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Combining Laplace transform and Adomian decomposition method for solving singular IVPs of Emden-Fowler of partial differential equations

Waleed Al-Hayani[1],* Fatima M. Yassin[2]

Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq
**Corresponding author. Email: waleedalhayani@uomosul.edu.iq / waleedalhayani@yahoo.es*
**https://orcid.org/0000-0001-9918-8573*
E-mail address: fatima.21csp40@student.uomosul.edu.iq[1]

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Abstract

In this paper, the time-dependent Emden-Fowler type partial differential equations and wave-type equations with singular behavior at $x = 0$ are analytically solved using the combined Laplace transform and Adomain decomposition method (LT-ADM). To avoid the singularity behavior for both models at $x = 0$, the benefit of this single global technique is used to present a solid framework. The method is shown to produce approximate-exact solutions to various kinds of problems in One-dimensional space. The results gained in each case demonstrate the dependability and effectiveness of this approach. To show the high accuracy of the approximate solution results (LT-ADM), compare the absolute errors obtained by the Padé approximation **(PA)** of order $[N/M]$ compared with the exact solution.

Keywords:
Emden-Fowler equation; Wave-type equation; Laplace Transform method; Adomian Decomposition Method; Adomian Polynomials.

Correspondence:
Author: Waleed Al-Hayani
Email: waleedalhayani@uomosul.edu.iq

1. INTRODUCTION

Many problems in the literature of the diffusion of heat perpendicular to the surfaces of parallel planes are modeled by the heat equation [1-3]

$$x^{-r}(x^r y_x)_x + af(x,t)g(y) + h(x,t) = y_t, \quad 0 < x \le L, 0 < t < T, r > 0, \quad \dots (1)$$

or equivalently

$$y_{xx} + \frac{r}{x}y_x + af(x,t)g(y) + h(x,t) = y_t, \quad 0 < x \le L, 0 < t < T, r > 0, \quad \dots (2)$$

where r, L and T are constants, $f(x,t)g(y) + h(x,t)$ is the linear heat source, $y(x,t)$ is the temperature, and t is the dimensionless time variable.

On the other hand, the wave type of equations with singular behavior of the form

$$x^{-r}(x^r y_x)_x + af(x,t)g(y) + h(x,t) = y_{tt}, \quad 0 < x \le L, 0 < t < T, r > 0, \quad \dots (3)$$

or equivalently

$$y_{xx} + \frac{r}{x}y_x + af(x,t)g(y) + h(x,t) = y_{tt}, \quad 0 < x \le L, 0 < t < T, r > 0, \quad \dots (4)$$

will be examined as well, where $f(x,t)g(y) + h(x,t)$ is a linear source, t is the dimensionless time variable, and $y(x,t)$ is the displacement of the wave at position x and at time t.

The singularity behavior that occurs at the point $x = 0$ is the main difficulty in the analysis of equations (2) and (4). In the recent literature there is many approximated methods to get an analytical solution for the time-dependent Emden-Fowler type of equations and wave-type equation with singular behavior were presented by [4-7].

At the beginning of the 1980's, Adomian [8-10] proposed a new and fruitful method (hereafter called the Adomian Decomposition Method or ADM) for solving linear and nonlinear (algebraic, differential, partial differential, integral, etc.) equations [11-16]. It has been shown that this method yields a rapid convergence of the solutions series to linear and nonlinear deterministic and stochastic equations. Recently, Mkhatshwa et al. [17] have presented a new modification to the bivariate spectral collocation method in solving Emden-Fowler equations. Naveen et al. [18] have studied of fractional Emden–Fowler (FEF) equations by utilizing a new adequate procedure, specifically the q-homotopy analysis transform method (q-HATM).

The main objective of this paper is to apply the combined Laplace transform method and Adomain decomposition method (LT-ADM) to obtain approximate-exact solutions for different models for the time-dependent Emden-Fowler type of equations and wave-type equation with singular behavior at $x = 0$. While the VIM [19] requires the determination of the Lagrange multiplier in its computational algorithm, LT-ADM is independent of any such requirements, LT-ADM handles linear and nonlinear terms in a simple and straightforward manner without any additional requirements.

II. APPLICATIONS OF LT-ADM TO EMDEN-FOWLER OF PDES

Taking the Laplace transform on both sides of Eqs. ...(2) and (4) gives

$$s\mathcal{L}[y(x,t)] = y(x,0) + \mathcal{L}[h(x,t)] + \mathcal{L}[y_{xx}] + \mathcal{L}\left[\frac{r}{x}y_x\right] + a\mathcal{L}[f(x,t)Ny(x,t)], \quad \dots (5)$$

and

$$s^2\mathcal{L}[y(x,t)] = sy(x,0) + y_t(x,0) + \mathcal{L}[h(x,t)] + \mathcal{L}[y_{xx}] + \mathcal{L}\left[\frac{r}{x}y_x\right] + a\mathcal{L}[f(x,t)Ny(x,t)], \quad \dots (6)$$

where $Ny(x,t) = g(y)$.

Simplifying and taking the inverse Laplace transform on both sides of Eqs. (5) and (6) we get

$$y(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)] + \mathcal{L}^{-1}s^{-1}\mathcal{L}[h(x,t)] + \mathcal{L}^{-1}s^{-1}\mathcal{L}[y_{xx}] + \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{r}{x}y_x\right] + a\mathcal{L}^{-1}s^{-1}\mathcal{L}[f(x,t)Ny(x,t)], \quad \dots (7)$$

and

$$y(x,t) = \mathcal{L}^{-1}s^{-1}y(x,0) + \mathcal{L}^{-1}s^{-2}y_t(x,0) + \mathcal{L}^{-1}s^{-2}\mathcal{L}[h(x,t)] + \mathcal{L}^{-1}s^{-2}\mathcal{L}[y_{xx}] + \mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{r}{x}y_x\right] + a\mathcal{L}^{-1}s^{-2}\mathcal{L}[f(x,t)Ny(x,t)]. \quad \dots (8)$$

The Adomian technique [7-9] consists of approximating the solution as an infinite series

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t), \quad \dots (9)$$

and decomposing the nonlinear operator N as

$$Ny(x,t) = \sum_{n=0}^{\infty} A_n, \quad \dots (10)$$

where A_n are polynomials (called Adomian polynomials) of $y_0, y_1, \dots, y_n$ [7-9] given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad n = 0,1,2, \dots .$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty} y_n$ and $\sum_{n=0}^{\infty} A_n$ are given in [20-24].

Substituting the equation (9) and (10) into the Eqs. (7) and (8) yields

$$\sum_{n=0}^{\infty} y_n(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)] + \mathcal{L}^{-1}s^{-1}\mathcal{L}[h(x,t)] + \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right] + \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{r}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right] + a\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[f(x,t)\sum_{n=0}^{\infty}A_n\right], \quad \dots (11)$$

and

$$\sum_{n=0}^{\infty} y_n(x,t) = \mathcal{L}^{-1}s^{-1}y(x,0) + \mathcal{L}^{-1}s^{-2}y_t(x,0) + \mathcal{L}^{-1}s^{-2}\mathcal{L}[h(x,t)] + \mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right] + \mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{r}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right] + a\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[f(x,t)\sum_{n=0}^{\infty}A_n\right]. \quad \dots (12)$$

From the Eq. (11), the iterates are then determined in the following recursive way

$$y_0(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)] + \mathcal{L}^{-1}s^{-1}\mathcal{L}[h(x,t)],$$
$$y_{n+1}(x,t) = \mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_n)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{r}{x}(y_n)_x(x,t)\right] + a\mathcal{L}^{-1}s^{-1}\mathcal{L}[f(x,t)A_n], \quad n \geq 0 \quad \dots (13)$$

and from Eq. (12), the iterates are then determined in the following recursive way

$$y_0(x,t) = \mathcal{L}^{-1}s^{-1}y(x,0) + \mathcal{L}^{-1}s^{-2}y_t(x,0) + \mathcal{L}^{-1}s^{-2}\mathcal{L}[h(x,t)],$$
$$y_{n+1}(x,t) = \mathcal{L}^{-1}s^{-2}\mathcal{L}[(y_n)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{r}{x}(y_n)_x(x,t)\right] + a\mathcal{L}^{-1}s^{-2}\mathcal{L}[f(x,t)A_n], \quad n \geq 0 \quad \dots (14)$$

Thus, all components of y can be calculated once the A_n are given for $n = 0,1, \dots$. We then define the n-term approximant to the solution y by $\phi_n[y] = \sum_{i=0}^{n-1} y_i$ with $\lim_{n\to\infty}\phi_n[y] = y$.

III. APPLICATIONS AND NUMERICAL RESULTS

In this section, we examine distinct models with singular behavior at $x = 0$, two linear time-dependent Emden-Fowler type of equations, two linear models of wave-type equation and one model nonlinear. To show the high accuracy of the approximate solution results (LT-ADM) and the Padé approximation **(PA)** of order $[N/M]$ compared with the exact solution, the absolute errors between them are defined as follows:

$AE_1 = |\text{Exact Solution} - (\text{LT} - \text{ADM})|$,

$AE_2 = |\text{Padé Approximation} - (\text{LT} - \text{ADM})|$.

With a precision of 20 digits, the computations related to the examples were carried out using the Maple 18 package.

A. TIME-DEPENDENT EMDEN-FOWLER TYPE

Problem 1. Solve the following linear Emden–Fowler type equation [4,5,7] by using LT-ADM

$$y_{xx}+\frac{2}{x}y_x-(6-4x^2-\cos t)y=y_t, \quad \dots(15)$$

with the initial condition

$$y(x,0)=e^{x^2}. \quad \dots(16)$$

The exact solution for this problem is

$$y_{Exact}(t)=e^{x^2+sint}. \quad \dots(17)$$

Taking the Laplace transform on both sides of equation (15) gives

$$s\mathcal{L}[y(x,t)]=y(x,0)+\mathcal{L}[y_{xx}]+\mathcal{L}\left[\frac{2}{x}y_x\right]$$
$$-\mathcal{L}[(6-4x^2-\cos t)y]. \quad \dots(18)$$

Simplifying and taking the inverse Laplace transform on both sides of equation (18) we get

$$y(x,t)=\mathcal{L}^{-1}s^{-1}[y(x,0)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}[y_{xx}]$$
$$+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}y_x\right]-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y], \quad \dots(19)$$

Substituting the equation (9) into the equation (19) yields

$$\sum_{n=0}^{\infty}y_n(x,t)=\mathcal{L}^{-1}s^{-1}[y(x,0)]$$
$$+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right]$$
$$+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[(6-4x^2-\cos t)\sum_{n=0}^{\infty}y_n(x,t)\right]. \quad \dots(20)$$

From the equation (20), the iterates are then determined in the following recursive way

$$y_0(x,t)=\mathcal{L}^{-1}s^{-1}[y(x,0)],$$
$$y_{n+1}(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_n)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}(y_n)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y_n(x,t)].\quad n\geq 0 \quad \dots(21)$$

Following the algorithm (21), the iterations are given by

$$y_0(x,t)=e^{x^2},$$
$$y_1(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_0)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}(y_0)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y_0(x,t)],$$
$$=e^{x^2}\sin t,$$
$$y_2(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_1)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}(y_1)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y_1(x,t)],$$
$$=\frac{1}{2}e^{x^2}-\frac{1}{2}e^{x^2}\cos^2(t),$$
$$y_3(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_2)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}(y_2)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y_2(x,t)],$$
$$=\frac{1}{6}e^{x^2}\sin^3(t),$$
$$y_4(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_3)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}(y_3)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y_3(x,t)],$$
$$=\frac{1}{24}e^{x^2}\sin^4(t),$$
$$y_5(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_4)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}(y_4)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(6-4x^2-\cos t)y_4(x,t)],$$
$$=\frac{1}{120}e^{x^2}\sin^5(t),$$
$$\vdots$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate solution is

$$\phi_6(x,t)=\sum_{n=0}^{5}y_n(x,t)$$
$$=\frac{1}{120}e^{x^2}\{[-22\cos^2(t)+\cos^4(t)+141]\sin(t)$$
$$+5\cos^4(t)-70\cos^2(t)+185\}.$$

This series has the closed form as $n\to\infty$ gives e^{x^2+sint}, i.e.,

$$y_{Exact}(x,t)=e^{x^2+sint},$$

which is the exact solution of the problem 1.

In Table 1 show a comparison of the numerical results applying the LT-ADM $(\boldsymbol{\phi_6(x,t)})$ and the Padé approximants (PA) of order [6/6] with the exact solution $(y_{Exact}(x,t))$ obtained

$$[6/6]=\frac{p_6(x)}{q_6(x)},$$

where

$$p_6(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6,$$
$$=1.6151282029+0.8075641014x^2$$
$$+0.1615128202x^4+0.0134594016x^6,$$
$$q_6(x)=1+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6,$$
$$=1.00-0.5000000000x^2+0.1000000000x^4$$
$$-0.0083333333x^6.$$

As shown in **(Cherruault and Adomian, 1993)** [23], the necessary condition for the convergence of the method is that $\|y_{n+1}\|_2<\|y_n\|_2$ for all n. In Figure 1, we represent the plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n=0,1,\dots,8$. In Figure 2, a very good agreement is shown between the exact solution $(y_{Exact}(x,t))$ with a continuous line and the LT-ADM $(\phi_6(x,t))$ with the symbol ∘. In Figure 3, we present the contour plot in $2D$ on the $(x,t)-$plane for the exact solution $(y_{Exact}(x,t))$ and the LT-ADM $(\phi_6(x,t))$.

Problem 2. Solve the following linear Emden–Fowler type equation [4,5,7] by using LT-ADM

$$y_{xx}+\frac{2}{x}y_x-(5+4x^2)y=y_t+(6-5x^2-4x^4), \quad \dots(22)$$

with the initial condition

$$y(x,0) = x^2 + e^{x^2}. \qquad \dots (23)$$

The exact solution for this problem is

$$y_{Exact}(t) = x^2 + e^{x^2+t}. \qquad \dots (24)$$

Taking the Laplace transform on both sides of equation (22) gives

$$s\mathcal{L}[y(x,t)] = y(x,0) - \mathcal{L}[6 - 5x^2 - 4x^4] + \mathcal{L}[y_{xx}] + \mathcal{L}\left[\frac{2}{x}y_x\right] - \mathcal{L}[(5+4x^2)y]. \qquad \dots (25)$$

Simplifying and taking the inverse Laplace transform on both sides of equation (25) we get

$$y(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)] - \mathcal{L}^{-1}s^{-1}\mathcal{L}[6 - 5x^2 - 4x^4] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[y_{xx}] + \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}y_x\right] - \mathcal{L}^{-1}s^{-1}\mathcal{L}[(5+4x^2)y]. \qquad \dots (26)$$

Substituting the equation (9) into the equation (26) yields

$$\sum_{n=0}^{\infty} y_n(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)] - \mathcal{L}^{-1}s^{-1}\mathcal{L}[6 - 5x^2 - 4x^4] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right] + \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{2}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\mathcal{L}\left[(5+4x^2)\sum_{n=0}^{\infty}y_n(x,t)\right]. \qquad \dots (27)$$

From the equation (27), the iterates by the ADM are then determined in the following recursive way

$$y_0(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)] - \mathcal{L}^{-1}s^{-1}\mathcal{L}[6 - 5x^2 - 4x^4],$$

$$y_{n+1}(x,t) = \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_n)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_n)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_n(x,t)]. \quad n \ge 0 \qquad \dots (28)$$

From the equation (27), the iterates by the MADM are then determined in the following recursive way

$$y_0(x,t) = \mathcal{L}^{-1}s^{-1}[y(x,0)],$$

$$y_1(x,t) = -\mathcal{L}^{-1}s^{-1}\mathcal{L}[6 - 5x^2 - 4x^4] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_0)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_0)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_0(x,t)],$$

$$y_{n+1}(x,t) = \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_n)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_n)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_n(x,t)]. \quad n \ge 1 \qquad \dots (29)$$

Following the algorithm (29), the iterations are given by

$$y_0(x,t) = x^2 + e^{x^2},$$

$$y_1(x,t) = te^{x^2},$$

$$y_2(x,t) = \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_1)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_1)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_1(x,t)] = \frac{1}{2}t^2e^{x^2},$$

$$y_3(x,t) = \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_2)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_2)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_2(x,t)] = \frac{1}{6}t^3e^{x^2},$$

$$y_4(x,t) = \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_3)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_3)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_3(x,t)] = \frac{1}{24}t^4e^{x^2},$$

$$y_5(x,t) = \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_4)_{xx}(x,t)] + \mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_4)_x(x,t)\right] - \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_4(x,t)] = \frac{1}{120}t^5e^{x^2},$$

$\vdots$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate solution is

$$\phi_6(x,t) = \sum_{n=0}^{5} y_n(x,t) = x^2 + e^{x^2}\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5\right).$$

This series has the closed form as $n \to \infty$ gives $x^2 + e^{x^2+t}$, i.e., $y_{Exact}(x,t) = x^2 + e^{x^2+t}$, which is the exact solution of the problem 2.

In Table 2 show a comparison of the numerical results applying the LT-ADM $\left(\boldsymbol{\phi_6(x,t)}\right)$ and the Padé approximants (PA) of order [6/6] with the exact solution $\left(y_{Exact}(x,t)\right)$ obtained

$$[6/6] = \frac{p_6(x)}{q_6(x)},$$

where

$$p_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6, = 1.64869791666 + 2.00167795950x^2 - 0.12117580801x^4 + 0.09731569963x^6,$$

$$q_6(x) = 1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 = 1.0 - 0.392443000392x^2 + 0.056977200156x^4 - 0.002955483352x^6.$$

As it is shown in (**Cherruault and Adomian, 1993)** [23], the necessary condition for the convergence of the method is that $\|y_{n+1}\|_2 < \|y_n\|_2$ for all n. In Figure 4, we represent the plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \dots, 8$. In Figure 5, a very good agreement is shown between the exact solution $(y_{Exact}(x,t))$ with a continuous line and the LT-ADM $(\phi_6(x,t))$ with the symbol ∘. In Figure 6, we present the contour plot in $2D$ on the $(x,t)-$plane for the exact solution $\left(y_{Exact}(x,t)\right)$ and the LT-ADM $\left(\phi_6(x,t)\right)$.

B. SINGULAR WAVE-TYPE EQUATIONS

Problem 3. Solve the following linear linear inhomogeneous singular wave-type equation [4,5,7] by using LT-ADM

$$y_{xx} + \frac{2}{x}y_x - (5+4x^2)y = y_{tt} + (12x - 5x^3 - 4x^5), \dots (30)$$

with the initial conditions

$$y(x,0) = x^3 + e^{x^2}, \quad y_t(x,0) = e^{x^2} \qquad \dots (31)$$

The exact solution for this problem is

$$y_{Exact}(t) = x^2 + e^{x^2-t}. \qquad \dots (32)$$

Taking the Laplace transform on both sides of equation (30) gives

$$s^2\mathcal{L}[y(x,t)] = sy(x,0) + y_t(x,0) - \mathcal{L}[12x - 5x^3 - 4x^5]$$

$$+\mathcal{L}[y_{xx}]+\mathcal{L}\left[\frac{2}{x}y_x\right]-\mathcal{L}[(5+4x^2)y]. \quad \ldots (33)$$

Simplifying and taking the inverse Laplace transform on both sides of equation (33) we get

$$\begin{aligned} y(x,t) &= \mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0) \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[12x-5x^3-4x^5] \\ &+\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[y_{xx}]+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{2}{x}y_x\right] \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(5+4x^2)y]. \quad \ldots (34) \end{aligned}$$

Substituting the equation (9) into the equation (34) yields

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x,t) &= \mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0) \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[12x-5x^3-4x^5] \\ &+\mathcal{L}^{-1}s^{-2}\,\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right] \\ &+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{2}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right] \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[(5+4x^2)\sum_{n=0}^{\infty}y_n(x,t)\right]. \quad \ldots (35) \end{aligned}$$

From the equation (35), the iterates by the ADM are then determined in the following recursive way

$$\begin{aligned} y_0(x,t) &= \mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0) \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[12x-5x^3-4x^5], \\ y_{n+1}(x,t) &= \mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_n)_{xx}(x,t)] \\ &+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{2}{x}(y_n)_x(x,t)\right] \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(5+4x^2)y_n(x,t)],\ n\geq 0. \quad \ldots (36) \end{aligned}$$

and from the equation (35), the iterates by the MADM are then determined in the following recursive way

$$\begin{aligned} y_0(x,t) &= \mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0), \\ y_1(x,t) &= -\mathcal{L}^{-1}s^{-2}\mathcal{L}[12x-5x^3-4x^5] \\ &+\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_0)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{2}{x}(y_0)_x(x,t)\right] \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(5+4x^2)y_0(x,t)], \\ y_{n+1}(x,t) &= \mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_n)_{xx}(x,t)] \\ &+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{2}{x}(y_n)_x(x,t)\right] \\ &-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(5+4x^2)y_n(x,t)],\quad n\geq 1. \quad \ldots (37) \end{aligned}$$

Following the algorithm (37), the iterations are given by

$$\begin{aligned} y_0(x,t) &= x^3+e^{x^2}(1-t), \\ y_1(x,t) &= e^{x^2}\left(\frac{1}{2!}t^2-\frac{1}{3!}t^3\right), \\ y_2(x,t) &= \mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(y_1)_{xx}(x,t)] \\ &+\mathcal{L}^{-1}s^{-1}\,\mathcal{L}\left[\frac{2}{x}(y_1)_x(x,t)\right]-\mathcal{L}^{-1}s^{-1}\,\mathcal{L}[(5+4x^2)y_1(x,t)], \\ &= e^{x^2}\left(\frac{1}{4!}t^4-\frac{1}{5!}t^5\right), \\ &\vdots \end{aligned}$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate solution is

$$\phi_3(x,t)=\sum_{n=0}^{2}y_n(x,t)$$

$$= x^3+e^{x^2}\left(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5\right).$$

This series has the closed form as $n\to\infty$ gives $x^3+e^{x^2-t}$, i.e., $y_{Exact}(x,t)=x^3+e^{x^2-t}$, which is the exact solution of the problem 3.

In Table 3 show a comparison of the numerical results applying the LT-ADM $(\boldsymbol{\phi_3(x,t)})$ and the Padé approximants (PA) of order [8/8] with the exact solution $(y_{Exact}(x,t))$ obtained

$$[8/8]=\frac{p_8(x)}{q_8(x)},$$

where

$$\begin{aligned} p_8(x) &= a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6 \\ &+a_7x^7+a_8x^8, \\ &= 0.606510416666+0.262060995314x \\ &+0.391636911511x^2+1.185271981523x^3 \\ &+0.547567047241x^4-0.293159163274x^5 \\ &-0.106761586249x^6+0.056786924510x^7 \\ &+0.013384260296x^8, \end{aligned}$$

and

$$\begin{aligned} q_8(x) &= 1+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6 \\ &+b_7x^7+b_8x^8, \\ &= 1.0+0.432079957925x-0.354278342549x^2 \\ &-0.126607905949x^3+0.044690718235x^4 \\ &+0.011339781178x^5-0.001496072086x^6 \\ &-0.000105216313x^7-0.000098674559x^8. \end{aligned}$$

As it is shown in (**Cherruault and Adomian, 1993)** [23], the necessary condition for the convergence of the method is that $\|y_{n+1}\|_2<\|y_n\|_2$ for all n. In Figure 7, we represent the plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n=0,1,\ldots,8$. In Figure 8, a very good agreement is shown between the exact solution $(y_{Exact}(x,t))$ with a continuous line and the LT-ADM $(\phi_3(x,t))$ with the symbol ∘. In Figure 9, we present the contour plot in $2D$ on the $(x,t)-$plane for the exact solution $(y_{Exact}(x,t))$ and the LT-ADM $(\phi_3(x,t))$.

Problem 4. Solve the following linear Emden–Fowler type equation [4,5,7] by using LT-ADM

$$y_{xx}+\frac{4}{x}y_x-(18x+9x^4)y=y_{tt}-2-(18x+9x^4)t^2, (38)$$

with the initial conditions

$$y(x,0)=e^{x^3}\quad y_t(x,0)=0 \quad \ldots (39)$$

The exact solution for this problem is

$$y_{Exact}(t)=t^2+e^{x^3}. \quad \ldots (40)$$

Taking the Laplace transform on both sides of equation (38) gives

$$\begin{aligned} s^2\mathcal{L}[y(x,t)] &= sy(x,0)+y_t(x,0)+\mathcal{L}[2] \\ &+\mathcal{L}[(18x+9x^4)t^2]+\mathcal{L}[y_{xx}]+\mathcal{L}\left[\frac{4}{x}y_x\right] \\ &-\mathcal{L}[(18x+9x^4)y]. \quad \ldots (41) \end{aligned}$$

Simplifying and taking the inverse Laplace transform on both sides of equation (41) we get

$$\begin{aligned} y(x,t) &= \mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0)+\mathcal{L}^{-1}s^{-2}\mathcal{L}[2] \\ &+\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)t^2]+\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[y_{xx}] \end{aligned}$$

$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}y_x\right]-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y]. \quad ...(42)$$

Substituting the equation (9) into the equation (42) yields

$$\sum_{n=0}^{\infty}y_n(x,t)=\mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0)$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}[2]+\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)t^2]$$
$$+\mathcal{L}^{-1}s^{-2}\,\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[(18x+9x^4)\sum_{n=0}^{\infty}y_n(x,t)\right]. \quad ...(43)$$

From the equation (43), the iterates ADM are then determined in the following recursive way

$$y_0(x,t)=\mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0)+\mathcal{L}^{-1}s^{-2}\mathcal{L}[2]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)t^2],$$
$$y_{n+1}(x,t)=\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_n)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}(y_n)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y_n(x,t)],\quad n\geq 0 \quad ...(44)$$

and from the equation (43), the iterates MADM are then determined in the following recursive way

$$y_0(x,t)=\mathcal{L}^{-1}s^{-1}y(x,0)+\mathcal{L}^{-1}s^{-2}y_t(x,0),$$
$$y_1(x,t)=\mathcal{L}^{-1}s^{-2}\mathcal{L}[2]+\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)t^2]$$
$$+\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_0)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}(y_0)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y_0(x,t)],$$
$$y_{n+1}(x,t)=\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_n)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}(y_n)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y_n(x,t)],\qquad n\geq 1 \quad ...(45)$$

Following the algorithm (45), the iterations are given by

$$y_0(x,t)=e^{x^3},$$
$$y_1(x,t)=\frac{3}{2}t^4x+\frac{3}{4}t^4x^4+t^2,$$
$$y_2(x,t)=\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_1)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}(y_1)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y_1(x,t)],$$
$$=-\frac{2}{5}t^6x^2+\frac{1}{10}\frac{t^6}{x}-\frac{9}{10}t^6x^5-\frac{3}{2}t^4x-\frac{9}{40}t^6x^8-\frac{3}{4}t^4x^4,$$
$$y_3(x,t)=\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_2)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}(y_2)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y_2(x,t)],$$
$$=-\frac{3}{40}t^8-\frac{207}{560}t^8x^3+\frac{9}{140}t^8x^6+\frac{2}{5}t^6x^2-\frac{1}{10}\frac{t^6}{x}$$
$$+\frac{243}{1120}t^8x^9+\frac{9}{10}t^6x^5+\frac{81}{2240}t^8x^{12}+\frac{9}{40}t^6x^8,$$
$$y_4(x,t)=\mathcal{L}^{-1}s^{-2}\,\mathcal{L}[(y_3)_{xx}(x,t)]$$
$$+\mathcal{L}^{-1}s^{-2}\mathcal{L}\left[\frac{4}{x}(y_3)_x(x,t)\right]$$
$$-\mathcal{L}^{-1}s^{-2}\mathcal{L}[(18x+9x^4)y_3(x,t)],$$
$$=-\frac{6}{175}t^{10}x+\frac{39}{350}t^{10}x^4+\frac{3}{40}t^8+\frac{27}{112}t^{10}x^7$$
$$+\frac{207}{560}t^8x^3+\frac{9}{700}t^{10}x^{10}-\frac{9}{140}t^8x^6-\frac{81}{2800}t^{10}x^{13}$$
$$-\frac{243}{1120}t^8x^9-\frac{81}{22400}t^{10}x^{16}-\frac{81}{2240}t^8x^{12},$$
$$\vdots$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate solution is

$$\phi_5(x,t)=\sum_{n=0}^{4}y_n\,(x,t)$$
$$=t^2+e^{x^3}-\frac{6}{175}xt^{10}+\frac{39}{350}x^4t^{10}+\frac{27}{112}x^7t^{10}$$
$$+\frac{9}{700}x^{10}t^{10}-\frac{81}{2800}x^{13}t^{10}-\frac{81}{22400}x^{16}t^{10}.$$

This series has the closed form as $n\to\infty$ gives $t^2+e^{x^3}$, i.e., $y_{Exact}(x,t)=t^2+e^{x^3}$, which is the exact solution of the problem 4.

In Table 4 show a comparison of the numerical results applying the LT-ADM $\left(\boldsymbol{\phi_5(x,t)}\right)$ and the Padé approximants (PA) of order [5/5] with the exact solution $\left(y_{Exact}(x,t)\right)$ obtained

$$[6/6]=\frac{p_6(x)}{q_6(x)}$$

where

$$p_6(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6$$
$$=\ 1.250-0.002811453703x+0.000006112998x^2$$
$$+0.374999986804x^3-0.000930611750x^4$$
$$+0.000002065597x^5+0.104166662198x^6,$$

and

$$q_6(x)=1+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6$$
$$=1.0-0.002222377248x+0.000004830870x^2$$
$$-0.500000010426x^3+0.000932965969x^4$$
$$-0.000001993762x^5+0.083333337626x^6.$$

In Figure 10, a very good agreement is shown between the exact solution ($y_{Exact}(x,t)$) with a continuous line and the LT-ADM ($\phi_5(x,t)$) with the symbol ∘. In Figure 11, we present the contour plot in $2D$ on the $(x,t)-$plane for the exact solution $\left(y_{Exact}(x,t)\right)$ and the LT-ADM $\left(\phi_8(x,t)\right)$.

C. NONLINEAR MODELS

Problem 5. Solve the following nonlinear Emden–Fowler type equation [4,5,7] by using LT-ADM

$$y_{xx}+\frac{5}{x}y_x-(24t+16t^2x^2)e^y-2x^2e^{\frac{y}{2}}=y_t,$$
$$0<t\leq 1 \quad ...(46)$$

with the initial conditions

$$y(x,0)=0 \quad ...(47)$$

The exact solution for this problem is

$$y_{Exact}(x,t)=-2ln(1+tx^2). \quad ...(48)$$

Taking the Laplace transform on both sides of equation (46) gives

$$s\mathcal{L}[y(x,t)]=y(x,0)+\mathcal{L}[y_{xx}]+\mathcal{L}\left[\frac{5}{x}y_x\right]$$

$$-\mathcal{L}[(24t+16t^2x^2)N_1y(t)]-\mathcal{L}[2x^2N_2y(t)]. \quad (49)$$

where $N_1y(t)=e^{y(t)}$ and $N_2y(t)=e^{y(t)/2}$.

Simplifying and taking the inverse Laplace transform on both sides of equation (49) we get

$$y(x,t)=\mathcal{L}^{-1}s^{-1}[y(x,0)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}[y_{xx}] +\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}y_x\right]-\mathcal{L}^{-1}s^{-1}\mathcal{L}[(24t+16t^2x^2)N_1y(t)] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[2x^2N_2y(t)]. \quad \dots(50)$$

Substituting the equations (9) and (10) into equation (50) yields

$$\sum_{n=0}^{\infty}y_n(x,t)=\mathcal{L}^{-1}s^{-1}[y(x,0)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\sum_{n=0}^{\infty}(y_n)_{xx}(x,t)\right] +\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}\sum_{n=0}^{\infty}(y_n)_x(x,t)\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[(24t+16t^2x^2)\sum_{n=0}^{\infty}A_n\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[2x^2\sum_{n=0}^{\infty}B_n\right]. \quad \dots(51)$$

From the equation (51), the iterates by the ADM are then determined in the following recursive way

$$y_0(x,t)=\mathcal{L}^{-1}s^{-1}[y(x,0)],$$

$$y_{n+1}(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_n)_{xx}(x,t)] +\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}(y_n)_x(x,t)\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[(24t+16t^2x^2)A_n] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[2x^2B_n]. \quad \dots(52)$$

For the nonlinear terms

$$N_1y=e^y=\sum_{n=0}^{\infty}A_n \quad \text{and} \quad N_2y=e^{y/2}=\sum_{n=0}^{\infty}B_n,$$

the Adomian polynomials are given in [8-10] as follows:

$$A_0=e^{y_0},$$
$$A_1=y_1e^{y_0},$$
$$A_2=\left(y_2+\frac{1}{2!}y_1^2\right)e^{y_0},$$
$$A_3=\left(y_3+y_1y_2+\frac{1}{3!}y_1^3\right)e^{y_0},$$
$$A_4=\left(y_4+\frac{1}{2!}y_2^2+y_1y_3+\frac{1}{2!}y_1^2y_2+\frac{1}{4!}y_1^4\right)e^{y_0},$$
$$\vdots$$

and

$$B_0=e^{y_0/2},$$
$$B_1=\frac{1}{2}y_1e^{y_0/2},$$
$$B_2=\left(\frac{1}{2}y_2+\frac{1}{2^2}\frac{1}{2!}y_1^2\right)e^{y_0/2},$$
$$B_3=\left(\frac{1}{2}y_3+\frac{1}{2^2}y_1y_2+\frac{1}{2^3}\frac{1}{3!}y_1^3\right)e^{y_0/2},$$
$$B_4=\left[\frac{1}{2}y_4+\frac{1}{2^2}\left(\frac{1}{2!}y_2^2+y_1y_3\right)+\frac{1}{2^3}\frac{1}{2!}y_1^2y_2 +\frac{1}{2^4}\frac{1}{4!}y_1^4\right]e^{y_0/2},$$
$$\vdots$$

Following the algorithm (52), the iterations are given by

$$y_0(x,t)=0,$$

$$y_1(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_0)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}(y_0)_x(x,t)\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[(24t+16t^2x^2)A_0]-\mathcal{L}^{-1}s^{-1}\mathcal{L}[2x^2B_0], = -2x^2t+12t^2+\frac{16}{3}x^2t^3,$$

$$y_2(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_1)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}(y_1)_x(x,t)\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[(24t+16t^2x^2)A_1]-\mathcal{L}^{-1}s^{-1}\mathcal{L}[2x^2B_1], = -12t^2+x^4t^2-20x^2t^3+88t^4-\frac{28}{3}x^4t^4 +64x^2t^5+\frac{128}{9}x^4t^6,$$

$$y_3(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_2)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}(y_2)_x(x,t)\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[(24t+16t^2x^2)A_2]-\mathcal{L}^{-1}s^{-1}\mathcal{L}[2x^2B_2], =\frac{44}{3}x^2t^3-\frac{2}{3}x^6t^3-132t^4+26x^4t^4 -\frac{5012}{15}x^2t^5+\frac{188}{15}x^6t^5-\frac{640}{3}x^4t^6+768t^6 +\frac{54784}{63}x^2t^7-\frac{1024}{21}x^6t^7+\frac{1024}{3}x^4t^8 +\frac{4096}{81}x^6t^9,$$

$$y_4(x,t)=\mathcal{L}^{-1}s^{-1}\mathcal{L}[(y_3)_{xx}(x,t)]+\mathcal{L}^{-1}s^{-1}\mathcal{L}\left[\frac{5}{x}(y_3)_x(x,t)\right] -\mathcal{L}^{-1}s^{-1}\mathcal{L}[(24t+16t^2x^2)A_3]-\mathcal{L}^{-1}s^{-1}\mathcal{L}[2x^2B_3], =\frac{1}{2}x^8t^4+44t^4-\frac{50}{3}x^4t^4+\frac{1964}{5}x^2t^5 -\frac{154}{5}x^6t^5-\frac{26584}{15}t^6-\frac{76}{5}x^8t^6+\frac{11072}{15}x^4t^6 -\frac{534088}{105}x^2t^7+\frac{47752}{105}x^6t^7-\frac{1431364}{315}x^4t^8 +\frac{33476}{315}x^8t^8+\frac{160448}{21}t^8+\frac{2194432}{189}x^2t^9 -\frac{107648}{63}x^6t^9+\frac{6375424}{945}x^4t^{10}-\frac{243968}{945}x^8t^{10} +\frac{16384}{9}x^6t^{11}+\frac{16384}{81}x^8t^{12},$$

$$\vdots$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate solution is

$$\phi_9(x,t)=\sum_{n=0}^{8}y_n(x,t) =-2x^2t+x^4t^2-\frac{2}{3}x^6t^3+\frac{1}{2}x^8t^4-\frac{2}{5}x^{10}t^5 +\frac{1}{3}x^{12}t^6-\frac{2}{7}x^{14}t^7+O(x^{16}t^8).$$

This series has the closed form as $n\to\infty$ gives $-2ln(1+tx^2)$, i.e.,

$$y_{Exact}(x,t)=-2ln(1+tx^2),$$

which is the exact solution of the problem 5.

In Table 5 show a comparison of the numerical results applying the LT-ADM $\left(\boldsymbol{\phi_9(x,t)}\right)$ and the Padé approximants (PA) of order $[4/4]$ with the exact solution $\left(y_{Exact}(x,t)\right)$ obtained

$$[4/4] = \frac{p_4(x)}{q_4(x)}$$

$$p_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$
$$= -0.000005854846 - 0.200005222322x^2$$
$$-0.006872376973x^4,$$
$$q_4(x) = 1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4,$$
$$= 1.00 + 0.084428707272x^2 + 0.000906357179x^4,$$

In Figure 12, a very good agreement is shown between the exact solution $(y_{Exact}(x,t))$ with a continuous line and the LT-ADM $(\phi_9(x,t))$ with the symbol ∘. In Figure 13, we present the contour plot in $2D$ on the $(x,t)-$ plane for the exact solution $\left(y_{Exact}(x,t)\right)$ and the LT-ADM $\left(\phi_9(x,t)\right)$.

Table 1. Numerical results for problem 1 at $t = 0.5$

x	$y_{Exact}(x,t)$	$\phi_6(x,t)$	AE_1	PA $[6/6]$	AE_2
0.0	1.615146296	1.615128202	1.809E-05	1.615128202	1.809E-05
0.1	1.631378786	1.631360511	1.827E-05	1.631360511	1.827E-05
0.2	1.681061667	1.681042835	1.883E-05	1.681042835	1.883E-05
0.3	1.767251541	1.767231744	1.979E-05	1.767231744	1.979E-05
0.4	1.895391737	1.895370504	2.123E-05	1.895370504	2.123E-05
0.5	2.073888896	2.073865663	2.323E-05	2.073865665	2.323E-05
0.6	2.315036695	2.315010761	2.593E-05	2.315010779	2.591E-05
0.7	2.636429497	2.636399962	2.953E-05	2.636400142	2.935E-05
0.8	3.063094068	3.063059754	3.431E-05	3.063061112	3.295E-05
0.9	3.630700259	3.630659586	4.067E-05	3.630668038	3.222E-05
1.0	4.390422827	4.390373644	4.918E-05	4.390418917	3.910E-06

Table 2. Numerical results for problem 2 at $t = 0.5$

x	$y_{Exact}(x,t)$	$\phi_6(x,t)$	AE_1	PA $[6/6]$	AE_2
0.0	1.648721270	1.648697916	2.335E-05	1.648697916	2.335E-05
0.1	1.675291194	1.675267606	2.358E-05	1.675267606	2.358E-05
0.2	1.756006862	1.755982555	2.430E-05	1.755982555	2.430E-05
0.3	1.893988415	1.893962862	2.555E-05	1.893962862	2.555E-05
0.4	2.094792334	2.094764928	2.740E-05	2.094764928	2.740E-05
0.5	2.367000016	2.366970029	2.998E-05	2.366970028	2.998E-05
0.6	2.723160693	2.723127219	3.347E-05	2.723127211	3.348E-05
0.7	3.181234472	3.181196351	3.812E-05	3.181196272	3.819E-05
0.8	3.766768365	3.766724074	4.429E-05	3.766723507	4.485E-05
0.9	4.516173712	4.516121214	5.249E-05	4.516117882	5.583E-05
1.0	5.481689070	5.481625587	6.348E-05	5.481608881	8.018E-05

Table 3. Numerical results for problem 3 at $t = 0.5$

x	$y_{Exact}(x,t)$	$\phi_3(x,t)$	AE_1	PA $[8/8]$	AE_2
0.0	0.606530659	0.606510416	2.024E-05	0.606510416	2.024E-05
0.1	0.613626394	0.613605947	2.044E-05	0.613605947	2.044E-05
0.2	0.639283645	0.639262576	2.106E-05	0.639262576	2.106E-05
0.3	0.690650250	0.690628100	2.214E-05	0.690628100	2.214E-05
0.4	0.775770322	0.775746567	2.375E-05	0.775746567	2.375E-05
0.5	0.903800783	0.903774790	2.599E-05	0.903774790	2.599E-05
0.6	1.085358235	1.085329220	2.901E-05	1.085329220	2.901E-05
0.7	1.333049833	1.333016790	3.304E-05	1.333016790	3.304E-05
0.8	1.662273798	1.662235408	3.839E-05	1.662235401	3.839E-05
0.9	2.092425114	2.092379609	4.550E-05	2.092379557	4.555E-05
1.0	2.648721270	2.648666244	5.502E-05	2.648665905	5.536E-05

Table 4. Numerical results for problem 4 at $t = 0.5$

x	$y_{Exact}(x,t)$	$\phi_5(x,t)$	AE_1	PA $[6/6]$	AE_2
0.0	1.250000000	1.250000000	0.0	1.250000000	0.0
0.1	1.251000500	1.250997162	3.337E-06	1.250997162	3.337E-06
0.2	1.258032085	1.258025566	6.519E-06	1.258025566	6.519E-06
0.3	1.277367802	1.277358691	9.111E-06	1.277358691	9.111E-06
0.4	1.316092398	1.316082178	1.022E-05	1.316082176	1.022E-05
0.5	1.383148453	1.383140361	8.092E-06	1.383140306	8.146E-06
0.6	1.491102379	1.491103020	6.417E-07	1.491102126	2.524E-07
0.7	1.659168761	1.659190908	2.214E-05	1.659180849	1.208E-05
0.8	1.918625110	1.918691962	6.685E-05	1.918604294	2.081E-05
0.9	2.323006564	2.323156969	1.504E-04	2.322516691	4.898E-04
1.0	2.968281828	2.968573358	2.915E-04	2.964422966	3.858E-03

Table 5. Numerical results for problem 5 at $t = 0.1$

x	$y_{Exact}(x,t)$	$\phi_9(x,t)$	AE_1	PA $[4/4]$	AE_2
0.0	0.000000000	-0.000005854	5.854E-06	-0.000005854	5.854E-06
0.1	-0.001999000	-0.002004901	5.900E-06	-0.002004901	5.900E-06
0.2	-0.007984042	-0.007990064	6.021E-06	-0.007990064	6.021E-06
0.3	-0.017919482	-0.017925649	6.167E-06	-0.017925649	6.167E-06
0.4	-0.031746698	-0.031752948	6.250E-06	-0.031752948	6.250E-06
0.5	-0.049385225	-0.049391374	6.149E-06	-0.049391374	6.148E-06
0.6	-0.070734287	-0.070739994	5.707E-06	-0.070739990	5.702E-06
0.7	-0.095674658	-0.095679407	4.748E-06	-0.095679387	4.728E-06
0.8	-0.124070781	-0.124073881	3.099E-06	-0.124073810	3.028E-06
0.9	-0.155773077	-0.155773712	6.353E-07	-0.155773497	4.201E-07
1.0	-0.190620359	-0.190617702	2.657E-06	-0.190617129	3.230E-06

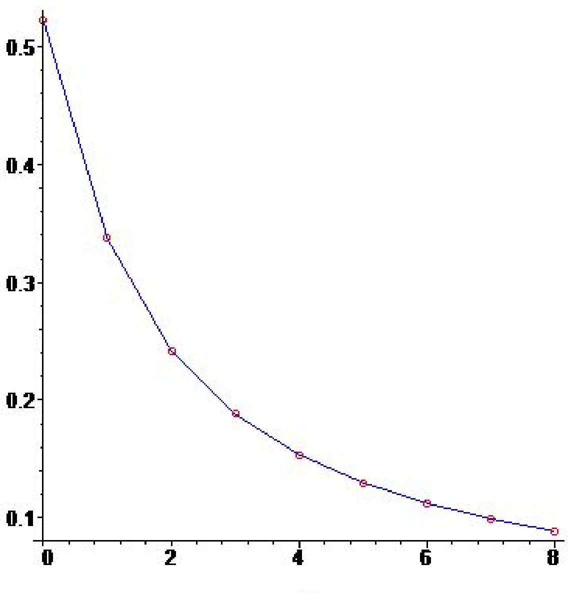

Fig. 1. Plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 8$ at $t = 0.5$

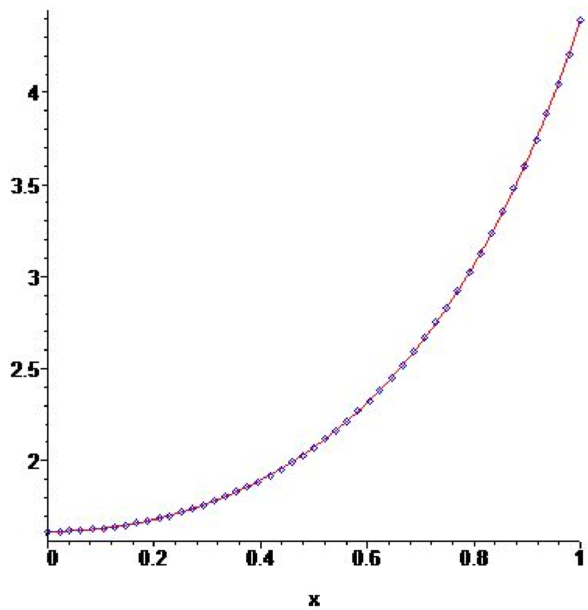

Fig. 2. Plot of $y_{Exact}(x,t)$ and $\phi_6(x,t)$ at $t = 0.5$

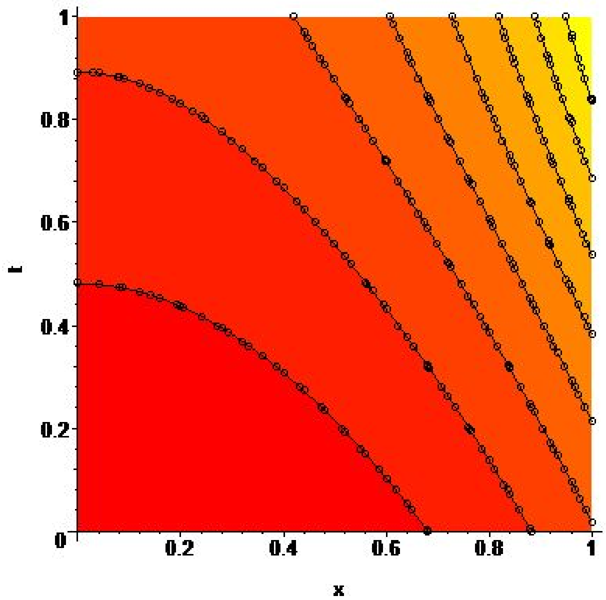

Fig. 3. The contour plot in $2D$ $(x, t) -$ plane for

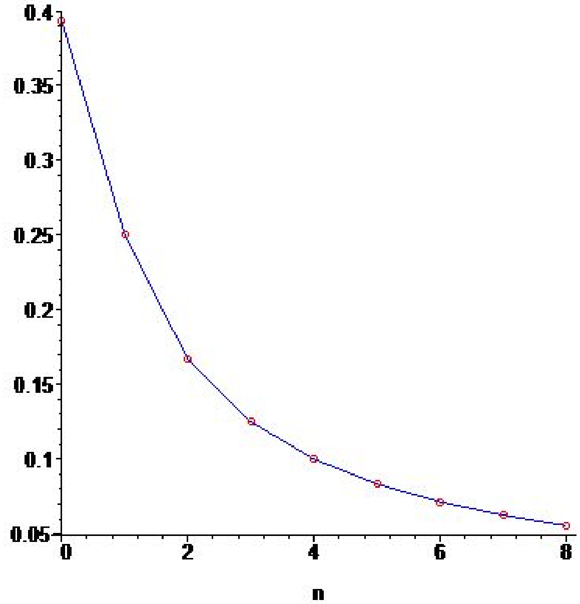

Fig. 4. Plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 8$ at $t = 0.5 y_{Exact}(x, t)$ and $\phi_6(x, t)$

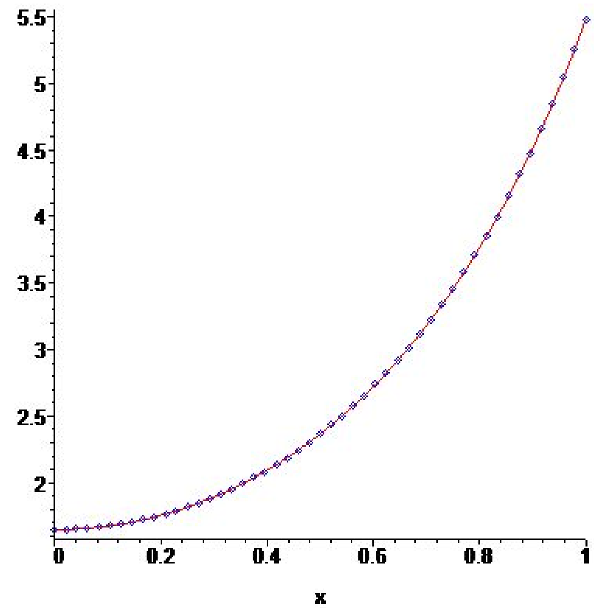

Fig. 5. Plot the $y_{Exact}(x, t)$ and $\phi_6(x, t)$ at $t = 0.5$

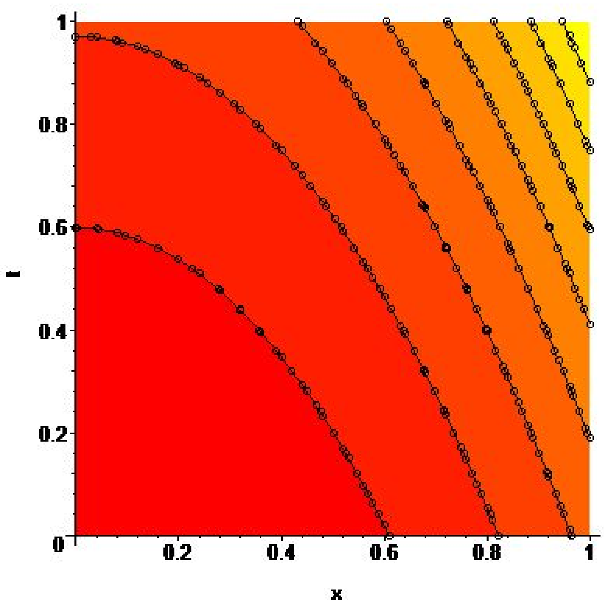

Fig. 6. The contour plot in $2D$ $(x, t) -$ plane for $y_{Exact}(x, t)$ and $\phi_6(x, t)$

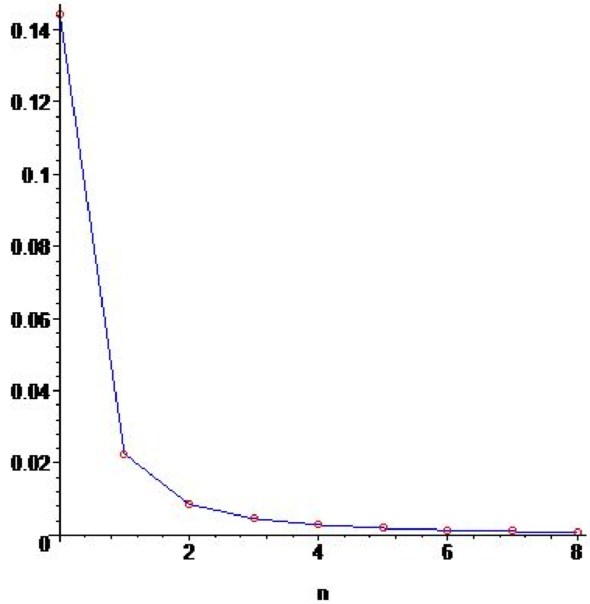

Fig. 7. Plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 8$ at $t = 0.5$

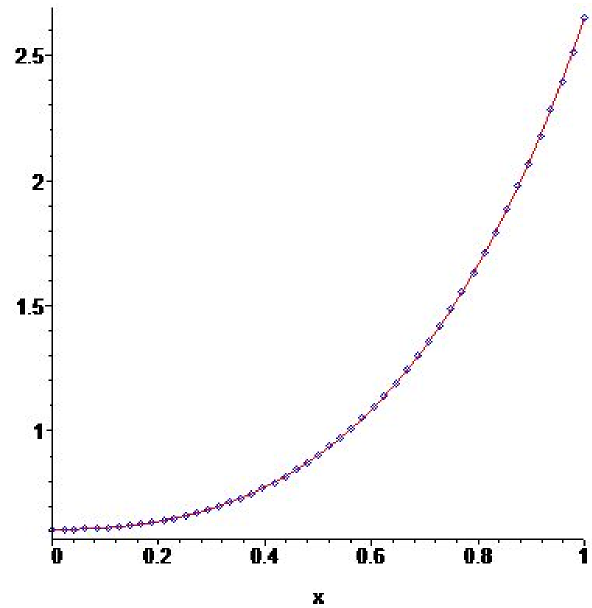

Fig. 8. Plot of $y_{Exact}(x,t)$ and $\phi_3(x,t)$ at $t = 0.5$

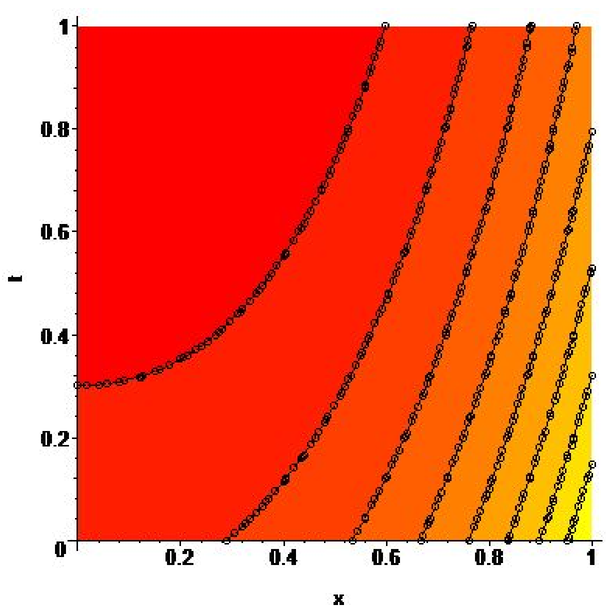

Fig. 9. The contour plot in $2D$ $(x,t)-$plane for $y_{Exact}(x,t)$ and $\phi_3(x,t)$

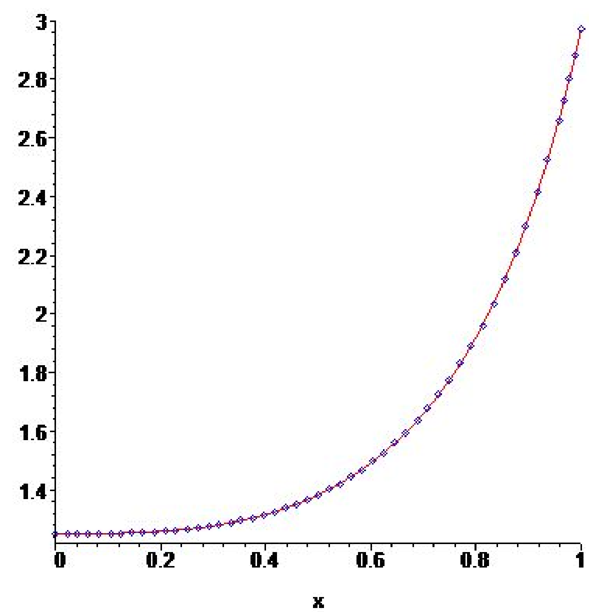

Fig. 10. Plot of $y_{Exact}(x,t)$ and $\phi_5(x,t)$ at $t = 0.5$

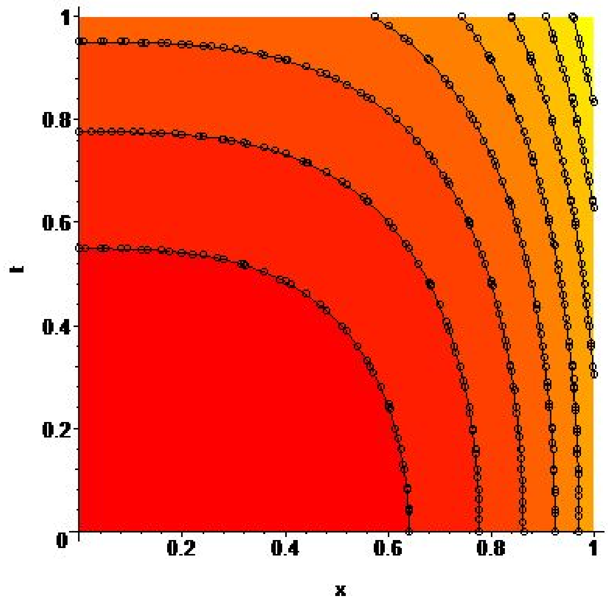

Fig. 11. The contour plot in $2D$ $(x,t)-$ plane for $y_{Exact}(x,t)$ and $\phi_8(x,t)$

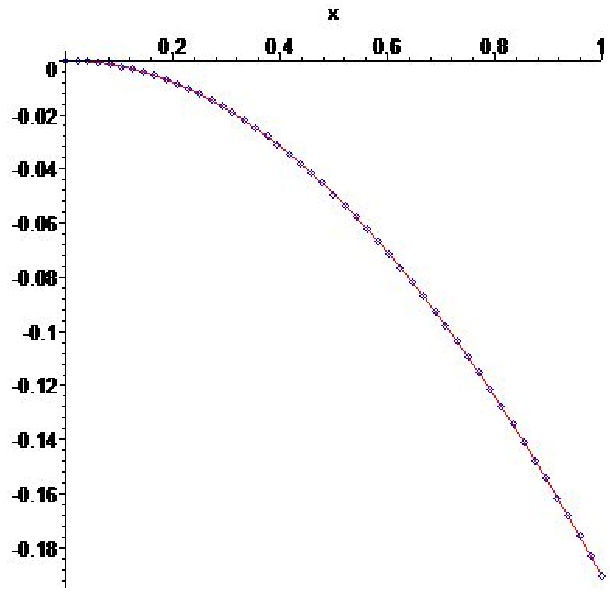

Fig. 12. Plot of $y_{Exact}(x,t)$ and $\phi_9(x,t)$ at $t=0.1$

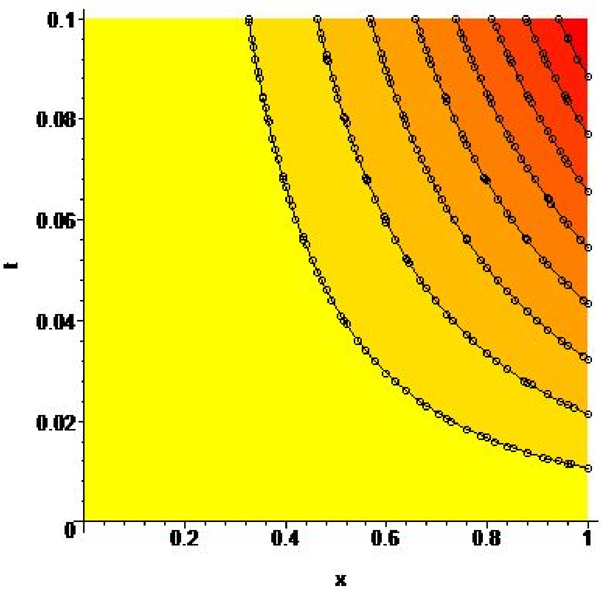

Fig. 13. The contour plot in $2D$ $(x,t)-$ plane for $y_{Exact}(x,t)$ and $\phi_9(x,t)$

IV. CONCLUSIONS

Models of singular IVPs of Emden-Fowler of partial differential equations have been successfully solved using the Laplace Transform-Adomian decomposition method. The LT-ADM has a larger range of applications because of how well it handled these models. Without the need of transformation formulae or constraining assumptions, the LT-ADM proposal has been applied directly. The LT-ADM solution procedure is compatible with approaches that provide analytical approximation in the literature. The LT-ADM method has been put to the test by using it to find approximate-exact answers for five problems. The outcomes gained in each situation show how reliable and effective this strategy is. It has been demonstrated that error monotically decreases with the integer n being increased.

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الجمع بين تحويل لابلاس وطريقة انحلال ادوميان لحل مسائل القيم الاولية الشاذة ايمدن-فويلر في المعادلات التفاضلية الجزئية

وليد الحياني   فاطمة محمد ياسين

كلية علوم الحاسوب والرياضيات/ جامعة الموصل

fatima.21csp40@student.uomosul.edu.iq
waleedalhayani@uomosul.edu.iq

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الملخص

في هذا البحث ، المعادلات التفاضلية الجزئية من نوع Emden-Fowler المعتمدة على الوقت والمعادلات من النوع الموجي ذات السلوك الشاذ عند $x = 0$ تم حلها تحليليًا باستخدام تحويل لابلاس وطريقة انحلال ادوميان. لتجنب سلوك التفرد لكلا النموذجين عند $x = 0$ ، يتم استخدام فائدة هذه التقنية العامة الفردية لتقديم إطار عمل صلب. تظهر هذه الطريقة لإنتاج حلول تقريبية دقيقة لأنواع مختلفة من المشاكل في مساحة ذات بعد واحد. تظهر النتائج المكتسبة في كل حالة موثوقية وفعالية هذا النهج.

الكلمات المفتاحية: معادلة ايمدن-فويلر، معادلة من نوع الموجه، طريقة تحويل لابلاس، طريقة أنحلال ادوميان، كثيرات حدود ادوميان