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Use Various Mathematical Methods to Solve Three Dimensional Conduction Heat Equation in Cylindrical Coordinate

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Abstract

In this paper we use the separation of variables method and L_{24c} transform method to solve three-dimensional conduction heat equation in cylindrical coordinate and results plotted by using Matlab. It was concluded that L_{24c} transform method is better than the method of separating the variables because it is a method that reaches the solution with fewer steps .

Keywords:
3D Conduction heat equation, Separation of variables, L_{24c} transform, Cylindrical coordinate .

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I. INTRODUCTION

The problem of heat transfer is crucial in many environmental and industrial matters. Beforehand, in output energy and transformation applications [5].The main goal of heat transfer by conduction is to determine the temperature distribution inside solids [4].Conduction heat transfer problems appear in a variety of engineering applications, including the following: Design, Glaciology, Re-entry Shield, Cryosurgery, Rocket Nozzle, Casting, Food Processing [3].In a general, a heat conduction equation has been solved, as the following :

Finite Hankel transform used by V. S. Kulkarni, K. C. Deshmukh and P. H.Munjankar to solve steady state temperature of the cylinder satisfies the heat conduction equation(r,z,t coordinate) [6]. The solution of steady state heat equation(r,θ,t coordinate) was found by using Mellin transform [2]. Also the solution of one dimension heat equation in cylindrical coordinate got by Laplace Transform [4].Ahmed S.Jalal , Ahmed M.J.Jassim used L_{24c} transform method to solve three dimensional conduction heat equation in cylindrical coordinate [1] .Also separation of variables method used to solve three dimensional conduction heat equation in cylindrical coordinate [4]. We will solve conduction heat equation in cylindrical coordinate by L_{24c} transform method and separation of variables method and We will compare between these two methods .

II. The model and mathematical methods to solve conduction heat equation

Consider conduction heat equation in cylindrical coordinate [4] :

$$\left(\frac{\rho c_p}{k}\right)\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \qquad \ldots(1)$$

ρ : density of fluid

c_p : fluid specific heat

k : thermal conductivity of fluid

r : radius

θ : angle

z : cylinder length

With boundary and initial conditions:

$$\mathrm{T(a,\theta,z,t)} = 0 \,, \mathrm{T(r,0,z,t)} = 0 \,, \mathrm{T}\left(\mathrm{r}, \frac{\pi}{2}, \mathrm{z,t}\right) = T_1 \,, \mathrm{T(r,\theta,0,t)} = 0, \mathrm{T(r,\theta,b,t)} = 0 \,, \mathrm{T(r,\theta,z,0)} = T_2$$

$$0 < r \le a \,, 0 \le z \le \mathrm{b} \,, 0 \le \theta \le \frac{\pi}{2} \,, t \ge 0 \,, \quad T_1 \,, T_2 = constants$$

We will use the following methods to solve (1) :

(i) Separation of variables method [4]:

By using separation of variables :

$$T(r,\theta,z,t) = \phi(r,\theta,z).f(t) \qquad \dots(2)$$

Substitution (2) in (1) we get :

$$\left(\frac{\rho c_p}{k}\right)\phi\, f' = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \phi}{\partial r}\,f\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\,f + \frac{\partial^2\phi}{\partial z^2}\,f\right)$$

Divide both sides by f :

$$\left(\frac{\rho c_p}{k}\right)\left(\frac{f'}{f}\right) = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}\right)\frac{1}{\phi}$$

By equal the above equation to negative term $-\lambda^2$:

$$f' + \frac{\lambda^2\,k}{\rho c_p}\,f = 0 \quad ,t \ge 0 \qquad \dots(3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = -\lambda^2\phi \qquad ,0 < r \le a \,, 0 \le \theta \le \frac{\pi}{2} \,, \quad 0 \le z \le b \qquad \dots(4)$$

Assume that:

$$\phi(r,\theta,z) = R(r)\,Q(\theta)\,Z(z)$$

Substitution value of $\phi(r,\theta,z)$ in (4) we get :

$$\frac{(r\,R')'}{r\,R} + \frac{Q''}{r^2\,Q} + \frac{Z''}{Z} = -\lambda^2$$

Now let that :

$$\frac{(r\,R')'}{r\,R} = -\beta^2 \quad ,\frac{Q''}{Q} = -\mu^2 \quad ,\frac{Z''}{Z} = -v^2 \quad ,$$

$-\beta^2\,, -\mu^2, -v^2$: negative terms

After simplify we get :

$$(r\,R')' - \frac{\mu^2}{r}\,R + \beta^2\,rR = 0 \,, \quad 0 < r \le a \,\,, R(a) = 0 \qquad \dots(5)$$

Which is Bessel's equation .

$$Q'' + \mu^2 Q = 0 \;, \quad 0 \le \theta \le \frac{\pi}{2} \,, Q(0) = 0 \,, Q\left(\frac{\pi}{2}\right) = T_1 \qquad \dots(6)$$

$$Z'' + v^2 Z = 0 \;\,, 0 \le z \le b \,, Z(0) = 0 \,, Z(b) = 0 \qquad \dots(7)$$

The solution of (3),(5),(6) and (7) are :

$$f(t) = e^{\left(\frac{-\lambda^2 kt}{\rho c_p}\right)}$$

$$R(r) = J_\mu\left(\frac{j_{mn}}{a}\,r\right) \qquad ,j_{mn}{:}\,nth\ zero\ of\ J_\mu(\beta r)\,, \quad m = 0,1,\dots \;, \quad n = 1,2,\dots$$

$J_\mu(\beta r)$: $Bessel\ function\ of\ order\ \mu\ of\ the\ first\ kind$.

$$Q(\theta) = \frac{T_1}{\sin(mc_1\pi)}\sin(2mc_1\,\theta) \qquad ,m = 1,2,\dots \quad , \quad 0 < c_1 \le 2$$

$$Z(z) = \sin\left(\frac{n\pi}{b}\,z\right) \qquad , \qquad n = 1,2,\dots$$

$$\lambda^2_{mn} = v^2_n + \beta^2_{mn}$$

Substitution above solutions in (2) we get :

$$T(r,\theta,z,t) = J_\mu\left(\frac{j_{mn}}{a}\,r\right).\frac{T_1}{\sin(mc_1\pi)}\sin(2mc_1\,\theta).\sin\left(\frac{n\pi}{b}\,z\right).e^{\left(\frac{-\lambda^2 kt}{\rho c_p}\right)}$$

$$for\;\; m = 1,2,\dots \;\,,n = 1,2,\dots \;\;, \qquad 0 < c_1 \le 2\,, \quad \lambda^2_{mn} = v^2_n + \beta^2_{mn}$$

Which is solution of equation (1) .

(ii) L_{24c} transform method [1]:

L_{24c} transform method is defined by :

$$\boldsymbol{L_{24c}(f(r,\theta,z,t))} = \int_0^\infty\int_0^\infty\int_0^{\pi/2}\int_0^\infty (\boldsymbol{r^2 sin\theta\, cos\theta\, z\, t})\, \boldsymbol{e^{-r^2(p^2cos^2\theta+q^2sin^2\theta)}e^{-v^2z^2-s^2t^2}f(r,\theta,z,t)\, rdr\, d\theta\, dz\, dt}$$

p,q,v,s $are\ parameters$

By take L_{24c} to equation (1) :

$$\left(\frac{\rho c_p}{k}\right) L_{24c}\left[\frac{\partial T}{\partial t}\right] = L_{24c}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)\right] + L_{24c}\left[\frac{1}{r^2}\,\frac{\partial^2 T}{\partial \theta^2}\right] + L_{24c}\left[\frac{\partial^2 T}{\partial z^2}\right]$$

We get :

$$L_{24c}[4r^2((p^2\cos^2\theta + q^2\sin^2\theta)^2\,T)] - L_{24c}[8(p^2\cos^2\theta + q^2\sin^2\theta)\,T] - L_{24c}[6(p^2\cos^2\theta + q^2\sin^2\theta)\,T] + L_{24c}\left[6\left(\frac{1}{r^2}\right)T\right] + L_{24c}[2(p^2cos^2\theta + q^2sin^2\theta)\,T] - L_{24c}\left[\left(\frac{2}{r^2}\right)T\right] + L_{24c}[(4sin^2\theta\,cos^2\theta(p^2-q^2)^2)\,T] - L_{24c}[6(p^2-q^2)sin^2\theta\,T] + L_{24c}[6(p^2-q^2)cos^2\theta\,T] + \left(\frac{1}{2q^2}\right)\left(\frac{1}{2v^2}\right)\left(\frac{1}{2s^2}\right)T\left(\frac{\pi}{2}\right) - L_{24c}\left[4\left(\frac{1}{r^2}\right)T\right] + L_{24c}[4z^2v^4\,T] - L_{24c}[4v^2\,T] - L_{24c}[2v^2\,T] = L_{24c}\left[2s^2\left(\frac{\rho c_p}{k}\right)t\,T\right] - L_{24c}\left[\left(\frac{1}{t}\right)\left(\frac{\rho c_p}{k}\right)T\right]$$

After simplicity and applying initial and boundary condition we get :

$$L_{24c}\left[2r^2\left(\frac{1}{p^2}\right)((p^2\cos^2\theta + q^2\sin^2\theta)^2\,T)\right] - L_{24c}\left[\left(\frac{4}{p^2}\right)(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right] - L_{24c}\left[\left(\frac{3}{p^2}\right)(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right] + L_{24c}\left[\left(\frac{3}{p^2}\right)\left(\frac{1}{r^2}\right)T\right] + L_{24c}\left[\left(\frac{1}{p^2}\right)(p^2cos^2\theta + q^2sin^2\theta)\,T\right] - L_{24c}\left[\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)T\right] + L_{24c}\left[\left(\frac{1}{p^2}\right)(2sin^2\theta\,cos^2\theta(p^2-q^2)^2)\,T\right] - L_{24c}\left[\left(\frac{3}{p^2}\right)(p^2-q^2)sin^2\theta\,T\right] + L_{24c}\left[\left(\frac{3}{p^2}\right)(p^2-q^2)cos^2\theta\,T\right] - L_{24c}\left[\left(\frac{2}{p^2}\right)\left(\frac{1}{r^2}\right)T\right] + L_{24c}\left[\left(\frac{2}{p^2}\right)z^2v^4\,T\right] - L_{24c}\left[\left(\frac{2}{p^2}\right)v^2\,T\right] - L_{24c}\left[\left(\frac{1}{p^2}\right)v^2\,T\right] - L_{24c}\left[s^2\left(\frac{\rho c_p}{k}\right)\left(\frac{1}{p^2}\right)t\,T\right] + L_{24c}\left[\left(\frac{1}{t}\right)\left(\frac{1}{2p^2}\right)\left(\frac{\rho c_p}{k}\right)T\right] = -\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)(2p^2)}\right)$$

Take L_{24c}^{-1} to the above equation:

$$\left[2r^2\left(\frac{1}{p^2}\right)((p^2\cos^2\theta + q^2\sin^2\theta)^2\,T)\right] - \left[\left(\frac{4}{p^2}\right)(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right] - \left[\left(\frac{3}{p^2}\right)(p^2\cos^2\theta + q^2\sin^2\theta)\,T\right] + \left[\left(\frac{3}{p^2}\right)\left(\frac{1}{r^2}\right)T\right] + \left[\left(\frac{1}{p^2}\right)(p^2cos^2\theta + q^2sin^2\theta)\,T\right] - \left[\left(\frac{1}{p^2}\right)\left(\frac{1}{r^2}\right)T\right] + \left[\left(\frac{1}{p^2}\right)(2sin^2\theta\,cos^2\theta(p^2-q^2)^2)\,T\right] - \left[\left(\frac{3}{p^2}\right)(p^2-q^2)sin^2\theta\,T\right] + \left[\left(\frac{3}{p^2}\right)(p^2-q^2)cos^2\theta\,T\right] - \left[\left(\frac{2}{p^2}\right)\left(\frac{1}{r^2}\right)T\right] + \left[\left(\frac{2}{p^2}\right)z^2v^4\,T\right] - \left[\left(\frac{2}{p^2}\right)v^2\,T\right] - \left[\left(\frac{1}{p^2}\right)v^2\,T\right] - \left[s^2\left(\frac{\rho c_p}{k}\right)\left(\frac{1}{p^2}\right)t\,T\right] + \left[\left(\frac{1}{t}\right)\left(\frac{1}{2p^2}\right)\left(\frac{\rho c_p}{k}\right)T\right] = -L_{24c}^{-1}\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)(2p^2)}\right)$$

The above equation reduce to:

$$T\Big[\left(\frac{2r^2}{p^2}\right)(p^2cos^2\theta + q^2sin^2\theta) - \frac{6}{p^2}(p^2cos^2\theta + q^2sin^2\theta) + \frac{2}{p^2}sin^2\theta cos^2\theta(p^2-q^2)^2 + \frac{3}{p^2}(p^2-q^2)cos2\theta + \frac{2}{p^2}z^2v^4 - 3\left(\frac{1}{p^2}\right)v^2 - \frac{s^2}{p^2}\left(\frac{\rho c_p}{k}\right)t + \left(\frac{\rho c_p}{k}\right)\frac{1}{2p^2t}\Big] = -L_{24c}^{-1}\left(\frac{T\left(\frac{\pi}{2}\right)}{(2s^2)(2v^2)(2q^2)(2p^2)}\right)$$

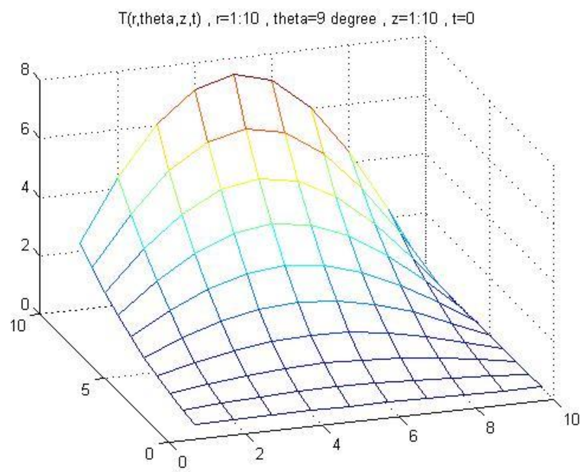
And finally we get :

$$T = T(r,\theta,z,t) = -T_1/\Big[\left(\frac{2r^2}{p^2}\right)(p^2cos^2\theta + q^2sin^2\theta) - \frac{6}{p^2}(p^2cos^2\theta + q^2sin^2\theta) + \frac{2}{p^2}sin^2\theta cos^2\theta(p^2-q^2)^2 + \frac{3}{p^2}(p^2-q^2)cos2\theta + \frac{2}{p^2}z^2v^4 - 3\left(\frac{1}{p^2}\right)v^2 - \frac{s^2}{p^2}\left(\frac{\rho c_p}{k}\right)t + \left(\frac{\rho c_p}{k}\right)\frac{1}{2p^2t}\Big]$$

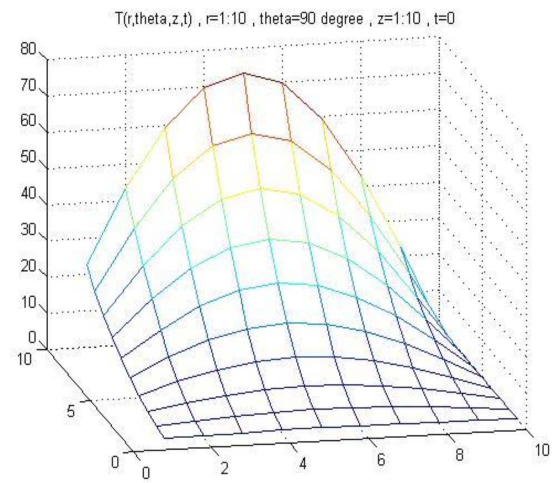
Which is also solution of equation (1) .

III. Results

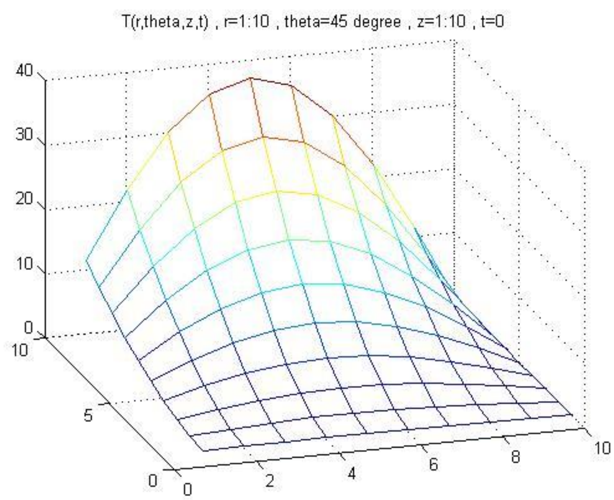
By using Matlab we get the following illustrations which explain the solution of equation (1) :



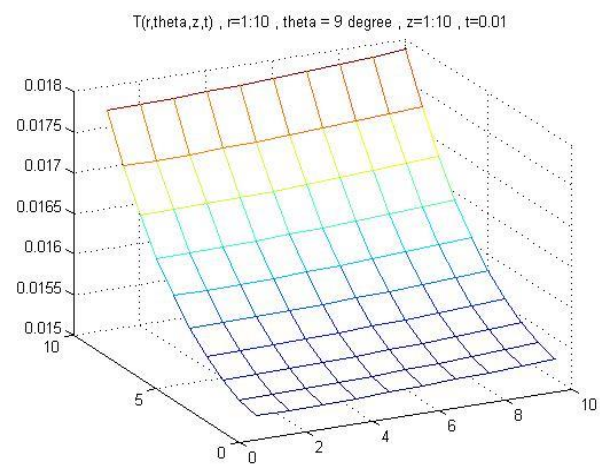
(1-1) Figure shows distribution of temperature by using Separation of Variables method for T(r, θ,z,t) , r=1:10 , $\theta = 9^{o}$, z=1:10 , t=0 .



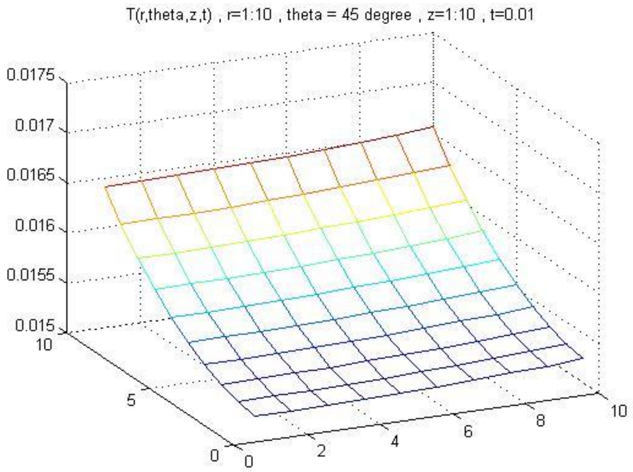
(1-3) Figure shows distribution of temperature by using Separation of Variables method for T(r, θ,z,t) , r=1:10 , $\theta = 90^{o}$, z=1:10 , t=0 .



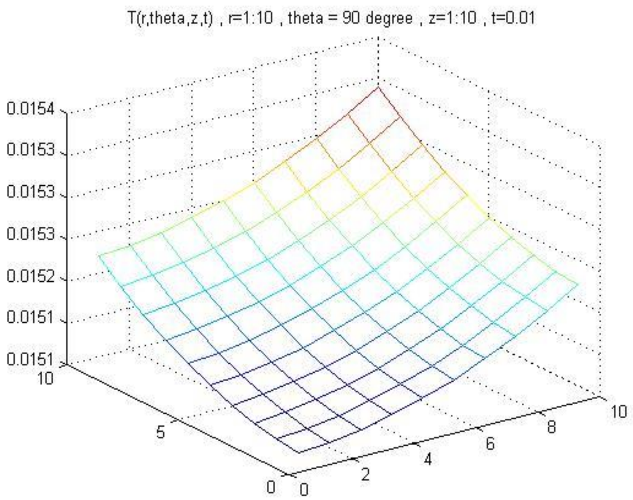
(1-2) Figure shows distribution of temperature by using Separation of Variables method for T(r, θ,z,t) , r=1:10 , $\theta = 45^{o}$, z=1:10 , t=0 .



(1-4) Figure shows distribution of temperature by using L_{24c} transform method for T(r, θ,z,t) , r=1:10 , $\theta = 9^{o}$, z=1:10 , t=0.01 .



(1-5) Figure shows distribution of temperature by using L_{24c} transform method for T(r, θ,z,t) , r=1:10 , θ = 45^o , z=1:10 , t=0.01 .



(1-6) Figure shows distribution of temperature by L_{24c} transform method for T(r, θ,z,t) , r=1:10 , θ = 90^o , z=1:10 , t=0.01 .

IIII. CONCLUSION

From the illustrations we note the following conclusions :

1- The difference in the solution methodology for the Separation of Variables method and for the leads to a difference in the result of the solution, as shown in the illustrations .

2- We note from figures (1-1),(1-2),(1-3) that the temperature distribution in method of separating the variables is in the form of a dome , as we notice an increase in temperatures as the value of θ increases in the same method .

3- We note from figures (1-4),(1-5),(1-6) that the temperature distribution in L_{24c} transform method is in an inclined plane , as we notice an decrease in temperatures as the value of θ increases in the same method .

4- We note that L_{24c} transform method reached to solution by fewer steps compared with separating the variables method .

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استخدام طرق رياضية مختلفة لحل معادلة الحرارة بالتوصيل ثلاثي ابعاد في احداثيات الكارتيزية

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الخلاصة:

في هذا البحث تم حل معادلة التوصيل الحراري ثلاثية الأبعاد في الإحداثيات الكارتيزية بطريقتين مختلفتين إحداهما تعتمد على فصل المتغيرات وأخر تعتمد على التحويل التكاملي. تم حصول على نتائج و رسمها باستخدام Matlab . النتائج تظهر الفرق بين الطريقتين اللتين تم استخدامهما في الحل. هذا الاختلاف واضح في الرسوم التوضيحية. وفقًا للنتائج تم استنتاج أن طريقة التحويل التكاملي هي الأفضل لأنها تحتوي على خطوات أقل للوصول الى الحل.

كلمات المفتاحية: معادلة الحرارة بالتوصيل ثلاثي ابعاد ، فصل المتغيرات ، تحويل لابلاس الرباعي ، احداثيات الكارتيزية .