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Use Different Mathematical Methods to Solve Three Dimensional Conduction Heat Equation in Cartesian Coordinate

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Abstract

In this paper three-dimensional heat conduction equation in cartesian coordinate has been solved in two different methods one of which depends on the separation of variables and the other depends on the integral transform .The results are got and plotted by using Matlab. And the results obtained showed the difference between the two methods that were used in the solution . That difference is evident in the illustrations . According to the results it was concluded that the integral transform method is the best because it has fewer steps to reached to the solution .

Keywords:
Three dimensional Conduction heat equation , Separation of variables , Quadruple Laplace transform , Cartesian coordinate .

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I. INTRODUCTION

Heat transfer problem has big importance in many problems of environmental and industrial. Beforehand , in output energy and transformation applications [6]. The Heat transfer by conduction is mainly concerned with determining the temperature distribution inside the solids [3]. The problems of conduction heat transfer are faced in several engineering applications like the following : Design , Nuclear Reactor Core , Glaciology , Re-entry Shield , Cryosurgery , Rocket Nozzle , Casting , Food Processing [2]. Generally , a heat conduction equation is solved in many cases such as :

Quadruple Laplace transform used by Hamood Ur Rehman, Muzammal Iftikhar, Shoaib Saleem, Muhammad Younis and Abdul Mueed to solve heat equation in cartesian coordinate [1].Separation of Variables used to solve conduction heat equation in cartesian coordinate [5] .Xiao-Jun Yang used a new integral transform operator to solve one dimensional heat-diffusion equation in Cartesian coordinate [7].Ranjit R. Dhunde and G.L.Waghmare used double Laplace transform to solve one dimensional heat equation [4]. Xiao-Jun Yang used a new integral transforms to solve a steady heat transfer problem [8].The aim of this paper is use quadruple Laplace transform method and separation of variables method to solve three dimensional heat conduction equation in cartesian coordinate and compare the solution that will be reached in each method and determine the best method .

II. The model and mathematical methods to solve conduction heat equation:

The model is consist of the following equation with initial and boundary conditions[2]:

$$\left(\frac{\rho c_p}{k}\right)\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \quad \ldots\ldots(1)$$

ρ : density of fluid

c_p : fluid specific heat

k : thermal conductivity of fluid

with boundary and initial conditions:

$T(0,y,z,t) = 0$, $T(a,y,z,t) = c1$, $T(x,0,z,t) = 0$, $T(x,b,z,t) = c2$, $T(x,y,0,t) = 0$,
$T(x,y,c,t) = c3$, $T(x,y,z,0) = xyz$
$0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, $t \ge 0$,

$$c1, c2, c3 = constants$$

We will use the following methods to solve equation(1) :

(a) Using separation of variables [5]:

The basic idea of the method of separation of variables is separate the solution $T(x,y,z,t)$ into two functions which is $\phi(x,y,z)$ and $f(t)$. And $\phi(x,y,z)$ also separate into $X(x), Y(y) and\ Z(z)$. Each of $X(x), Y(y), Z(z) and\ f(t)$ has equation we will solve it and finally substitution all solution in $T(x,y,z,t)=\phi(x,y,z).f(t)$ which is solution of equation (1) .

Now applying separation of variables to equation(1) :

$T(x,y,z,t)=\phi(x,y,z).f(t)\ \dots\dots(2)$

Substitution (2) in (1) we get :

$$\left(\frac{\rho c_p}{k}\right)\phi f'=\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)f$$

Divide both sides by f :

$$\left(\frac{\rho c_p}{k}\right)\left(\frac{f'}{f}\right)=\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)\frac{1}{\phi}$$

By equal the above equation to $-\lambda^2$:

$$f'+\frac{\lambda^2 k}{\rho c_p}f=0\quad ,t\geq 0\quad \dots\dots(3)$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=-\lambda^2\phi\quad ,0\leq x\leq a\ ,0\leq y\leq b\ ,0\leq z\leq c\ \dots\dots(4)$$

After simplify we get :

$$X''+\mu^2X=0\ ,\quad 0\leq x\leq a\ ,X(0)=0\ ,X(a)=c1\quad \dots\dots\ (5)$$

$$Y''+v^2Y=0\ ,\quad 0\leq y\leq b\ ,Y(0)=0\ ,Y(b)=c2\quad \dots\dots\ (6)$$

$$Z''+w^2Z=0\ ,\quad 0\leq z\leq c\ ,Z(0)=0\ ,Z(c)=c3\quad \dots\dots\ (7)$$

and $\lambda^2=\mu^2+v^2+w^2$, $\mu,v,w: constants$

The solution of (3),(5),(6) and (7) are :

$$f(t)=e^{\left(\frac{-\lambda^2kt}{\rho c_p}\right)}$$

$$X(x)=\frac{c1}{\sin(\mu a)}\sin(\mu x)$$

$$Y(y)=\frac{c2}{\sin(vb)}\sin(vy)$$

$$Z(z)=\frac{c3}{\sin(cw)}\sin(wz)$$

Substitution above solutions in (2) we get :

$$T(x,y,z,t)=\frac{c1}{\sin(\mu a)}\sin(\mu x).\frac{c2}{\sin(vb)}\sin(vy).\frac{c3}{\sin(cw)}\sin(wz).e^{\left(\frac{-\lambda^2kt}{\rho c_p}\right)}$$

And we find that :

$$\mu=\frac{m\pi c4}{a}\quad ,m=1,2,\dots\quad ,0\leq c4\leq 2$$

$$v=\frac{n\pi c4}{b}\quad ,n=1,2,\dots\quad ,0\leq c4\leq 2$$

$$w=\frac{q\pi c4}{c}\quad ,q=1,2,\dots\quad ,0\leq c4\leq 2$$

$c_4: constant$

Then the final solution is :

$$T(x,y,z,t)=\left[\frac{c1}{\sin(m\pi c4)}\sin\left(\frac{m\pi c4}{a}x\right)\right]\times\left[\frac{c2}{\sin(n\pi c4)}\sin\left(\frac{n\pi c4}{b}y\right)\right]\times\left[\frac{c3}{\sin(q\pi c4)}\sin\left(\frac{q\pi c4}{c}z\right)\right].e^{\left(\frac{-\lambda_{mnq}^2kt}{\rho c_p}\right)}$$

$$\lambda_{mnq}^2=\left(\frac{m\pi c4}{a}\right)^2+\left(\frac{n\pi c4}{b}\right)^2+\left(\frac{q\pi c4}{c}\right)^2$$

$m=1,2,\dots\quad ,n=1,2,\dots\quad ,q=1,2,\dots$

$0\leq c4\leq 2$

Which is solution of equation (1) .

(b) Using Quadruple Laplace transform [1]:

Quadruple Laplace transform is defined as :

$$L_{xyzt}[f(x,y,z,t)]=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f(x,y,z,t)e^{-px}e^{-qy}e^{-rz}e^{-st}\,dx\,dy\,dz\,dt$$

$p,q,r,s: parameters$

By take L_{xyzt} to both sides of equation(1) :

$$\left(\frac{\rho c_p}{k}\right)L_{xyzt}\left[\frac{\partial T}{\partial t}\right]=L_{xyzt}\left[\frac{\partial^2T}{\partial x^2}\right]+L_{xyzt}\left[\frac{\partial^2T}{\partial y^2}\right]+L_{xyzt}\left[\frac{\partial^2T}{\partial z^2}\right]$$

We get :

$$p^2L_{xyzt}\,T(x,y,z,t)-p\,L_{yzt}\,T(0,y,z,t)-\frac{\partial}{\partial x}L_{yzt}\,T(0,y,z,t)+q^2L_{xyzt}\,T(x,y,z,t)-q\,L_{xzt}\,T(x,0,z,t)-\frac{\partial}{\partial y}L_{xzt}\,T(x,0,z,t)+r^2L_{xyzt}\,T(x,y,z,t)-r\,L_{xyt}\,T(x,y,0,t)-\frac{\partial}{\partial z}L_{xyt}\,T(x,y,0,t)=\left(\frac{\rho c_p}{k}\right)[s\,L_{xyzt}\,T(x,y,z,t)-L_{xyz}\,T(x,y,z,0)]$$

Applying initial and boundary condition to above equation we get :

$$[p^2+q^2+r^2-w1\,s]\,L_{xyzt}\,T(x,y,z,t)=\frac{-w1}{p^2\,q^2\,r^2}=\frac{\rho c_p}{k}$$

Then :

$$L_{xyzt}\,T(x,y,z,t)=\frac{-w1}{[p^2+q^2+r^2-w1\,s]\,p^2\,q^2\,r^2}$$

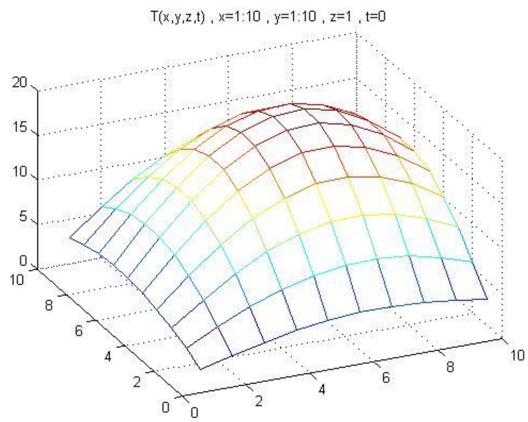
By taking L_{xyzt}^{-1} :

$$T(x,y,z,t)=xyzt.\frac{-w1.s^2}{[p^2+q^2+r^2-w1\,s]}$$

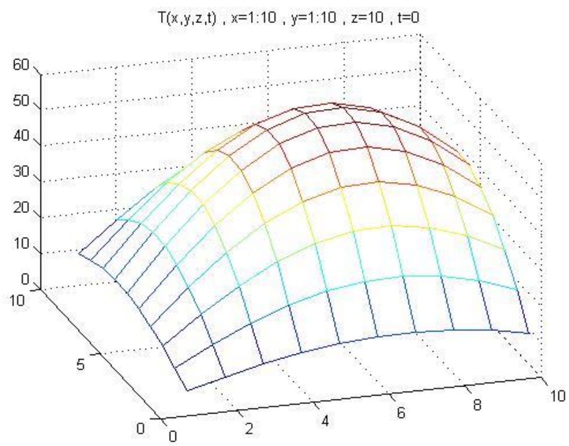
Which is also solution of equation (1) .

III. Results and Data:

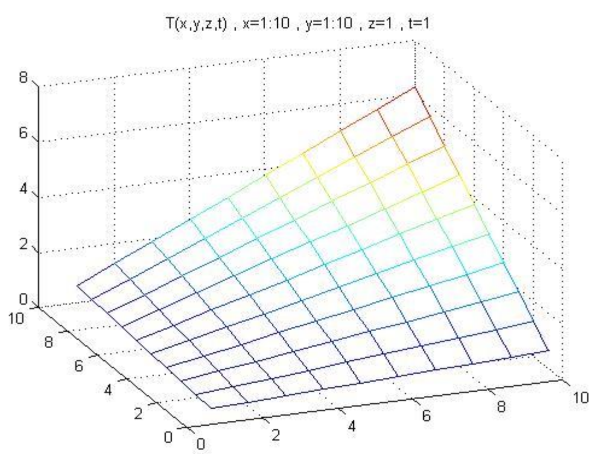
By using Matlab we get the following illustrations which explain the solution of equation (1) :



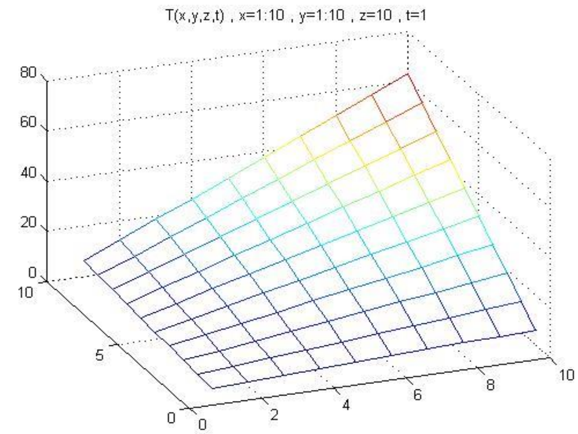
(1-1) Figure shows distribution of temperature by using Separation of Variables method for $T(x,y,z,t)$, x=1:10 , y=1:10 , z=1 , t=0 .



(1-2) Figure shows distribution of temperature by using Separation of Variables method for $T(x,y,z,t)$, x=1:10 , y=1:10 , z=10 , t=0 .



(1-3) Figure shows distribution of temperature by using Quadruple Laplace transform method for $T(x,y,z,t)$, x=1:10 , y=1:10 , z=1 , t=1 .



(1-4) Figure shows distribution of temperature by using Quadruple Laplace transform method for $T(x,y,z,t)$, x=1:10 , y=1:10 , z=10 , t=1 .

IIII. Conclusions:

From the illustrations we note the following conclusions :

1- The difference in the solution methodology for the separation of variables method and for the quadruple Laplace transform method leads to a difference in the result of the solution, as shown in the illustrations (1-1),(1-2),(1-3)and(1-4).

2- We note that the temperature distribution in the method of separating the variables is in the form of a dome , also we notice an increase in temperatures as the value of z increases in the same method as shown in the illustrations(1-1)and(1-2) .

3- We note that the temperature distribution in the quadruple Laplace transform method is in an inclined plane , as we notice an increase in temperatures as the value of z increases in the same method as shown in the illustrations(1-3)and(1-4) .

4- We note that quadruple Laplace transform method is better because it reached to the solution by fewer steps compared with separation the variables method .

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استخدام طرق رياضية مختلفة لحل معادلة الحرارة بالتوصيل ثلاثي ابعاد في احداثيات الكارتيزية

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الملخص

في هذا البحث تم حل معادلة التوصيل الحراري ثلاثية الأبعاد في الإحداثيات الكارتيزية بطريقتين مختلفتين إحداهما تعتمد على فصل المتغيرات وأخر تعتمد على التحويل التكاملي. تم حصول على نتائج و رسمها باستخدام Matlab . النتائج التي تم الحصول عليها اظهرت الفرق بين الطريقتين اللتين تم استخدامهما في الحل. هذا الاختلاف واضح في الرسوم التوضيحية. وفقًا للنتائج تم استنتاج أن طريقة التحويل التكاملي هي الأفضل لأنها تحتوي على خطوات أقل للوصول الى الحل.

كلمات المفتاحية: معادلة الحرارة بالتوصيل ثلاثي ابعاد , فصل المتغيرات , تحويل لابلاس الرباعي , احداثيات الكارتيزية .