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Strongly Nil* Clean Ideals

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Abstract

An element a is known a strongly nil* clean element if $a = \mathrm{e}_1 - \mathrm{e}_1\mathrm{e}_2 + n$, where e_1, e_2 are idempotents and n is nilpotent, that commute with one another. An ideal I of a ring R is called a strongly nil* clean ideal if each element of I is strongly nil* clean element. We investigate some of its fundamental features, as well as its relationship to the nil clean ideal.

Keywords:
Nilpotent element, Jacobson radical, Strongly nil* clean ideal, Idempotent element.

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1. INTRODUCTION

In this paper, a ring R is associative with unity $1 \neq 0$ unless otherwise expressed. $J(R)$, $U(R), Id(R)$ *and* $N(R)$ are respectively Jacobson radical, the set of unit, idempotent and nilpotent elements of respectively. "An ideal I of a unital ring R is clean in case every element in I is a sum of an idempotent and a unit of R"[4]. In [1] Sharma and Basnet defined the concept nil clean ideal (henceforth: NCI) as for each $a \in I$, there is a nilpotent element n in R and an idempotent element e *in* R then $a = e + n$. We call I is strongly nil clean ideal (henceforth: SNCI) if each a in I are written as $a = e + n$ where $e \in Id(R), n \in N(R)$ and $en = ne$ [1]. An element t in a ring R is called tripotent if $t^3 = t$ [6], "An ideal I is called strongly tri nil clean ideal (henceforth: STNCI) if for each element $a \in I$ can be expressed as $a = t + n$ where t is tripotent and n is nilpotent elements with $tn = nt$"[5].

This paper introduces the concept of strongly nil* clean ideal (henceforth: SN*CI).We give some of it's properties, and find it's relationships with NC ideals.

2. Strongly nil* clean ideals:

In this section we introduce the concept of the SN*CI. Some of it's characteristics are discussed as well as some examples.

Definition 2.1:

An element a of a ring R is said to be strongly nil clean if $a = e + n$, where $e \in Id(R)$ and $n \in N(R)$ and $en = ne$. A ring R is said to be strongly nil clean if every element of R is strongly nil clean[2].

Example 2.2:

The ring of integers modulo 4, Z_4 is SNC ring.

Definition 2.3:

An element a of a ring R is known SN*C element if for each $a \in R$ there exist two idempotent elements e_1, e_2 in R and a nilpotent element n in R that commute with one another, such that $a = e_1 - e_1 e_2 + n$. A ring R is said to be SN*C ring if each element of R is SN*C element.

Example 2.4:

The ring of integers modulo 8, Z_8 is SN*C ring.

Definition 2.5:

An ideal I of a ring R is known SN*CI if each element of I is SN*C element.

Example 2.6:

Consider the ring of integers modulo 16, the ring of Z_{16} contained three proper ideals namely:
$\mathrm{I}_1 = \{0, 2, 4, 6, 8, 10, 12, 14\}$, $\mathrm{I}_2 = \{0, 4, 8, 12\}$ and $\mathrm{I}_3 = \{0, 8\}$. The ideals I_1, I_2 and I_3 of a ring Z_{16} are SN*C ideals.

Lemma 2.7:

Let $e_1, e_2 \in Id(R)$, with $e_1(e_1e_2) = (e_1e_2)e_1$. Then $e_1 - e_1e_2$ is an idempotent.

Proof:

Since $e_1(e_1e_2) = (e_1e_2)e_1$, then $e_1e_2 = e_1e_2e_1$.
Note that:
$(e_1 - e_1e_2)^2 = e_1 - e_1e_2 - e_1e_2 + e_1e_2 = e_1 - e_1e_2$.
Hence $e_1 - e_1e_2$ is an idempotent.

Proposition 2.8:

Let I is a SN*CI. Then I is a SNCI.

Proof:

Let I be strongly nil* clean ideal. Then for all $a \in I$, we have $a = e_1 - e_1e_2 + n$, where e_1, e_2 are idempotent elements and n is a nilpotent that commute with one another. By (lemma 2.7) $e_1 - e_1e_2$ is idempotent.
Then we get $(e_1 - e_1e_2)^2 = e_1 - e_1e_2$ is idempotent, since $(e_1 - e_1e_2)n = n(e_1 - e_1e_2)$.
Hence $a = e + n$ where $e = e_1 - e_1e_2$ and $en = ne$.
Hence I is strongly nil – clean ideal.

Next, we give the following results:

Lemma 2.9:

If $u \in U(R)$ *and* $n \in N(R)$ *with* $un = nu$, then $u + n$ is a unit.

Proof:

Since $n \in N(R)$, then $n^r = 0$, for some positive integer r. If we set
$1 = 1 + n^r = (1 + n)(1 - n + \cdots + (-1)^{r-1}n^{r-1})$,
showing that $1 + n$ is a unit. Since $u^{-1}n \in N(R)$, hence $1 + u^{-1}n \in U(R)$. So $u + n$ is also unit.

Proposition 2.10:

If I is SN*CI, and if $2 \in I$, then 2 is a nilpotent.

Proof:

Let I be SN*CI such that $a \in I$, then
$a = e_1 - e_1e_2 + n$ *where* $e_1, e_2 \in Id(R)$ *and,* $n \in N(R)$
that commute with one another.
By (lemma2.7) $(e_1 - e_1e_2)$ is an idempotent.
Then $a = e + n$ where $e = e_1 - e_1e_2$.
Let $2 = e + n$ this implies $1 + 1 = e + n$.
We get $1 - e = n - 1$, since $n - 1$ is unit.
Let $1 - e = u$. Suppose $1 - e = 1$, hence $e = 0$.
Then $2 = 0 + n$. So $2 \in N(R)$. Then 2 is nilpotent.

Proposition 2.11:

If R is SN*C ring, then $J(R)$ is a nil ideal.

Proof:

Suppose $a \in J(R)$, then $1 - a$ is a unit. Since R is SN*C ring, then $a = e_1 - e_1e_2 + n$.
Now $1 - a = 1 - (e_1 - e_1e_2) - n$, this implies
$u = 1 - (e_1 - e_1e_2) - n$.
Hence $u + n = (1 - (e_1 - e_1e_2))$.
This implies $u_1 = 1 - (e_1 - e_1e_2)$ where u_1 is a unit.
Thus $1 = 1 - (e_1 - e_1e_2)$, but $e_1 - e_1e_2$ is an idempotent. Then $e_1 - e_1e_2 = 0$.
So $e_1 = e_1e_2$. Therefore $a = n \in J(R)$.

Proposition 2.12:

Let I be an ideal of a ring R with every $a \in I$, $a = f - gf + u, fg = gf$ *and* if $a^2 - a$ is nilpotent, then I is SNCI.

Proof:

Since $fg = gf$ then $(f - gf)$ is an idempotent we get $a = e + u$. Now $a^2 = (e + u)^2$.
Then $a^2 = e + 2eu + u^2$.
Now $a^2 - a = e + 2eu + u^2 - e - u$.
Thus $a^2 - a = (2e + u - 1)u \in N(R)$.
On the other hand, $a = 1 - e + (2e - 1 + u)$.
Since $2e - 1 + u \in N(R)$. Then I is SNCI.

Proposition 2.13:

If R is local ring, and I is SN*CI of R, then I is a nil ideal.

Proof:

Let R be a local ring, then either a or $1 - a$ is a unit. Let I be a SN*CI of R, and let $a \in I$, if a is a unit. Then $I = R$. Let $1 - a$ is a unit. Since I is a SN*CI, then $a = e_1 - e_1 e_2 + n$, where $e_1 e_2 \in Id(R)$ and $n \in N(R)$, that commute with one another.
Now $1 - a = 1 - (e_1 - e_1 e_2) - n$ then
$(1 - a + n) = 1 - (e_1 - e_1 e_2)$.
Since $1 - a$ is a unit we get $u + n$ also is unit, say u_1. Then $u_1 = 1 - (e_1 - e_1 e_2)$. Since $e_1 - e_1e_2$ is idempotent By (lemma2.7). Then $1 - (e_1 - e_1e_2)$ is also idempotent. Hence $1 - (e_1 - e_1e_2) = 1$, this implies $e_1 = e_1e_2$. Therefore $a = n \in I$. Hence I is a nil ideal.

Lemma 2.14:

Let R be a ring, with $2 \in U(R)$, and if e is idempotent element, then $1 + e$ is a unit.

Proof:

Let $e = e^2 \in R$.
Then $(1 + e)(2 - e) = 2 - e + 2e - e = 2$.
Therefore $1 + e$ is a unit.

Theorem 2.15:

Let R be a ring, with $2 \in U(R)$, and I be a SN*CI, then each element of I, can be written as a sum of two units.

Proof:

Let I be a SN*CI and $a \in I$, then
$a = e_1 - e_1 e_2 + n$, Where $e_1, e_2 \in Id(R)$ and $n \in N(R)$, that commute with one another.
Consider $a = 1 + (e_1 - e_1e_2) + n - 1$. Since $e_1 - e_1e_2$ is an idempotent. Then by (lemma2.14) $1 + (e_1 - e_1e_2)$ is a unit, say u_1 and $n - 1$ is a unit, say u_2, then $a = u_1 + u_2$.

3. Tri nil clean ideal

In this section we give the definition of the tri nil clean ideal. We investigate some of its properties and provide some examples.

Definition 3.1:

An ideal I is known TNCI if for each element $a \in I$ can be expressed as $a = t + n$ where $t = t^3$ and $n \in N(R)$ if further $tn = nt$, then I is called STNCI[5]. Clearly every NCI is TNCI.

Example 3.2:

In the ring of integers modulo 6, the ideals of Z_6 are I_1={0, 2, 4} and I_2={ 0, 3} are TNC ideals.

The next results shows the relation between TNCI with strongly clean ideal and nil ideals.

Proposition 3.3:

If I is an ideal with every $a \in I,\ a = t + n,$ $tn = nt\ \ and\ \ t^3 = t$. Then I is a strongly clean ideal.

Proof:

Let $a \in I, then\ \ a = t + n, tn = nt, t^3 = t.$
Consider $t^2 + t - 1$, then $(t^2 + t - 1)^2 = t^2 + t - t^2 + t + t^2 - t - t^2 - t + 1 = 1$. Hence $t^2 + t - 1$ is a unit.
This implies $a = (1 - t^2) + (t^2 + t\ - 1) + n$.
Since $t^2 + t - 1$ is a unit. Then $a = (1 - t^2) + u + n$ where $u = t^2 + t - 1$,
by (lemma2.9) $u + n \in U(R)$. We get $a = (1 - t^2) + u^*$, where $u^*= u + n$, since $(1 - t^2)$ is an idempotent.
Then $a = e^*+u^*$ where $e^*= 1 - t^2$. Hence I is strongly clean ideal.

Proposition 3.4:

Let I be an ideal of a ring R and $2 \in N(R)$, If every element of I, $a = t - t^2 + n$ where $t^3 = t$ and $tn = nt$. Then I is a nil ideal of R.

Proof:

Let $a = t - t^2 + n$ where $t^3 = t$ and $tn = nt$.
Now $(t - t^2)^2 = t^2 - 2t + t^2 = 2(t^2 - t)$. Since $2 \in N(R)$. Then $2(t^2 - t) \in N(R)$.
Let $-t^2 = n_1$, hence $a = n_1 - n$. Then $a \in N(R)$.
Thus I is a nil ideal of R.

Proposition 3.5:

Let $t = t^3$, and let $t \in J(R)$, then $t = 0$.

Proof:

Let $t \in J(R)$ then $t^2 \in J(R)$ then $1 - t^2$ is unit.
Let $1 - t^2 = u$. Since $(t = t^3)$.
Then it follows $t^2 - t^2 = t^2u$, then $t^2u = 0$. So $t^2 = 0$.
Hence $t = 0$.

Proposition 3.6:

If I is strongly tripotent ideal $a \in I$,
$a = t + n,\ t^3 = t, n \in N(R)$, if $2 \in N(R)$. Then $I \cap J(R)$ is a nil ideal.

Proof:

Let $a \in I, a = t + n, tn = nt$, and let $a \in I \cap J(R)$.
Since $a \in J(R)$ then also $a^2 \in J(R)$, hence $1 - a^2$ is a unit, let $1 - a^2 = u$.
Now $a^2 = (\mathrm{t} + \mathrm{n})^2 = t^2 + 2tn + n^2$. Since $2tn + n^2$ is nilpotent. Then $a^2 = t^2 + ń$ where $ń = 2tn + n^2 \in N(R)$.
Now $1 - a^2 = 1 - t^2 - ń$. Then $u = 1 - t^2 - ń$.
This implies $u + ń = 1 - t^2$ since $u + ń$ is a unit. Then $ú = 1 - t^2$ where $ú = u + ń$. Since $1 - \mathrm{t}^2$ is idempotent.
Then $1 - t^2 = 1$, we get $t^2 = 0$, hence $\mathrm{t} = 0$. We get $a = n$. Thus $I \cap J$ is nil ideal.

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مثاليات المعدومة* النقية بقوة

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الملخص

يقال للعنصر a في الحلقة R بأنها نقية معدومة بقوة إذا كانت $a = e + n, en = ne$ حيث e عنصر متحايد, و n عنصر معدوم القوى. ويقال للمثالي I بأنه مثالي نقي معدوم بقوة إذا كان كل عنصر في I, نقي معدوم بقوة.

في هذا البحث اعطينا خواص جديدة لهذا النوع من المثاليات. اعطينا تعريفا جديداً للمثاليات النقية* المعدومة بقوة بالشكل التالي: يقال للمثالي I بأنه نقي* معدوم بقوة اذا كان كل عنصر $a \in R$ يكتب بالشكل التالي $a = e_1 - e_1e_2 + n$ حيث e_1, e_2 عناصر متساوية القوى و n عنصر معدوم القوى وتبادلية مع بعضها. اعطينا بعض الخواص الاساسية لهذا النوع من المثاليات, ووجدنا بعض العلاقات لهذه المثاليات مع مثاليات اخرى.

الكلمات المفتاحية: عنصر معدوم القوى، جاكوبسون راديكال، مثالية معدومة* نقية بقوة، عنصر متساوي القوى.