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Spectral Extension Property of Perturbed Triple Product on Semisimple Commutative Banach Algebras

Marwan A. Jardo [1]* and Amir A. Mohammed [2]

[1, 2] Department of Mathematics, College of Education for Pure Sciences, University of Mosul, Mosul, Iraq
**Corresponding author Email: [1] marwanjardo@uomosul.edu.iq*

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Correspondence:
Author: Marwan A. Jardo
Email: marwanjardo@uomosul.edu.iq

Abstract

Under a perturbed triple product defined on three semisimple commutative Banach algebras with the influence of two homomorphisms defined on two of them and that carry certain characteristics, we proved that the spectral extension property is stable.

I. INTRODUCTION

The spectrum of an element is without a doubt the most foundational concept in Banach algebra theory, and it has generated interest in harmonic analysis for spectral properties such as the spectral extension property (SEP) (for a systematic presentation of this property, see [4] and [10]). As a result, many researchers chose to investigate the SEP's stability in the product of Banach algebra (B. a.). We refer to some of them, Dabhi and Patel see [1], [4], [5], [11], and Dedanin and Kanani see [6], [7], and [8].

Now, assume that each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is B. a. Let $h \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$ (where $\mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$ is the set of all homomorphisms from $\mathfrak{B}$ into $\mathfrak{A}$) and $g \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{C})$ such that $\|h\| \leq 1$ and $\|g\| \leq 1$. Then the product space $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is a B. a. under the following product defined by Jardo and Mohammed in [9]:

$$(m_1, n_1, r_1)(m_2, n_2, r_2) = (m_1 m_2 + m_1 h(n_2) + h(n_1) m_2, n_1 n_2, r_1 r_2 + r_1 g(n_2) + g(n_1) r_2)$$

$\forall (m_1, n_1, r_1), (m_2, n_2, r_2) \in \mathfrak{A} \times \mathfrak{B} \times \mathfrak{C}$

with a norm defined as follows:

$$\|(m, n, r)\|_{\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}} = \|m\|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|r\|_{\mathfrak{C}}$$

$\forall (m, n, r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

This B. a. is called (h, g)-perturbed product of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$.

In this paper, we discuss the stability of SEP of $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

II. Spectral Extension Property (SEP) of $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$

For our goal, we need the following propositions, and the proofs of these propositions can be found in [9].

Proposition 2.1 Assume that $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are B. a.s. Let $h \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$ and $g \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{C})$ with $\|h\| \leq 1$ and $\|g\| \leq 1$. Then,

1. $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is a commutative Banach algebra (c. B. a.) if and only if $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are c. B. a.s.
2. $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is an untial Banach algebra (u. B. a.) if and only if $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are u. B. a.s.
3. $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is a semisimple Banach algebra (ss. B. a.) if and only if $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are ss. B. a.s.

Proposition 2.2 Assume that $\mathfrak{A}$ and $\mathfrak{C}$ are c. B. a.s with the Gel'fand spaces $\Delta(\mathfrak{A})$ and $\Delta(\mathfrak{C})$ respectively, and let $\mathfrak{B}$ is a Banach algebra with the Gel'fand space $\Delta(\mathfrak{B})$, let $h \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$ and $g \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{C})$ such that $\|h\| \leq 1$ and $\|g\| \leq 1$. Then the Gel'fand spaces of $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is denoted by $\Delta(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})$ and

$$\Delta(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}) = \{(\alpha, \alpha \circ h, 0) : \alpha \in \Delta(\mathfrak{A})\} \uplus \{(0, \gamma \circ g, \gamma) : \gamma \in \Delta(\mathfrak{C})\} \uplus \{(0, \beta, 0) : \beta \in \Delta(\mathfrak{B})\},$$

a disjoint union, where

$E := \{(\alpha, \alpha \circ h, 0) : \alpha \in \Delta(\mathfrak{A})\} \uplus \{(0, \gamma \circ g, \gamma) : \gamma \in \Delta(\mathfrak{C})\}$ and $F := \{(0, \beta, 0) : \beta \in \Delta(\mathfrak{B})\}$ are clopen in $\Delta(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})$.

Recall that, a c. B. a. $\mathfrak{B}$ is said to be a commutative extension of $\mathfrak{A}$ if the algebra $\mathfrak{A}$ is a (not necessarily closed) subalgebra of $\mathfrak{B}$ [8]. And a c. B. a. $\mathfrak{A}$ has SEP if $r_{\mathfrak{B}}(x) = r_{\mathfrak{A}}(x)$ for every extension $\mathfrak{B}$ of $\mathfrak{A}$ and every $x \in \mathfrak{A}$ (where $r_{\mathfrak{A}}(x)$ is the spectral radius of an element $x \in \mathfrak{A}$)[10].

Lemma 2.3[10] Let $\mathfrak{A}$ be a c. B. a. Then $\mathfrak{A}$ has a SEP if and only if every algebra norm $|\cdot|$ on $\mathfrak{A}$ satisfies $r_{\mathfrak{A}}(x) \le |x|$ for all $x \in \mathfrak{A}$.

Now, we discuss the stability of SEP of $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$. First step, assume that $|\cdot|_{\mathfrak{A}}$ is an algebra norm on $\mathfrak{A}$. Identify $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}}: \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C} \to \mathbb{R}$ by

$$|(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}} = |m + h(n)|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|r + g(n)\|_{\mathfrak{C}}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Then $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}}$ is an algebra norm on $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$. Indeed, let $(m,n,r),(\grave{m},\grave{n},\grave{r}) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$. Then

$$\begin{aligned}
&|(m,n,r)(\grave{m},\grave{n},\grave{r})|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}} \\
&= |(m\grave{m} + m h(\grave{n}) + h(n)\grave{m}, n\grave{n}, r\grave{r} + r\, g(\grave{n}) + g(n)\grave{r})|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}} \\
&= |m\grave{m} + m h(\grave{n}) + h(n)\grave{m} + h(n\grave{n})|_{\mathfrak{A}} + \|n\grave{n}\|_{\mathfrak{B}} + \|r\grave{r} + r\, g(\grave{n}) + g(n)\grave{r} + g(n\grave{n})\|_{\mathfrak{C}} \\
&\le |m + h(n)|_{\mathfrak{A}} |\grave{m} + h(\grave{n})|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}}\|\grave{n}\|_{\mathfrak{B}} + \|r + g(n)\|_{\mathfrak{C}} \|\grave{r} + g(\grave{n})\|_{\mathfrak{C}} \\
&\le (|m + h(n)|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|r + g(n)\|_{\mathfrak{C}}) (|\grave{m} + h(\grave{n})|_{\mathfrak{A}} + \|\grave{n}\|_{\mathfrak{B}} + \|\grave{r} + g(\grave{n})\|_{\mathfrak{C}}) \\
&= |(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}} |(\grave{m},\grave{n},\grave{r})|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}}.
\end{aligned}$$

Therefor $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}}$ is an algebra norm on $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Similarly, if we assume that $|\cdot|_{\mathfrak{B}}$ is an algebra norm on $\mathfrak{B}$. Identify $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{B}} : \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C} \to \mathbb{R}$ by

$$|(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{B}} = \|m + h(n)\|_{\mathfrak{A}} + |n|_{\mathfrak{B}} + \|r + g(n)\|_{\mathfrak{C}}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$

and if we assume that $|\cdot|_{\mathfrak{C}}$ is a norm on $\mathfrak{C}$.

Identify $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{C}} : \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C} \to \mathbb{R}$ by

$$|(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{C}} = \|m + h(n)\|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + |r + g(n)|_{\mathfrak{C}}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

By similar way of above method we can show that $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{B}}$ and $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{C}}$ are also algebra norms on $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Proposition 2.4 Assume that $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are c. B. a.s. Let $h \in Hom(\mathfrak{B},\mathfrak{A})$ and $g \in Hom(\mathfrak{B},\mathfrak{C})$ with $\|h\| \le 1$ and $\|g\| \le 1$. If $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP, then each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ has SEP.

Proof. Assume that $|\cdot|_{\mathfrak{A}}$ is an algebra norm on $\mathfrak{A}$. Identify $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}}: \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C} \to \mathbb{R}$ by

$$|(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}} = |m + h(n)|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|r + g(n)\|_{\mathfrak{C}}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Since $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP,

$$r_{\mathfrak{A}}(m) = r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}(m,0,0) \le |m,0,0|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{A}} = |m|_{\mathfrak{A}}\ (m \in \mathfrak{A}).$$

Also, Assume that $|\cdot|_{\mathfrak{B}}$ is an algebra norm on $\mathfrak{B}$. Identify $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathcal{B}}: \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C} \to \mathbb{R}$ by

$$|(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{B}} = \|m + h(n)\|_{\mathfrak{A}} + |n|_{\mathfrak{B}} + \|r + g(n)\|_{\mathfrak{C}}$$

$\forall (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Since $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP,

$$r_{\mathfrak{B}}(n) = r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}(-h(n), n, -g(n)) \le |(-h(n), n, -g(n))|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{B}} = |n|_{\mathfrak{B}}.$$

Finally, assume that $|\cdot|_{\mathfrak{C}}$ is an algebra norm on $\mathfrak{C}$. Identify $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{C}}: \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C} \to \mathbb{R}$ by

$$|(m,n,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{C}} = \|m + h(n)\|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + |r + g(n)|_{\mathfrak{C}}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Since $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP,

$$r_{\mathfrak{C}}(r) = r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}(0,0,r) \le |(0,0,r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C},\mathfrak{C}} = |r|_{\mathfrak{C}}.$$

Hence, each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ has SEP. ■

To prove the opposite of proposition 2.4 under the assumption of semi simplicity, we need the following lemmas.

Lemma 2.5 Assume that $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are c. B. a.s. Let $h \in Hom(\mathfrak{B},\mathfrak{A})$ and $g \in Hom(\mathfrak{B},\mathfrak{C})$ with $\|h\| \le 1$ and $\|g\| \le 1$. Then,

$$r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}(m,n,r) = \max\{r_{\mathfrak{A}}(m + h(n)), r_{\mathfrak{B}}(n), r_{\mathfrak{C}}(r + g(n))\}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

Proof. Since each of $\mathfrak{A}, \mathcal{B}$ and $\mathfrak{C}$ is c. B. a. Then by proposition 2.1(1), $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is a c. B. a., therefor

$$r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}((m,n,r)) = \sup\{|\zeta((m,n,r))| : \zeta \in \Delta(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})\}$$

$\forall\, (m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$.

If $(m,n,r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$, then

$$\begin{aligned}
&r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}((m,n,r)) \\
&= \sup\left\{\begin{matrix} |\alpha(m + h(n))|, |\beta(n)|, |\gamma(r + g(n))| : \\ \alpha \in \Delta(\mathfrak{A}), \beta \in \Delta(\mathfrak{B}), \gamma \in \Delta(\mathfrak{C}) \end{matrix}\right\} \\
&= \max\{r_{\mathfrak{A}}(m + h(n)), r_{\mathfrak{B}}(n), r_{\mathfrak{C}}(r + g(n))\}. ■
\end{aligned}$$

Lemma 2.6 Assume that each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is B. a. Let $h \in Hom(\mathfrak{B},\mathfrak{A})$ and $g \in Hom(\mathfrak{B},\mathfrak{C})$, then $\{(-h(n), n, -g(n)) : n \in \mathfrak{B}\} \cong \mathfrak{B}$ as a B. a.

Proof. Define $\psi: \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\} \to \mathfrak{B}$ by

$$\psi((-h(n), n, -g(n))) = n$$

$\forall\, (-h(n), n, -g(n)) \in \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\}$.

Then,

1. ψ is well define, Suppose that

$(-h(n_1), n_1, -g(n_1)), (-h(n_2), n_2, -g(n_2)) \in \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\}$ Such that,

$$(-h(n_1), n_1, -g(n_1)) = (-h(n_2), n_2, -g(n_2))$$

$$\Rightarrow h(n_1) = h(n_2), n_1 = n_2 \text{ and } g(n_1) = g(n_2)$$

Since $n_1 = n_2$, we have

$$\psi\left((-h(n_1), n_1, -g(n_1))\right) = \psi\left((-h(n_2), n_2, -g(n_2))\right).$$

2. ψ is one to one, Suppose that

$\left(-h(n_1), n_1, -g(n_1)\right), \left(-h(n_2), n_2, -g(n_2)\right) \in \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\}$ Such that, $\psi\left(\left(-h(n_1), n_1, -g(n_1)\right)\right) = \psi\left(\left(-h(n_2), n_2, -g(n_2)\right)\right)$ $\Rightarrow n_1 = n_2 \Rightarrow h(n_1) = h(n_2)$ and $g(n_1) = g(n_2)$ (h and g are well define)

$$\Rightarrow \left(-h(n_1), n_1, -g(n_1)\right) = \left(-h(n_2), n_2, -g(n_2)\right).$$

3. ψ is onto, Suppose that $n \in \mathfrak{B}$, then there exists $(-h(n), n, -g(n)) \in \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\}$ such that $\psi((-h(n), n, -g(n))) = n$.

4. ψ is homomorphism, Suppose that

$\left(-h(n_1), n_1, -g(n_1)\right), \left(-h(n_2), n_2, -g(n_2)\right) \in \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\}$ and $\sigma \in \mathbb{C}$ Then,

- $\psi\left(\left(-h(n_1), n_1, -g(n_1)\right) + \left(-h(n_2), n_2, -g(n_2)\right)\right)$

$$= \psi\left(\left(-h(n_1) - h(n_2), n_1 + n_2, -g(n_1) - g(n_2)\right)\right)$$
$$= \psi\left(\left(-h(n_1 + n_2), n_1 + n_2, -g(n_1 + n_2)\right)\right)$$
$$= n_1 + n_2$$
$$= \psi\left(\left(-h(n_1), n_1, -g(n_1)\right)\right) + \psi\left(\left(-h(n_2), n_2, -g(n_2)\right)\right).$$

- $\psi\left(\left(-h(n_1), n_1, -g(n_1)\right)\left(-h(n_2), n_2, -g(n_2)\right)\right)$

$$= \psi\Big(\left(-h(n_1)\right)\left(-h(n_2)\right) - h(n_1)h(n_2) - h(n_1)h(n_2), n_1 n_2, \left(-g(n_1)\right)\left(-g(n_2)\right) - g(n_1)g(n_2) - g(n_1)g(n_2)\Big)$$
$$= \psi\left(-h(n_1 n_2), n_1 n_2, -g(n_1 n_2)\right) = n_1 n_2$$
$$= \psi\left(\left(-h(n_1), n_1, -g(n_1)\right)\right)\psi\left(\left(-h(n_2), n_2, -g(n_2)\right)\right).$$

- $\psi\left(\sigma\left(-h(n_1), n_1, -g(n_1)\right)\right) =$

$$\psi\left(\left(-h(\sigma n_1), \sigma n_1, -g(\sigma n_1)\right)\right) = \sigma n_1$$
$$= \sigma\psi\left(\left(-h(n_1), n_1, -g(n_1)\right)\right)$$

It follows, ψ is a bijective algebra homomorphism.

Now, if $(-h(n), n, -g(n)) \in \{(-h(n), n, -g(n)) : n \in \mathfrak{B}\}$, then

$$\left\|\psi\left(\left(-h(n), n, -g(n)\right)\right)\right\|_{\mathfrak{B}} = \|n\|_{\mathfrak{B}}$$
$$\leq \|h(n)\|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|g(n)\|_{\mathfrak{C}}$$
$$= \|(-h(n), n, -g(n))\|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$

As a result, ψ is continuous. Hence, ψ is a Banach algebra isomorphism according to the open mapping theorem. ■

Lemma 2.7 Assume that each of $e_{\mathfrak{A}}, e_{\mathfrak{B}}$ and $e_{\mathfrak{C}}$ is the identity of c. B. a.s $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$, respectively, and let $h \in Hom(\mathfrak{B}, \mathfrak{A})$ and $g \in Hom(\mathfrak{B}, \mathfrak{C})$ with $\|h\| \leq 1$ and $\|g\| \leq 1$. If each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ has SEP, then $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP.

Proof. By Lemma 2.6 we note that $\{(-h(n), n, -g(n)) : n \in \mathfrak{B}\} \cong \mathfrak{B}$ as a B. a.

Assume that $|\cdot|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$ is a norm on $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$. Then $|m|_{\mathfrak{A}} = |(m, 0,0)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} (m \in \mathfrak{A})$ is a norm on $\mathfrak{A}$, $|n|_{\mathfrak{B}} = |(-h(n), n, -g(n))|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} (n \in \mathfrak{B})$ is a norm on $\mathfrak{B}$ and $|r|_{\mathfrak{C}} = |(0,0, r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} (r \in \mathfrak{C})$ is a norm on $\mathfrak{C}$. Since $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ have SEP, then

$$r_{\mathfrak{A}}(m) \leq |m|_{\mathfrak{A}} = |(m, 0,0)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} (m \in \mathfrak{A}),$$
$$r_{\mathfrak{B}}(n) \leq |n|_{\mathfrak{B}} = |(-h(n), n, -g(n))|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} (n \in \mathfrak{B}),$$
$$\text{and, } r_{\mathfrak{C}}(r) \leq |r|_{\mathfrak{C}} = |(0,0, r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} (r \in \mathfrak{C}).$$

Let $(m, n, r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ and let $k \in \mathbb{N}$. Then,

$$(m + h(n), 0,0)^k = (m, n, r)^k (e_{\mathfrak{A}}, 0,0),$$
$$\left(-h(n), n, -g(n)\right)^k = (m, n, r)^k (-h(e_{\mathfrak{B}}), e_{\mathfrak{B}}, -g(e_{\mathfrak{B}})),$$
$$\text{and } (0,0, r + g(n))^k = (m, n, r)^k (0,0, e_{\mathfrak{C}}).$$

Since the spectral radius is a uniform semi norm [4, Lemma 2.26], then

$$r_{\mathfrak{A}}((m + h(n))^{2^k} = r_{\mathfrak{A}}\left(\left(m + h(n)\right)^{2^k}\right)$$
$$\leq \left|(m + h(n), 0,0)^{2^k}\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$
$$= \left|(m, n, r)^{2^k}(e_{\mathfrak{A}}, 0,0)\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$
$$\leq |(m, n, r)^{2^k}|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} |(e_{\mathfrak{A}}, 0,0)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}},$$
$$r_{\mathcal{B}}(n)^{2^k} = r_{\mathcal{B}}(n^{2^k}) \leq \left|\left(-h(n), n, -g(n)\right)^{2^k}\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$
$$= \left|(m, n, r)^{2^k}((-h(e_{\mathfrak{B}}), e_{\mathfrak{B}}, -g(e_{\mathfrak{B}})\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$
$$\leq |(m, n, r)|^{2^k}{}_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} |(-h(e_{\mathfrak{B}}), e_{\mathfrak{B}}, -g(e_{\mathfrak{B}})|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}},$$
$$\text{and } r_{\mathfrak{C}}((r + g(n))^{2^k} = r_{\mathfrak{C}}\left(\left(r + g(n)\right)^{2^k}\right)$$
$$\leq \left|\left(0,0, r + g(n)\right)^{2^k}\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$
$$= \left|(m, n, r)^{2^k}(0,0, e_{\mathfrak{C}})\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$$
$$\leq \left|(m, n, r)^{2^k}\right|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} |(0,0, e_{\mathfrak{C}})|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}},$$

Since $k \in \mathbb{N}$ is arbitrary, hence, $r_{\mathfrak{A}}\left(m + h(n)\right) \leq |(m, n, r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$, $r_{\mathfrak{B}}(n) \leq |(m, n, r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$ and $r_{\mathfrak{C}}\left(r + g(n)\right) \leq |(m, n, r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$.

Now, by lemma 2.5, if $(m, n, r) \in \mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$, then,

$$r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}(m, n, r)$$
$$= \max\{r_{\mathfrak{A}}\left(m + h(n)\right), r_{\mathfrak{B}}(n), r_{\mathfrak{C}}\left(r + g(n)\right)\}$$

Hence, $r_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}(m, n, r) \leq |(m, n, r)|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}$ and therefore $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP. ■

Lemma 2.8 [4] Let I be a closed ideal of c. B. a. $\mathfrak{A}$. If $\mathfrak{A}$ has SEP and $mI = \{0\}$ $(m \in \mathfrak{A})$ lead to $m = 0$, then I has SEP.

Remember that the unitization of a Banach algebra $\mathfrak{A}$ is $\mathfrak{A}_e = \mathfrak{A} \times \mathbb{C}$. Where $\mathfrak{A}_e$ is a Banach algebra with the product

$$(m + \lambda 1_{\mathfrak{A}})(n + \mu 1_{\mathfrak{A}})$$
$$= mn + \mu m + \lambda n + \lambda\mu 1_{\mathfrak{A}} \ (m + \lambda 1_{\mathfrak{A}}, n + \mu 1_{\mathfrak{A}} \in \mathfrak{A}_e)$$

and the norm

$$\|m + \lambda 1_{\mathfrak{A}}\|_1 = \|m\|_{\mathfrak{A}} + |\lambda| \ (m + \lambda 1_{\mathfrak{A}} \in \mathfrak{A}_e).$$

We see that $\mathfrak{A}$ is a closed ideal of $\mathfrak{A}_e$ and $\mathfrak{A}_e$ is a commutative if and only if $\mathfrak{A}$ is a commutative. For a systematic presentation of this topic, see [3].

Lemma 2.9 Assume that each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is c. B. a. Let $\mathrm{h} \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$ and $\mathrm{g} \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{C})$ with $\|\mathrm{h}\| \leq 1$ and $\|\mathrm{g}\| \leq 1$. Then,

1. If $\mathfrak{A}$ and $\mathfrak{C}$ are non-unitals and $\mathfrak{B}$ with identity $e_{\mathfrak{B}}$, then $\mathfrak{A}_e \times_h \mathfrak{B} \times_g \mathfrak{C}_e \cong (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$ as a B. a.
2. If $e_{\mathfrak{A}}$ and $e_{\mathfrak{C}}$ are identities of $\mathfrak{A}$ and $\mathfrak{C}$, respectively, and $\mathfrak{B}$ be non-unital, then $\mathfrak{A} \times_{h_e} \mathfrak{B}_e \times_{g_e} \mathfrak{C} \cong (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$ as a B. a.
3. If $e_{\mathfrak{A}}$ and $e_{\mathfrak{B}}$ are identities of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $\mathfrak{C}$ be non-unital, then $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}_e \cong (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$ as a B. a.
4. If $e_{\mathfrak{B}}$ and $e_{\mathfrak{C}}$ are identities of $\mathfrak{B}$ and $\mathfrak{C}$ respectively, and $\mathfrak{A}$ be non-unital, then $\mathfrak{A}_e \times_h \mathfrak{B} \times_g \mathfrak{C} \cong (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$ as a B. a.
5. If $\mathfrak{A}$ and $\mathfrak{B}$ are non-unitals and $\mathfrak{C}$ with identity $e_{\mathfrak{C}}$, then $\mathfrak{A}_e \times_{h^e} \mathfrak{B}_e \times_{g_e} \mathfrak{C} \cong (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$ as a B. a.
6. If $\mathfrak{B}$ and $\mathfrak{C}$ are non-unitals and $\mathfrak{A}$ with identity $e_{\mathfrak{A}}$, then $\mathfrak{A} \times_{h_e} \mathfrak{B}_e \times_{g^e} \mathfrak{C}_e \cong (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$ as a B. a.

Where,

$h_e: \mathfrak{B}_e \to \mathfrak{A}$ defined by $h_e(n + \lambda 1_{\mathfrak{B}}) = h(n) + \lambda e_{\mathfrak{A}}\ (\forall n + \lambda 1_{\mathfrak{B}} \in \mathfrak{B}_e)$, $g_e: \mathfrak{B}_e \to \mathfrak{C}$ defined by $g_e(n + \lambda 1_{\mathfrak{B}}) = g(n) + \lambda e_{\mathfrak{C}}\ (\forall n + \lambda 1_{\mathfrak{B}} \in \mathfrak{B}_e)$, $h^e: \mathfrak{B}_e \to \mathfrak{A}_e$ defined by $h^e(n + \lambda 1_{\mathfrak{B}}) = h(n) + \lambda 1_{\mathfrak{A}}\ (\forall n + \lambda 1_{\mathfrak{B}} \in \mathfrak{B}_e)$, and $g^e: \mathfrak{B}_e \to \mathfrak{C}_e$ defined by $g_e(n + \lambda 1_{\mathfrak{B}}) = g(n) + \lambda 1_{\mathfrak{C}}\ (\forall n + \lambda 1_{\mathfrak{B}} \in \mathfrak{B}_e)$,

Proof.

1. Define $\Psi : (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e \to \mathfrak{A}_e \times_h \mathfrak{B} \times_g \mathfrak{C}_e$ by

$$\Psi((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}) = (m,n,r) + \lambda(1_{\mathfrak{A}} - h(e_{\mathfrak{B}}), e_{\mathfrak{B}}, 1_{\mathfrak{C}} - g(e_{\mathfrak{B}})).$$
$$\forall\, (m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} \in \left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$$

Then Ψ is a bijective algebra homomorphism.

If $(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} \in (\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C})_e$, then

$$\left\|\Psi\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right)\right\|_{\mathfrak{A}_e\times_h\mathfrak{B}\times_g\mathfrak{C}_e}$$
$$= \left\|\left(m - \lambda h(e_{\mathfrak{B}})\right) + \lambda 1_{\mathfrak{A}}, n + \lambda e_{\mathfrak{B}}, (r - \lambda g(e_{\mathfrak{B}})) + \lambda 1_{\mathfrak{C}})\right\|_{\mathfrak{A}_e\times_h\mathfrak{B}\times_g\mathfrak{C}_e}$$
$$\leq \|m\|_{\mathfrak{A}} + |\lambda|\|h(e_{\mathfrak{B}})\|_{\mathfrak{A}} + |\lambda| + \|n\|_{\mathfrak{B}} + |\lambda|\|e_{\mathfrak{B}}\|_{\mathfrak{B}} + \|r\|_{\mathfrak{C}} + |\lambda|\|g(e_{\mathfrak{B}})\|_{\mathfrak{C}} + |\lambda|$$
$$\leq \|m\|_{\mathfrak{A}} + |\lambda|\|h\|\|e_{\mathfrak{B}}\|_{\mathfrak{B}} + |\lambda| + \|n\|_{\mathfrak{B}} + |\lambda|\|e_{\mathfrak{B}}\|_{\mathfrak{B}} + \|r\|_{\mathfrak{C}} + |\lambda|\|g\|\|e_{\mathfrak{B}}\|_{\mathfrak{B}} + |\lambda|$$
$$\leq 5\|e_{\mathfrak{B}}\|_{\mathfrak{B}}(\|m\|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|r\|_{\mathfrak{C}} + |\lambda|)$$
$$= 5\|e_{\mathfrak{B}}\|_{\mathfrak{B}}\left(\|(m,n,r)\|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} + |\lambda|\right)$$
$$= 5\|e_{\mathfrak{B}}\|_{\mathfrak{B}}\left\|(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right\|_1.$$

As a result, Ψ is continuous. Hence, Ψ is a Banach algebra isomorphism according to the open mapping theorem.

2. Define $\Psi : \left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e \to \mathfrak{A} \times_{h_e} \mathfrak{B}_e \times_{g_e} \mathfrak{C}$ by

$$\Psi\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right) = (m, n + \lambda 1_{\mathfrak{B}}, r)$$

$\forall\, (m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} \in \left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$.

Then Ψ is a bijective algebra homomorphism.

if $(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} \in \left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$. Then,

$$\left\|\Psi\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right)\right\|_{\mathfrak{A}\times_{h_e}\mathfrak{B}_e\times_{g_e}\mathfrak{C}}$$
$$= \|(m, n + \lambda 1_{\mathfrak{B}}, r)\|_{\mathfrak{A}\times_{h_e}\mathfrak{B}_e\times_{g_e}\mathfrak{C}}$$
$$= \|m\|_{\mathfrak{A}} + \|n\|_{\mathfrak{B}} + \|r\|_{\mathfrak{C}} + |\lambda|$$
$$= \|(m,n,r)\|_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} + |\lambda|$$
$$= \left\|(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right\|_1.$$

As a result, Ψ is continuous. Hence, Ψ is a Banach algebra isomorphism according to the open mapping theorem.

the proof of other cases by similar way. ∎

Theorem 2.10 Assume that each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is ss. c. B. a. Let $h \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$ and $g \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{C})$ with $\|h\| \leq 1$ and $\|g\| \leq 1$. If each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ has SEP, then $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP.

Proof.

Case 1. If each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is untial, then by lemma 2.7 $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP.

Case 2. If each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is non-unital. Since each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ has SEP and each of them is ss. c. B. a. by a assumption, then by corollary 3.2 [2], each of $\mathfrak{A}_e, \mathfrak{B}_e$ and $\mathfrak{C}_e$ has SEP.

Now, define $h_e : \mathfrak{B}_e \to \mathfrak{A}_e$ as

$$h_e(n + \lambda 1_{\mathfrak{B}}) = h(n) + \lambda 1_{\mathfrak{A}}\ (n + \lambda 1_{\mathfrak{B}} \in \mathfrak{B}_e)$$

and $g_e : \mathfrak{B}_e \to \mathfrak{C}_e$ as

$$g_e(n + \lambda 1_{\mathfrak{B}}) = g(n) + \lambda 1_{\mathfrak{C}}\ (n + \lambda 1_{\mathfrak{B}} \in \mathfrak{B}_e).$$

Then h_e and g_e are algebra homomorphisms with $\|h_e\| \leq 1$ and $\|g_e\| \leq 1$, as stated in theorem 2.30 [4]. Therefore $\mathfrak{A}_e \times_{h_e} \mathfrak{B}_e \times_{g_e} \mathfrak{C}_e$ is a Banach algebra and by lemma 2.7 $\mathfrak{A}_e \times_{h_e} \mathfrak{B}_e \times_{g_e} \mathfrak{C}_e$ has SEP.

Now, $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is a closed ideal of $\mathfrak{A}_e \times_{h_e} \mathfrak{B}_e \times_{g_e} \mathfrak{C}_e$.

Let $(m + \lambda 1_A, n + \mu 1_{\mathfrak{B}}, r + \eta 1_{\mathfrak{C}}) \in \mathfrak{A}_e \times_{h_e} \mathfrak{B}_e \times_{g_e} \mathfrak{C}_e$ such that

$$(m + \lambda 1_{\mathfrak{A}}, n + \mu 1_{\mathfrak{B}}, r + \eta 1_{\mathfrak{C}})\left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right) = \{(0,0,0)\}.$$

Since each of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ is non-unital and each of them is ss., we have

$$(m + \lambda 1_{\mathfrak{A}}, n + \mu 1_{\mathfrak{B}}, r + \eta 1_{\mathfrak{C}}) = (0,0,0).$$

Hence, by Lemma 2.8, $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ has SEP.

Case 3. if $\mathfrak{A}$ and $\mathfrak{C}$ are non-unitals and $\mathfrak{B}$ is unital, then $\mathfrak{A}_e \times_h \mathfrak{B} \times_g \mathfrak{C}_e$ is unital. By Lemma 2.7, it has SEP. From lemma 2.9 (1) $\mathfrak{A}_e \times_h \mathfrak{B} \times_g \mathfrak{C}_e \cong \left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$, $\left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$ has SEP.

Since $\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}$ is a closed ideal of $\left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$.

Let $(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g C} \in \left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right)_e$ such that

$$\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right)\left(\mathfrak{A} \times_h \mathfrak{B} \times_g \mathfrak{C}\right) = (0,0,0).$$

Then,

$$\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right)(\grave{m}, \grave{n}, \grave{r})$$
$$= (m\grave{m} + m h(\grave{n}) + h(n)\grave{m} + \lambda\grave{m}, n\grave{n} + \lambda\grave{n}, r\grave{r} + r\, g(\grave{n}) + g(n)\grave{r} + \lambda\grave{r}) = (0,0,0)$$

$\forall(\grave{m},\grave{n},\grave{r}) \in \mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$.

Suppose that $\lambda \neq 0$ and taking $\grave{n} = 0$, we get $-\frac{1}{\lambda}(m + h(n))\grave{m} = \grave{m}$ for all $\grave{m} \in \mathfrak{A}$ and $-\frac{1}{\lambda}(r + g(n))\grave{r} = \grave{r}$ for all $\grave{r} \in \mathfrak{C}$. These are not possible as $\mathfrak{A}$ and $\mathfrak{C}$ are non-unitals.

Thus, $(m\grave{m} + mh(\grave{n}) + h(n)\grave{m}, n\grave{n}, r\grave{r} + r\,g(\grave{n}) + g(n)\grave{r}) = (0,0,0)$ for all $(\grave{m},\grave{n},\grave{r}) \in \mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$. Since $\mathfrak{B}$ is unital and $n\grave{n} = 0$ for all $\grave{n} \in \mathfrak{B}$, we get $n = 0$. By taking $\grave{n} = 0$, we get $m\grave{m} = 0$ for all $\grave{m} \in \mathfrak{A}$ and $r\grave{r} = 0$ for all $\grave{r} \in \mathfrak{C}$. In particular, $m^2 = 0$ and $r^2 = 0$. This gives $r_{\mathfrak{A}}(m)^2 = r_{\mathfrak{A}}(m^2) = 0$ and $r_{\mathfrak{C}}(r)^2 = r_{\mathfrak{C}}(r^2) = 0$. Since each of $\mathfrak{A}$ and $\mathfrak{C}$ is semisimple, $m = 0$ and $r = 0$. Thus $(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} = (0,0,0)$. Hence, by lemma 2.8 $\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$ has SEP.

Case 4. If $\mathfrak{A}$ and $\mathfrak{C}$ are unitals and $\mathfrak{B}$ is non-unital, then $\mathfrak{A}\times_{h_e} \mathfrak{B}_e\times_{g_e} \mathfrak{C}$ is untial. By Lemma 2.7, it has SEP. From lemma 2.9 (2) $\mathfrak{A}\times_{h_e} \mathfrak{B}_e\times_{g_e} \mathfrak{C} \cong (\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C})_e$, $(\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C})_e$ has SEP.

Since $\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$ is a closed ideal of $(\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C})_e$.

Let $(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} \in (\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C})_e$ such that

$$\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right)(\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}) = (0,0,0).$$

Then,

$$\left((m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}}\right)(\grave{m},\grave{n},\grave{r})$$
$$= (m\grave{m} + mh(\grave{n}) + h(n)\grave{m} + \lambda\grave{m}, n\grave{n} + \lambda\grave{n}, r\grave{r} + r\,g(\grave{n}) + g(n)\grave{r} + \lambda\grave{r}) = (0,0,0)$$

$\forall(\grave{m},\grave{n},\grave{r}) \in \mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$.

Suppose that $\lambda \neq 0$. we get $-\frac{1}{\lambda}n\grave{n} = \grave{n}$ for all $\grave{n} \in \mathfrak{B}$. This is not possible as $\mathfrak{B}$ is non-unital. Thus, $(m\grave{m} + mh(\grave{n}) + h(n)\grave{m}, n\grave{n}, r\grave{r} + r\,g(\grave{n}) + g(n)\grave{r}) = (0,0,0)$ for all $(\grave{m},\grave{n},\grave{r}) \in \mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$. Thus, $n\grave{n} = 0$ for all $\grave{n} \in \mathfrak{B}$. In particular, $n^2 = 0$. This gives $r_{\mathfrak{B}}(n)^2 = r_{\mathfrak{B}}(n^2) = 0$. Since $\mathfrak{B}$ is semisimple, $n = 0$. Thus $(m\grave{m} + mh(\grave{n}), 0, r\grave{r} + r\,g(\grave{n})) = (0,0,0)$ for all $(\grave{m},\grave{n},\grave{r}) \in \mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$. Thus $m(\grave{m} + h(\grave{n})) = 0$ and $r(\grave{r} + g(\grave{n})) = 0$, in particular taking $\grave{n} = 0$, then $m\grave{m} = 0$ for all $\grave{m} \in \mathfrak{A}$ and $r\grave{r} = 0$ for all $\grave{r} \in \mathfrak{C}$, and Since $\mathfrak{A}$ and $\mathfrak{C}$ are unital, we get $m = 0$ and $r = 0$. Thus $(m,n,r) + \lambda 1_{\mathfrak{A}\times_h\mathfrak{B}\times_g\mathfrak{C}} = (0,0,0)$. Hence, by Lemma 2.8 $\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$ has SEP.

The other cases can be proved by using the similar arguments as above, it follows $\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$ SEP.■

Conclusion

Assume that $\mathfrak{A},\mathfrak{B}$ and $\mathfrak{C}$ are c. B. a.s Let $h \in Hom(\mathfrak{B},\mathfrak{A})$ and $g \in Hom(\mathfrak{B},\mathfrak{C})$ with $\|h\| \leq 1$ and $\|g\| \leq 1$. Then,

1. If $\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$ has SEP, then each of $\mathfrak{A},\mathfrak{B}$ and $\mathfrak{C}$ has SEP.
2. If each of $\mathfrak{A},\mathfrak{B}$ and $\mathfrak{C}$ has SEP and each of them is ss., then $\mathfrak{A}\times_h \mathfrak{B}\times_g \mathfrak{C}$ has SEP. In another word, the SEP is stable with respect to the (h,g)-perturbed product defined on three semisimple commutative Banach.

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خاصية الامتداد الطيفي للضرب الثلاثي المشوش على جبور باناخ الابدالية شبه البسيطة

مروان عزيز جردو و عامر عبد الاله محمد

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة الموصل، الموصل، العراق

amirabdulillah@uomosul.edu.iq marwanjardo@uomosul.edu.iq

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الملخص

تحت عملية ضرب ثلاثي مشوش معرفة على ثلاثة جبور باناخ ابدالية شبه بسيطة مع تأثير تشاكلان معرفان على اثنين منهم و يحملان صفات معينة ، برهنا ان خاصية الامتداد الطيفي مستقرة.

الكلمات المفتاحية: جبر باناخ، الضرب الثلاثي المشوش، خاصية الامتداد الطيفي.