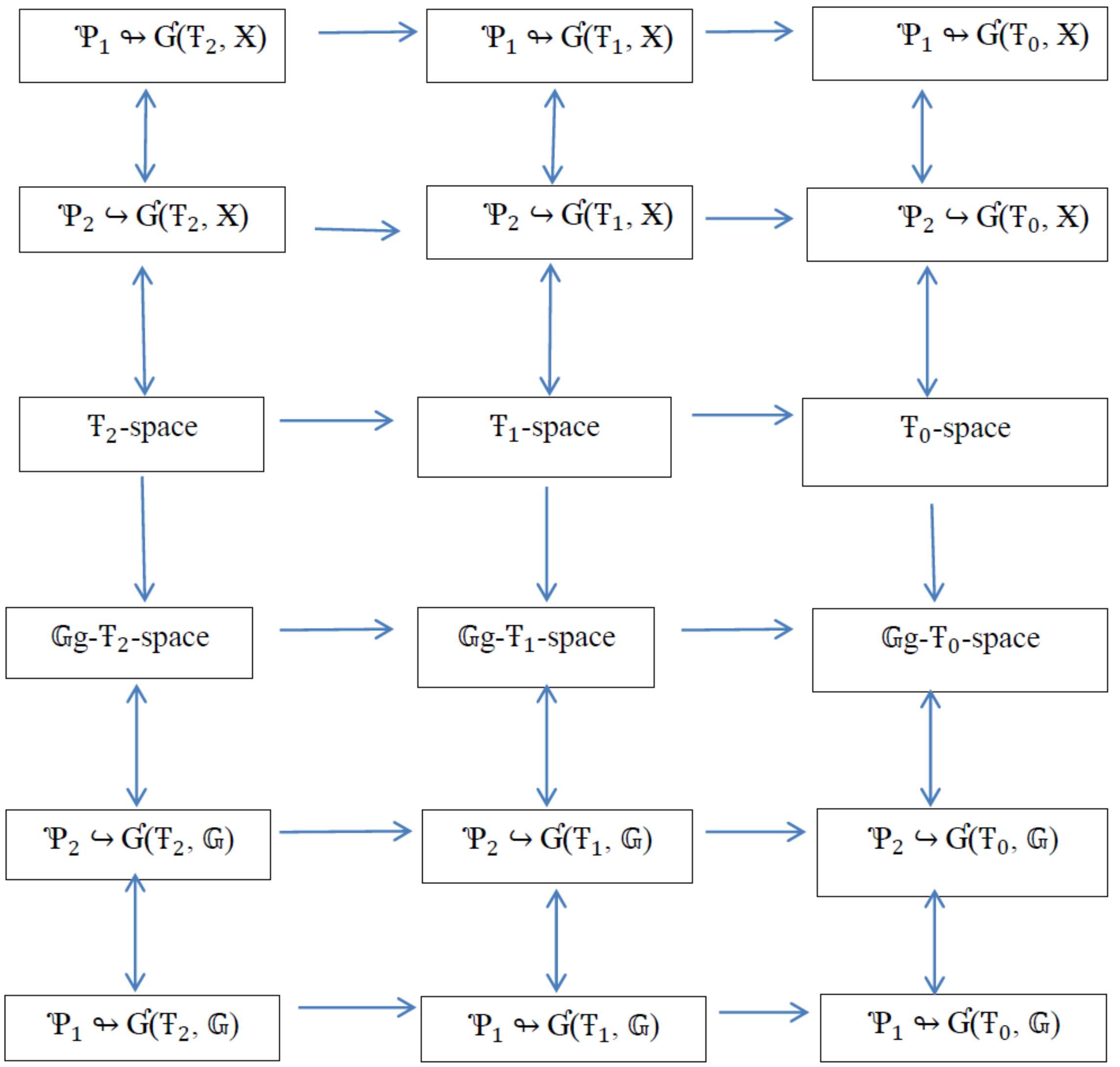

Diagram (3.1)

The winning and losing strategy for any player in $\acute{G}(Ŧ_i, 𝕏)$ and $\acute{G}(Ŧ_i, \mathbb{G})$.

Remark 3. 35: For any space (𝕏, ŧ, $\mathbb{G}$):

i. If $Ƥ_1 \hookrightarrow \acute{G}(Ŧ_i, \mathbb{G})$, then $Ƥ_1 \hookrightarrow \acute{G}(Ŧ_{i+1}, \mathbb{G})$, whenever i= {0, 1}.
ii. If $Ƥ_2 \looparrowright \acute{G}(Ŧ_i, \mathbb{G})$, then $Ƥ_2 \looparrowright \acute{G}(Ŧ_{i+1}, \mathbb{G})$, whenever i= {0, 1}.
iii. If $Ƥ_1 \hookrightarrow \acute{G}(Ŧ_i, \mathbb{G})$, then $Ƥ_1 \hookrightarrow \acute{G}(Ŧ_i, 𝕏)$, whenever i= {0, 1, 2}.

The following Diagram 3. 2 clarifies the relationships given in the Remark 3. 35.

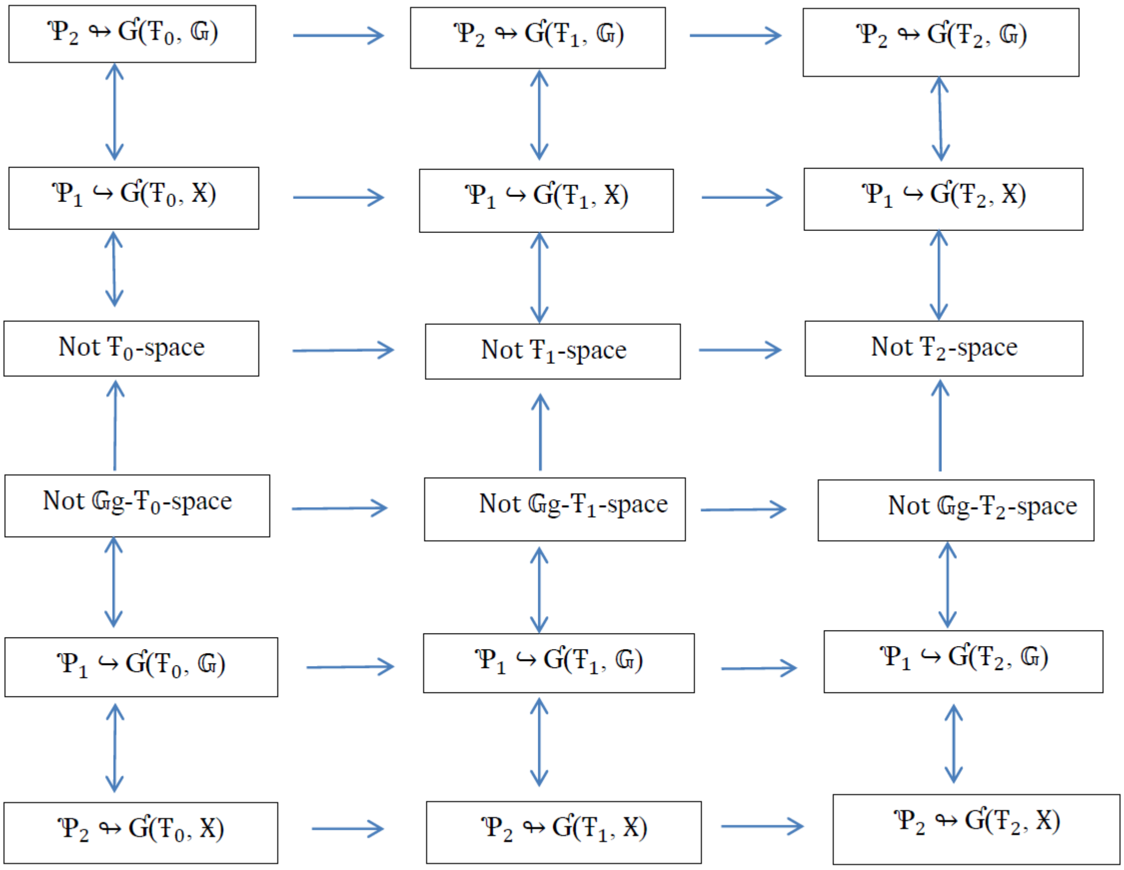

Diagram (3.2)

The winning and losing strategy whenever Ӿ is not 𝔾g-Ŧ$_i$-space and not Ŧ$_i$-space

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