Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM), Vol. 16*, No. 2, 2022* (61-70)

Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM)

www.csmj.mosuljournals.com

The Homotopy Perturbation Method to Solve Initial Value Problems of First Order with Discontinuities

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Article information

Article history:
Received : 26/6/2022
Accepted :17/8/2022
Available online :

Abstract

In this work, the homotopy perturbation method (HPM) is used to solve initial value problems of first order with various types of discontinuities. The numerical results obtained (are compared) using the traditional HPM, and the integral equation of the *nth* equation with the solution numerical obtained using Simpson and Trapezoidal Rules to demonstrate that the solution results are extremely accurate when compared to the exact solution. The maximum absolute error, $\|\cdot\|_2$, maximum relative error, maximum residual error, and expected convergence order are also provided.

Keywords:
Homotopy perturbation method; He's polynomials*;* Initial value problems (IVPs); Unit step function; Unit impulse function; Simpson Rule; Trapezoidal Rule

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1. INTRODUCTION

Ji-Huan He (1998, 1999) published the so-called HPM [1-3] six years after Shijun Liao (1992) suggested the earliest homotopy analysis method (HAM) in his Ph.D. thesis. The HPM, such as the earliest HAM, is constructed Based on a homotopy equation

$$(1-p)L[\phi(x;p)-u_0(x)]+pN[\phi(x;p)]=0,\quad x\in\Omega,\quad p\in[0,1] \qquad (1)$$

where $u_0(x)$ is an initial guess and L is an auxiliary linear operator. The concept of homotopy (Hilton, 1953) in topology (Sen, 1983) theoretically gives us a lot of leeway in selecting the auxiliary linear operator L and the initial guess $u_0(x)$. The zeroth-order deformation equation is the same of Eq. (1).

The HPM is a semi-analytical method for solving both linear and non-linear differential and integral problems. A system of linear and non-linear differential equations may also be solved using this approach. Artificial parameters [4] were used to build this approach [5-8]. Almost every classic perturbation approach is predicated on the assumption of a small number of parameters. However, the vast majority of non-linear problems contain no tiny parameters at all, therefore determining small parameters appears to be a unique skill needing unique methodologies. Tiny changes in small factors might have a big impact on the outcomes. Unsuitable tiny parameter selection, on the other hand, has negative consequences, which can be severe.

Consider the general IVPs of first order [9,10]

$$u'+k^2u-g(u)=\mu f(x,u),\quad u(0)=\alpha,\quad 0\le x\le l \qquad (2)$$

where g is a linear / non-linear function of u and f is a function with some discontinuity, whereas k, μ analpha arere real constants.

Based on the currently available literature, Al-Hayani and Casasús [9,11,12] solved the IVP Eq. (2) with discontinuities by the Adomian decomposition method (ADM). Al-Hayani and Rasha Fahad [10,13] have been utilized the homotopy analysis method (HAM) for the Eq. (2). Ji-Huan He [14] utilized the HPM to solve non-linear oscillators.

The major purpose of this study is to test the HPM for solving first-order IVPs with a derivative discontinuous, unit step function and unit impulse function to achieve approximate-exact solutions in a variety of circumstances. Section 2 gives the fundamental idea of HPM, section 3 shows HPM Applied to an IVPs for linear and non-linear cases, and the conclusions in section 4.

2. The Fundamental Idea of HPM

Now, we demonstrate the fundamental concept of the HPM [1-3,15-18]. For this, we'll use the non-linear differential equation below:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{3}$$

subject to the boundary conditions

$$B\left(u, \frac{\partial u}{\partial r}\right) = 0, \quad r \in \Gamma \tag{4}$$

where A denotes a generic differential operator, B denotes a boundary operator, Γ denotes the domain Ω boundary, and $f(r)$ denotes a known analytic function. The linear L and non-linear N components of the operator A can be separated. As a result, Eq. (3) may be expressed as follows:

$$L(u) + N(u) - f(r) = 0, \tag{5}$$

In Eq. (5), a fake parameter p can be inserted as follows:

$$L(u) + p\big(N(u) - f(r)\big) = 0, \tag{6}$$

where $p \in [0,1]$ is a parameter for embedding (also named as an artificial parameter).

We create a homotopy by employing homotopy approaches [1-3,15-18].

$v(r,p): \Omega \times [0,1] \to R$ to Eq. (5) which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{7}$$

and

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{8}$$

Here, u_0 is an initial approximation of Eq. (8) it meets the requirements.

By substituting $p = 0$ and $p = 1$ in Eq. (8), The following equations may be obtained, respectively

$$H(v,0) = L(v) - L(u_0) = 0,$$

and

$$H(v,1) = A(v) - f(r) = 0.$$

when the value of p changes from 0 to 1, $v(r,p)$ changes from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are homotopic to each other. Because $p \in [0,1]$ is a tiny parameter, we consider Eq. (7) solution as a power series in p, as shown below.

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{9}$$

The approximation solution of Eq. (3) can then be acquired as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{10}$$

In [1] has given the convergence of the series solution (10).

3. HPM Applied to an IVPs

Applying the standard HPM as in [1-3], Eq. (2) can be written as

$$u' - v_0' + p[v_0' + k^2 u - g(u) - \mu f(x,u)] = 0, \tag{11}$$

We can utilize the embedding parameter p as a tiny parameter and suppose that the solution of Eq. (2) can be represented as a power series in p, according to the HPM.

$$u(x) = \sum_{n=0}^{\infty} p^n u_n(x), \tag{12}$$

and the non-linear term can be decomposed as

$$g(u) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{13}$$

for some He's polynomials $H_n(u)$ [18] that are given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2,\dots \tag{14}$$

Substituting (12) and (13) into Eq. (11) we get

$$\sum_{n=0}^{\infty} p^n u_n' - v_0' + p\left[v_0' + k^2 \sum_{n=0}^{\infty} p^n u_n - \sum_{n=0}^{\infty} p^n H_n(u) - \mu f(x,u)\right] = 0, \tag{15}$$

and equating the terms in the same power of p, we have a system IVPs of first order

$$\begin{aligned}
p^0:&\quad u_0' - v_0' = 0, \quad u_0(0) = \alpha\\
p^1:&\quad u_1' + v_0' + k^2 u_0 - H_0(u) - \mu f(x,u) = 0, \quad u_1(0) = 0\\
p^2:&\quad u_2' - k^2 u_1 - H_1(u) = 0, \quad u_2(0) = 0\\
p^3:&\quad u_3' - k^2 u_2 - H_2(u) = 0, \quad u_3(0) = 0\\
p^4:&\quad u_4' - k^2 u_3 - H_3(u) = 0, \quad u_4(0) = 0\\
&\vdots\\
p^n:&\quad u_n' - k^2 u_{n-1} - H_{n-1}(u) = 0, \quad u_n(0) = 0, \quad n = 1,2,\dots
\end{aligned} \tag{16}$$

Solving the system of Eqs. (16), we obtain the iterations $u_0, u_1, u_2, \dots, u_n$. Thus, the approximate solution in a series form is given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x).$$

3.1. Linear Case: Let $g(u) = 0$ and $\alpha = 1$.

Case 3.1.1. If we consider $k = 2$, $\mu = 10$ and $f(x,u)$ is a continuous function, but non-differentiable, for instance

$$f(x,u) = \begin{cases} -x + \frac{1}{2}, & x < \frac{1}{2} \\ x - \frac{1}{2}, & x \geq \frac{1}{2} \end{cases}$$

From the system (16) the initial iterations are then determined in the recursive manner described below:

$$u_0 = 1,$$

$$u_1(x) = \begin{cases} -5x^2 + x, & x < \frac{1}{2} \\ 5x^2 - 9x + \frac{5}{2}, & x \geq \frac{1}{2} \end{cases}$$

$$u_2(x) = \begin{cases} \frac{20}{3}x^3 - 2x^2, & x < \frac{1}{2} \\ -\frac{20}{3}x^3 + 18x^2 - 10x + \frac{5}{3} & x \geq \frac{1}{2} \end{cases}$$

$$u_3(x) = \begin{cases} -\frac{20}{3}x^4 + \frac{8}{3}x^3, & x < \frac{1}{2} \\ \frac{20}{3}x^4 - 24x^3 + 20x^2 - \frac{20}{3}x + \frac{5}{6}, & x \geq \frac{1}{2} \end{cases}$$

$$u_4(x) = \begin{cases} \frac{16}{3}x^5 - \frac{8}{3}x^4, & x < \frac{1}{2} \\ -\frac{16}{3}x^5 + 24x^4 - \frac{80}{3}x^3 + \frac{40}{3}x^2 - \frac{10}{3}x + \frac{1}{3}, & x \geq \frac{1}{2} \end{cases}$$

⋮

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x) = \sum_{n=0}^{14} u_n(x) = \begin{cases} h_1(x), & x < \frac{1}{2} \\ h_2(x), & x \geq \frac{1}{2} \end{cases}$$

where

$$h_1(x) = \frac{65536}{127702575}x^{15} - \frac{16384}{6081075}x^{14} + \frac{8192}{868725}x^{13} - \frac{2048}{66825}x^{12} + \frac{2048}{22275}x^{11} - \frac{512}{2025}x^{10} + \frac{256}{405}x^9 - \frac{64}{45}x^8 + \frac{128}{45}x^7 - \frac{224}{45}x^6 + \frac{112}{15}x^5 + \frac{28}{3}x^4 - 7x^2 + x + 1,$$

and

$$h_2(x) = -\frac{65536}{127702575}x^{15} + \frac{376832}{42567525}x^{14} - \frac{352256}{6081075}x^{13} + \frac{346112}{1403325}x^{12} - \frac{2048}{2475}x^{11} + \frac{100684}{42525}x^{10} - \frac{153344}{25515}x^9 + \frac{89792}{6615}x^8 - \frac{539264}{19845}x^7 + \frac{242720}{5103}x^6 - \frac{1011376}{14175}x^5 + \frac{1191988}{13365}x^4 - \frac{5006356}{56133}x^3 + \frac{27117767}{405405}x^2 - \frac{263452799}{8513505}x + \frac{1656639563}{255405150}.$$

The closed form of this series is as follows: $n \to \infty$

$$u_{Exact}(x) = \begin{cases} -\frac{5}{2}x + \frac{15}{8} - \frac{7}{8}e^{-4x}, & x < \frac{1}{2} \\ \frac{5}{2}x - \frac{15}{8} - \frac{7}{8}e^{-4x} + \frac{5}{4}e^{-4x+2}, & x \geq \frac{1}{2} \end{cases} \quad (17)$$

is the exact solution of the case **3.1.1.**

In Table 1 we compare numerical results produced using the HPM ($\boldsymbol{n = 15}$), the integral equation of the *nth*-Eq. for the system (16) (**IEI**), the numerical solution of the *nth*-Eq. for the system (16) using the Simpson rule (**SIMPR**) and trapezoidal rule (**TRAPR**) with the exact solution (17). We used twenty points in the Simpson and trapezoidal rules. **Table 2** shows the maximum absolute error **(MAE)**, $\|\cdot\|_2 = \sqrt{\int_0^1 [u_{Exact}(x) - u_n(x)]^2 dx}$, the maximum relative error **(MRE)** and the maximum residual error **(MRR)** obtained by the **HPM** with the exact solution (17) on [0,1]. **Table 3** shows the estimated orders of convergence (EOC) for various values of the constant k.

Fig. 1 gives the exact solution $u_{Exact}(x)$ with our approximation HPM ($\boldsymbol{n = 14}$) on $0 \leq x \leq 1$. The application of the HPM for $k \geq 3$, necessitates order $n \geq 20$ approximants if we want to get over the (at $x = \frac{1}{2}$) discontinuity.

Case 3.1.2. Taking $k = 1$, $\mu = 10$, and

$$f(x,u) = U(x-1) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

is the unit step function at $x = 1$. Following that, from the system (16) the first iterations are calculated recursively as manner below:

$u_0 = 1,$

$$u_1(x) = \begin{cases} -x, & x < 1 \\ 9x - 10 & x \geq 1 \end{cases}$$

$$u_2(x) = \begin{cases} \frac{x^2}{2}, & x < 1 \\ -\frac{9}{2}x^2 + 10x - 5, & x \geq 1 \end{cases}$$

$$u_3(x) = \begin{cases} -\frac{x^3}{6}, & x < 1 \\ -\frac{3}{2}x^3 - 5x^2 + 5x - \frac{5}{3}, & x \geq 1 \end{cases}$$

$$u_4(x) = \begin{cases} \frac{x^4}{24}, & x < 1 \\ -\frac{3}{8}x^4 + \frac{5}{3}x^3 - \frac{5}{2}x^2 + \frac{5}{3}x - \frac{5}{12}, & x \geq 1 \end{cases}$$

⋮

and etc, obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x) = \sum_{n=0}^{8} u_n(x) = \begin{cases} h_1(x), & x < 1 \\ h_2(x), & x \geq 1 \end{cases}$$

where

$$h_1(x) = -\frac{x^9}{362880}$$

and

$$h_2(x) = \frac{1}{40320}x^9 - \frac{1}{4032}x^8 + \frac{1}{1008}x^7 - \frac{1}{432}x^6 + \frac{1}{288}x^5 - \frac{1}{288}x^4 + \frac{1}{432}x^3 - \frac{1}{1008}x^2 + \frac{1}{4032}x - \frac{1}{36288}$$

The closed form of this series is as follows: $n \to \infty$

$$u_{Exact}(x) = \begin{cases} e^{-x}, & x < 1 \\ 10 - 10e^{1-x} + e^{-x}, & x \geq 1 \end{cases} \quad (18)$$

is the exact solution of the case 3.1.2.

Table 4 compares numerical results obtained by the HPM ($\boldsymbol{n = 9}$), the integral equation of the *nth*-Eq. for the system (16) (**IEI**), the numerical solution of the *nth*-Eq. for the system (16) using the Simpson rule (**SIMPR**) and trapezoidal rule (**TRAPR**) with the exact solution (18). We used twenty points in the Simpson and trapezoidal rules. **Table 5** shows the maximum absolute error **MAE,** $\|\cdot\|_2$, the **MRE** and the **MRR** obtained by the **HPM** with the exact solution (18) on the interval [0,2].

The EOC are 1.10504 at $x = 0.9$ and 1.12704 at $x = 1.1$.

In Fig. 2 we represent the exact solution $u_{Exact}(x)$ with our approximation HPM ($\boldsymbol{n = 8}$) on $0 \leq t \leq 2$.

The HPM is applicable until the value $k \simeq 2$ in this case.

Case 3.1.3. Let $k = 1$, $\mu = 1$ and $f(x,u) = \delta(x-1)$ is the unit impulse function at $x = 1$. From the system (16) the first iterations are calculated recursively as manner below:

$u_0(x) = 1,$
$u_1(x), = U(x-1) - x$
$u_2(x) = (1-x)U(x-1) + \frac{1}{2}x^2,$
$u_3(x) = \left(\frac{1}{2} - x + \frac{1}{2}x^2\right)U(x-1) - \frac{1}{6}x^3,$
$u_4(x) = \left(\frac{1}{6} - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)U(x-1) + \frac{1}{24}x^4,$
⋮
and etc, obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x) = \sum_{n=0}^{8} u_n(x),$$
$$= \left(\frac{685}{252} - \frac{1957}{720}x + \frac{163}{120}x^2 - \frac{65}{144}x^3 + \frac{1}{9}x^4 - \frac{1}{48}x^5 + \frac{1}{360}x^6 - \frac{1}{5040}x^7\right)U(x-1) + 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8.$$

The closed form of this series is as follows: $n \to \infty$

$$u_{Exact}(x) = U(x-1)e^{-x+1} + e^{-x}, \quad (19)$$

is the exact solution of the case 3.1.3.

In **Table 6** we show the **MAE,** $\|\cdot\|_2$, the **MRE** and the **MRR** obtained by the **HPM** with the exact solution (19) on $[0,2]$. The EOC are 1.10504 at $x = 0.9$ and 1.13506 at $x = 1.1$.

Fig. 3 gives the exact solution $u_{Exact}(x)$ with our approximation HPM ($\boldsymbol{n = 8}$) on $0 \le t \le 2$.

The HPM is applicable in this case when $k \lesssim 2.2$ for all values of μ.

Case 3.1.4. Now we take $k = 1,\ \mu = 1$ and

$$f(x,u) = \delta\left(x - \frac{1}{2}\right) + \delta(x-1) + \delta\left(x - \frac{3}{2}\right),$$

is the unit impulse function at $x = \frac{1}{2}, 1, \frac{3}{2}$. From the system (16) the first iterations are calculated recursively as manner below:

$u_0(x) = 1,$
$u_1(x) = U\left(x - \frac{1}{2}\right) + U(x-1) + U\left(x - \frac{3}{2}\right) - x,$
$u_2(x) = \left(\frac{1}{2} - x\right)U\left(x - \frac{1}{2}\right) + (1-x)U(x-1) + \left(\frac{3}{2} - x\right)U\left(x - \frac{3}{2}\right) + \frac{1}{2}x^2,$
$u_3(x) = \left(\frac{1}{8} - \frac{1}{2}x + \frac{1}{2}x^2\right)U\left(x - \frac{1}{2}\right) + \left(\frac{1}{2} - x + \frac{1}{2}x^2\right)U(x-1) + \left(\frac{9}{8} - \frac{3}{2}x + \frac{1}{2}x^2\right)U\left(x - \frac{3}{2}\right) - \frac{1}{6}x^3$
$u_4(x) = \left(\frac{1}{48} - \frac{1}{8}x + \frac{1}{4}x^2 - \frac{1}{6}x^3\right)U\left(x - \frac{1}{2}\right) + \left(\frac{1}{6} - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right)U(x-1) + \left(\frac{9}{16} - \frac{9}{8}x + \frac{3}{4}x^2 - \frac{1}{6}x^3\right)U\left(x - \frac{3}{2}\right) + \frac{1}{24}x^4$
⋮
and etc, obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x) = \sum_{n=0}^{14} u_n(x)$$
$$= \left(\frac{28034721508153}{17003918131200} - \frac{3234775558633}{1961990553600}x + \frac{134782314943}{163499212800}x^2 - \frac{2042156287}{7431782400}x^3 + \frac{306323443}{4459069440}x^4 - \frac{17017969}{1238630400}x^5 + \frac{354541}{154828800}x^6 - \frac{75973}{232243200}x^7 + \frac{6331}{154828800}x^8 - \frac{211}{46448640}x^9 + \frac{79}{174182400}x^{10} - \frac{13}{319334400}x^{11} + \frac{1}{319334400}x^{12} - \frac{1}{6227020800}x^{13}\right)U\left(x - \frac{1}{2}\right)$$
$$+ \left(\frac{8463398743}{3113510400} - \frac{260412269}{95800320}x + \frac{13563139}{9979200}x^2 - \frac{9864101}{21772800}x^3 + \frac{98641}{870912}x^4 - \frac{109601}{4838400}x^5 + \frac{137}{36288}x^6 - \frac{1957}{3628800}x^7 + \frac{163}{2419200}x^8 - \frac{13}{1741824}x^9 + \frac{1}{1360800}x^{10} - \frac{1}{15966720}x^{11} + \frac{1}{239500800}x^{12} - \frac{1}{6227020800}x^{13}\right)U(x-1)$$
$$+ \left(\frac{134402599609}{29989273600} - \frac{36185315027}{8074035200}x + \frac{904632821}{403701760}x^2 - \frac{205598269}{275251200}x^3 + \frac{10279877}{55050240}x^4 - \frac{5139817}{137625600}x^5 + \frac{321193}{51609600}x^6 - \frac{131}{147456}x^7 + \frac{5711}{51609600}x^8 - \frac{563}{46448640}x^9 + \frac{67}{58060800}x^{10} - \frac{29}{319334400}x^{11} + \frac{1}{191600640}x^{12} - \frac{1}{6227020800}x^{13}\right)U\left(x - \frac{3}{2}\right)$$
$$+ 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \frac{1}{40320}x^8 - \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} - \frac{1}{39916800}x^{11} + \frac{1}{479001600}x^{12} - \frac{1}{6227020800}x^{13} + \frac{1}{87178291200}x^{14}.$$

The closed form of this series is as follows: $n \to \infty$

$$u_{Exact}(x) = U\left(x - \frac{1}{2}\right)e^{\frac{1}{2}-x} + U(x-1)e^{1-x}$$

$$+U\left(x-\frac{3}{2}\right)e^{\frac{3}{2}-x}+e^{-x}, \quad (20)$$

is the exact solution of the case 3.1.4.

The EOC for both sides of the discontinuity are given in **Table 7**.

Figs. 4 and 5 show the exact solution $u_{Exact}(x)$ as well as our HPM ($\boldsymbol{n=15}$) approximation. The approximation HPM ($\boldsymbol{n=15}$) is only valid until the second discontinuity, as can be seen in **Fig. 5**.

3.2. Non-Linear Case: Let $g(u)=u^2$ and $\alpha=1$.

The non-linear term u^2 is calculated using He's polynomials [18] as follows:

$$u^2=\sum_{i=0}^{n}u_iu_{n-i}, \quad n\geq i, \quad n=0,1,2,\dots.$$

Case 3.2.1. If we take $k=1$, μ = 1 and

$$f(x,u)=U(x-1)=\begin{cases}0, & x<1\\ 1, & x\geq 1\end{cases}$$

the unit step function at $x=1$. From the system (16) the first iterations are calculated recursively as manner below:

$$u_0(x)=1,$$

$$u_1(x)=\begin{cases}0, & x<1\\ x-1, & x\geq 1\end{cases}$$

$$u_2(x)=\begin{cases}0, & x<1\\ \frac{1}{2}x^2-x+\frac{1}{2}, & x\geq 1\end{cases}$$

$$u_3(x)=\begin{cases}0, & x<1\\ \frac{1}{2}x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{1}{2}, & x\geq 1\end{cases}$$

$$u_4(x)=\begin{cases}0, & x<1\\ \frac{3}{8}x^4-\frac{3}{2}x^3+\frac{9}{2}x^2-\frac{3}{2}x+\frac{3}{8}, & x\geq 1\end{cases}$$

$$\vdots$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x)=\sum_{n=0}^{15}u_n(x)=\begin{cases}0, & x<1\\ h(x), & x\geq 1\end{cases}$$

where

$$h(x)=\frac{85863963}{1793792000}x^{15}-\frac{118413753}{179379200}x^{14}+\frac{219650349}{51251200}x^{13}-\frac{170883}{9856}x^{12}+\frac{38491597}{788480}x^{11}-\frac{90715067}{896000}x^{10}+\frac{57227327}{358400}x^{9}-\frac{122242643}{627200}x^{8}+\frac{93116567}{501760}x^{7}-\frac{4938851}{35840}x^{6}+\frac{141888429}{1792000}x^{5}-\frac{17049507}{492800}x^{4}+\frac{44616683}{3942400}x^{3}-\frac{13881059}{5125120}x^{2}+\frac{16964357}{14350336}x+\frac{76066623}{448448000}.$$

Fig. 6 depicts the numeric solution $u_N(x)$ with a very tiny error as well as our HPM approximation ($\boldsymbol{n=16}$) for $0\leq x\leq 2$.

The HPM is applicable in this case for all values of μ when $k\lesssim 1.5$.

Case 3.2.2. Taking $k=1$, μ = 1 and $f(x,u)=\delta\left(x-\frac{1}{2}\right)$, the unit impulse function *at* $x=\frac{1}{2}$. From the system (16) the first iterations are calculated recursively as manner below:

$$u_0(x)=1,$$

$$u_1(x)=U\left(x-\frac{1}{2}\right),$$

$$u_2(x)=\left(x-\frac{1}{2}\right)U\left(x-\frac{1}{2}\right),$$

$$u_3(x)=\left(\frac{1}{2}x^2+\frac{1}{2}x-\frac{3}{8}\right)U\left(x-\frac{1}{2}\right),$$

$$u_4(x)=\left(\frac{1}{6}x^3+\frac{5}{4}x^2-\frac{11}{8}x-\frac{17}{48}\right)U\left(x-\frac{1}{2}\right),$$

$$\vdots$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x)=\sum_{n=0}^{15}u_n(x)=1+\left(\frac{1}{87178291200}x^{14}+\frac{5461}{4151347200}x^{13}+\frac{4186001}{3832012800}x^{12}+\frac{34381987}{383201280}x^{11}+\frac{127942159}{92897280}x^{10}+\frac{3940482079}{1393459200}x^{9}-\frac{15845512463}{1857945600}x^{8}+\frac{5304427333}{1083801600}x^{7}+\frac{9658527373}{1486356480}x^{6}-\frac{9134472937}{891813888}x^{5}+\frac{222986673947}{29727129600}x^{4}-\frac{1054623527057}{490497638400}x^{3}+\frac{4539815287201}{3923981107200}x^{2}+\frac{1897379562929}{3400783626240}x+\frac{125252924948413}{285665824604160}\right)U\left(x-\frac{1}{2}\right)$$

In Fig. 7 we show our approximations by HPM ($\boldsymbol{n=16}$) and HPM ($\boldsymbol{n=15}$) on $0\leq x\leq 1$.

The HPM is applicable in this case when $k\lesssim 2.3$ for all values of μ.

Case 3.2.3. Lastly, we take $k=1$, μ = 1 and

$$f(x,u)=\delta\left(x-\frac{1}{4}\right)+\delta\left(x-\frac{1}{2}\right),$$

is the unit impulse function at $x=\frac{1}{4},\frac{1}{2}$. From the system (16) the first iterations are calculated recursively as manner below:

$$u_0(x)=1,$$

$$u_1(x)=U\left(x-\frac{1}{4}\right)+U\left(x-\frac{1}{2}\right),$$

$$u_2(x)=\left(x-\frac{1}{4}\right)U\left(x-\frac{1}{4}\right)+\left(x-\frac{1}{2}\right)U\left(x-\frac{1}{2}\right),$$

$$u_3(x)=\left(\frac{1}{2}x^2+\frac{3}{4}x-\frac{7}{32}\right)U\left(x-\frac{1}{4}\right)+\left(\frac{1}{2}x^2+\frac{5}{2}x-\frac{11}{8}\right)U\left(x-\frac{1}{2}\right),$$

$$u_4(x) = \left(\frac{1}{6}x^3 + \frac{11}{8}x^2 - \frac{23}{32}x + \frac{35}{384}\right)U\left(x-\frac{1}{4}\right) + \left(\frac{1}{6}x^3 + \frac{17}{4}x^2 - \frac{31}{8}x + \frac{41}{48}\right)U\left(x-\frac{1}{2}\right),$$

$$\vdots$$

and etc., obtaining the rest of the iterations in this manner. As a result, the series form of the approximate answer is

$$u(x) = \sum_{n=0}^{12} u_n(x)$$
$$= 1 + \left(\frac{1}{39916800}x^{11} + \frac{13}{46080}x^{10} + \frac{7473}{143360}x^9 + \frac{18495359}{15482880}x^8 + \frac{7264829}{1474560}x^7 + \frac{36195413}{29491200}x^6 - \frac{5120087}{14155776}x^5 + \frac{1667427029}{660602880}x^4 + \frac{688364861}{754974720}x^3 + \frac{236849951231}{190253629440}x^2 + \frac{48840731441}{46976204800}x + \frac{339044310157}{531502202880}\right)U\left(x-\frac{1}{4}\right)$$
$$+ \left(\frac{1}{39916800}x^{11} + \frac{877}{1036800}x^{10} + \frac{16619}{46080}x^9 + \frac{8285741}{483840}x^8 + \frac{256410067}{1935360}x^7 - \frac{362974489}{11059200}x^6 - \frac{5396729393}{22118400}x^5 + \frac{8437009321}{27525120}x^4 - \frac{5697570557}{35389440}x^3 + \frac{31199858569}{679477248}x^2 - \frac{1432995034841}{237817036800}x + \frac{280129663691}{523197480960}\right)U\left(x-\frac{1}{2}\right).$$

Fig. 8 represents our approximations by HPM ($n = 13$) and HPM ($n = 12$) for $0 \le x \le 0.6$.

Table 1. Numerically results of the case 3.1.1

x	$u_{Exact}(x)$	HPM	IEI	SIMPR	TRAPR
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.038469959	1.038469959	1.038469959	1.038469959	1.038469959
0.2	0.981837156	0.981837156	0.981837156	0.981837156	0.981837156
0.3	0.861455064	0.861455064	0.861455064	0.861455064	0.861455064
0.4	0.698340546	0.698340548	0.698340548	0.698340548	0.698340548
0.5	0.506581627	0.506581654	0.506581655	0.506581655	0.506581655
0.6	0.383521848	0.383522158	0.383522173	0.383522173	0.383522173
0.7	0.383452400	0.383454722	0.383454859	0.383454859	0.383454864
0.8	0.465825836	0.465838661	0.465839564	0.465839564	0.465839598
0.9	0.603462390	0.603517986	0.603522609	0.603522609	0.603522773
1.0	0.778142920	0.778339367	0.778358613	0.778358613	0778359252

Table 2. MAE, $\|\cdot\|_2$, MRE and MRR of the case 3.1.1

n	MAE	$\|\cdot\|_2$	MRE	MRR
8	4.0047E-02	6.8841E-03	5.1465E-02	7.8095E-01
9	4.1173E-03	1.7059E-03	7.5656E-03	1.7354E-01
10	3.8490E-03	1.4102E-03	5.4421E-03	2.8138E-2
11	2.9400E-03	7.8419E-04	3.7783E-03	2.6525E-02
12	1.4487E-03	3.3660E-04	1.8618E-03	1.7555E-02
13	5.7357E-04	1.2251E-04	7.3710E-04	8.0893E-03
14	1.9644E-04	3.9440E-05	2.5245E-04	3.0800E-03
15	6.0179E-05	1.14910E-4	7.7337E-05	1.0265E-03

Table 3. EOC of the case 3.1.1

k	$x = 0.4$	$x = 0.6$
1	1.04447	1.05406
2	1.07048	1.05309
3	1.14818	1.22488

Table 4. Numerically results of the case 3.1.2

x	$u_{Exact}(x)$	HPM	IEI	SIMPR	TRAPR
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.2	0.818730753	0.818730753	0.818730753	0.818730753	0.818730753
0.4	0.670320046	0.670320046	0.670320046	0.670320046	0.670320046
0.6	0.548811636	0.548811634	0.548811634	0.548811634	0.548811634
0.8	0.449328964	0.449328936	0.449328934	0.449328934	0.449328934
1.0	0.367879441	0.367879188	0.367879163	0.367879163	0.367879163
1.2	2.113886681	2.113885143	2.113884957	2.113884957	2.113884953
1.4	3.543396503	3.543389440	3.543388426	3.543388426	3.543388402
1.6	4.713780157	4.713753757	4.713749351	4.713749351	4.713749250
1.8	5.672009247	5.671925111	5.671909032	5.671909032	5.671908667
2.0	6.456540871	6.456305114	6.456254058	6.456254058	6.456252908

Table 5. MAE, $\|\cdot\|_2$, MRE and MRR of the case 3.1.2

n	MAE	$\|\cdot\|_2$	MRE	MRR
4	1.7151E-01	1.2827E-01	9.3906E-02	3.5622E-01
5	1.2679E-01	6.6716E-02	1.9637E-02	2.7074E-01
6	5.6540E-02	8.7571E-02	8.7571E-03	1.8333E-01
7	1.8459E-02	7.0568E-03	2.8589E-03	7.4999E-02
8	4.9535E-03	1.7437E-03	7.6721E-04	2.3412E-02
9	1.1476E-03	3.7838E-04	1.7774E-04	6.1011E-03

Table 6. MAE, $\|\cdot\|_2$, MRE and MRR of the case 3.1.3

n	MAE	$\|\cdot\|_2$	MRE	MRR
4	3.3654E-01	1.7947E-01	6.6879E-01	8.3333E-01
5	1.6345E-01	7.5229E-02	3.2481E-01	5.0000E-01
6	6.1548E-02	2.5328E-02	1.2230E-01	2.2499E-01
7	7.1587E-03	7.1587E-03	3.7772E-02	8.0555E-02
8	5.0004E-03	1.7508E-03	9.9369E-03	2.4007E-02
9	1.1503E-03	3.7860E-04	2.2860E-03	6.1507E-03

Table 7. EOC of the case 3.1.4

$x = 0.4$	$x = 0.6$	$x = 0.9$	$x = 1.1$	$x = 1.4$	$x = 1.6$
1.04182	1.05990	1.08885	1.07732	1.07485	1.03233

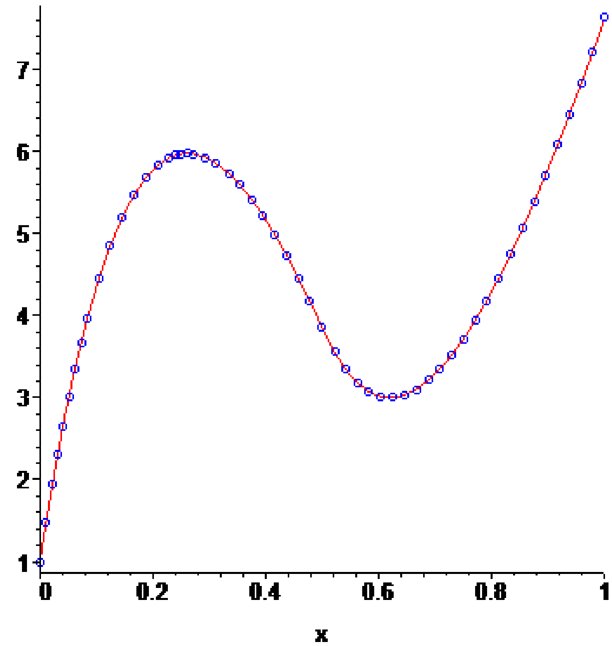

Fig. 1. Continuous line: $u_{Exact}(x)$, o: HPM, $k = 2$, $\mu = 100$

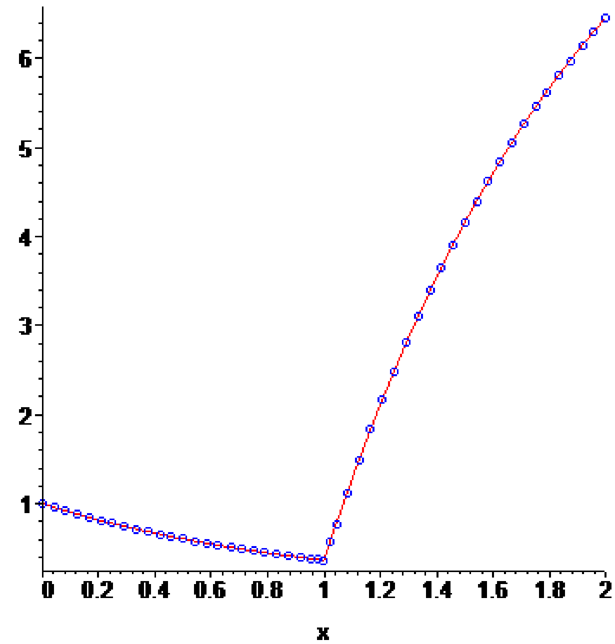

Fig. 2. Continuous line: $u_{Exact}(x)$, o: HPM, $k = 1$, $\mu = 10$

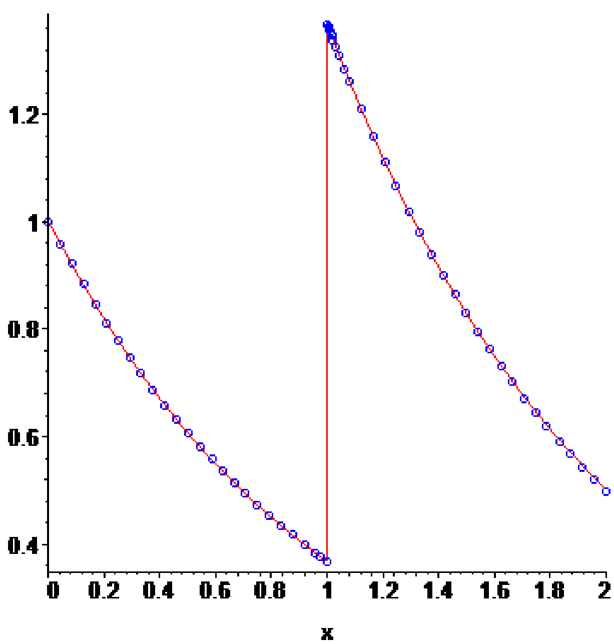

Fig. 3. Continuous line: $u_{Exact}(x)$, o: HPM, $k = 1$, $\mu = 1$

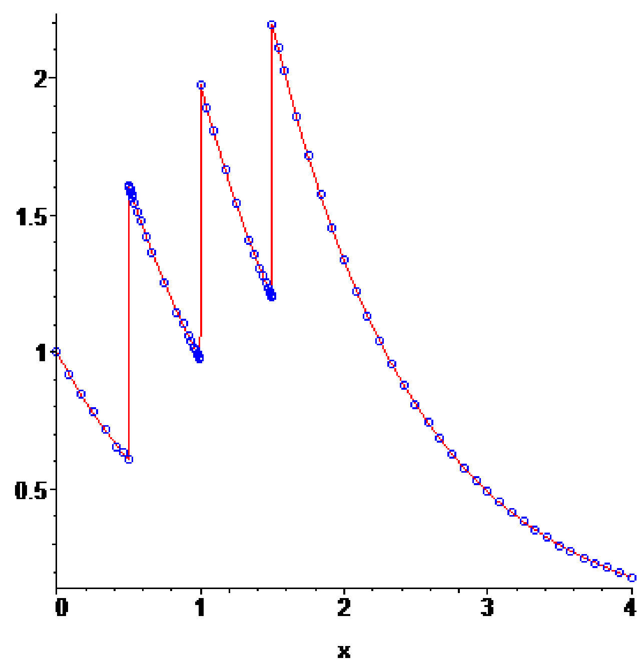

Fig. 4. Continuous line: $u_{Exact}(x)$, o: HPM, $k = 1$, $\mu = 1$

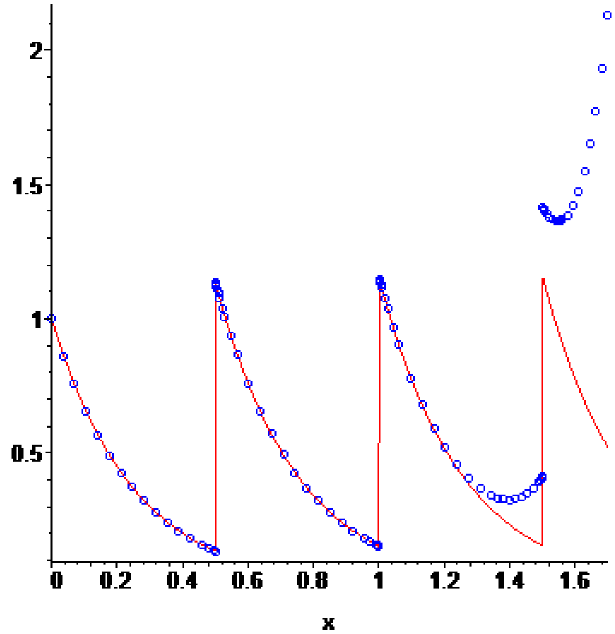

Fig. 5. Continuous line: $u_{Exact}(x)$, o: HPM, $k = 2$, $\mu = 1$

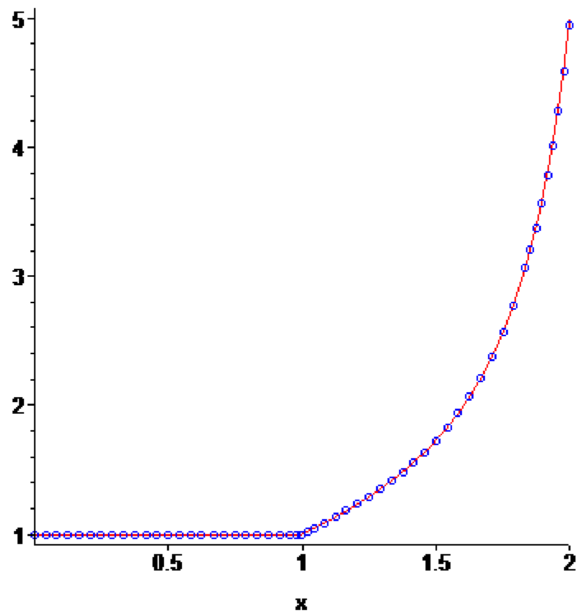

Fig. 6. Continuous line: $u_N(x)$, o: HPM, $k = 1$, $\mu = 1$

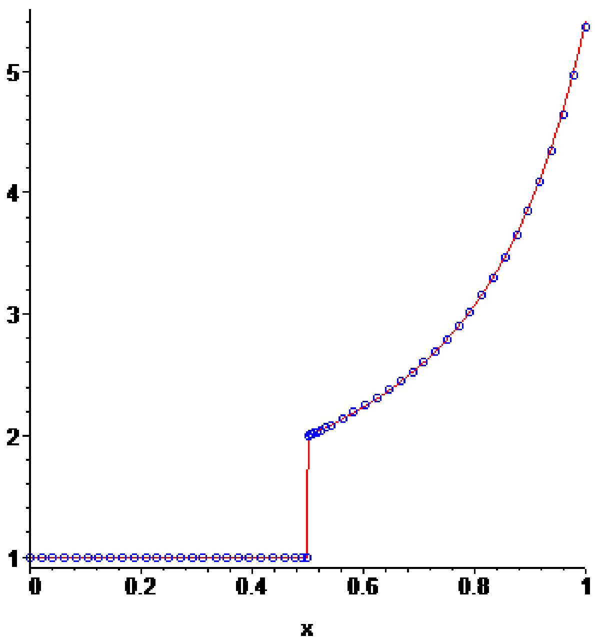

Fig. 7. Continuous line: HPM ($n = 16$), o: HPM ($n = 15$)

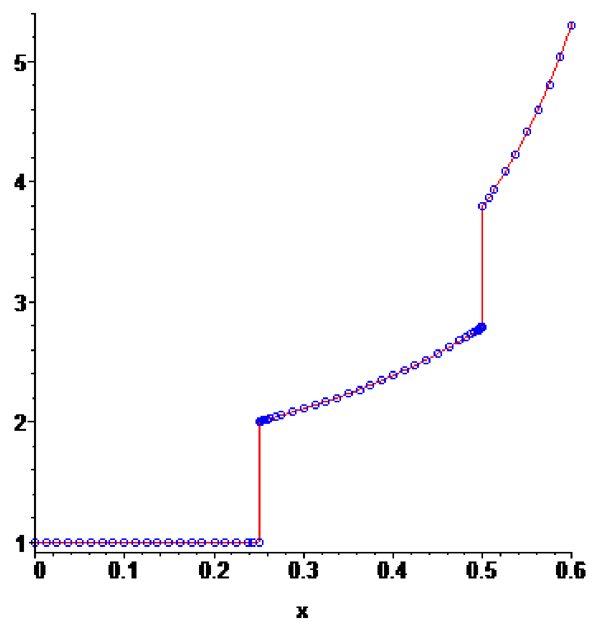

Fig. 8. Continuous line: HPM ($n = 13$), o: HPM ($n = 12$)

4. Conclusions

In this paper, the HPM was effectively used to solve first-order initial value problems with discontinuities. The size of the jump (given by μ), which performs equally well on both sides of the discontinuity, has no effect on the method's convergence. The HPM for $k = 3$ does not converge in these initial value problems even for small values of the parameter, such as $\mu = 10^{-3}$. In the non-linear cases with large values of μ, sometimes a computation with more digits is required in order to avoid unstable oscillations. The approximate solution obtained by HPM is compatible with analytical approximation approaches in the literature, such as Adomian decomposition method. The HPM has been confirmed by applying it to a linear situation to yield approximation exact results. The method's dependability is demonstrated by the outcomes obtained in all scenarios.

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طريقة هوموتوبي المضطربة لحل مسائل القيم الأولية من الرتبة الاولى مع عدم الاستمرارية

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تاريخ الاستلام:26/6/2022 تاريخ القبول:17/8/2022

الملخص

في هذا العمل تم استخدام طريقة هوموتوبي المضطربة (HPM) لحل مسائل القيم الأولية من الرتبة الاولى مع أنواع مختلفة من عدم الاستمرارية. تمت مقارنة النتائج العددية التي تم الحصول عليها باستخدام طريقة هوموتوبي المضطربة التقليدية ، والمعادلة التكاملية للمعادلة n مع الحل العددي الذي تم الحصول عليه باستخدام قاعدتي سمبسون وشبه المنحرف لإثبات أن نتائج الحل دقيقة للغاية عند مقارنتها مع الحل الدقيق. القيمة القصوى للخطأ المطلق ، المعيار من الرتبة 2 ، القيمة القصوى للخطأ النسبي ، القيمة القصوى للخطأ المتبقي ويتم أيضًا توفير ترتيب التقارب المتوقع.

الكلمات المفتاحية: طريقة هوموتوبي المضطربة، متعددات حدود هي، مسائل القيم الأولية، دالة الخطوة الأحادية (دالة هيفيسايد الدرجية)، دالة النبضة الأحادية (دالة ديراك)، قاعدة سمبسون، قاعدة شبه المنحرف.