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Schultz and Modified Schultz Polynomials for Vertex – Identification Chain and Ring – for Hexagon Graphs

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ABSTRACT

The aim of this paper is to find polynomials related to Schultz, and modified Schultz indices of vertex identification chain and ring for hexagonal rings (6 – cycles). Also to find index and average index of all of them.

Keywords: Schultz, modified Schultz, vertex identification chain and ring.

متعددات حدود شولتز وشولتز المعدلة لتطابق رأس لسلسلة وحلقة للبيانات السداسية

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الملخص

الهدف من هذا البحث هو ايجاد متعددات حدود شولتز وشولتز المعدلة لتطابق رأس لسلسلة وحلقة للحلقات السداسية، كما أيضاً وجدنا دليل شولتز وشولتز المعدلة ودليلهما.

الكلمات المفتاحية: شولتز، شولتز المعدلة، تطابق رؤوس لسلسلة وحلقة.

1. INTRODUCTION:

We will let all graphs in this paper to be connected, finite, undirected and simple, which means empty from loops and multiple edges. Let $G = (V, E)$ be a connected simple graph, and $V = V(G)$ and $E = E(G)$ denote the sets of vertices and edges, respectively, of G.

In any graph G represent the number of vertices the **order** of G and denoted that by symbol $p = p(G) = |V(G)|$, and we called the number of edges the **size** of G, and denoted that by symbol $q = q(G) = |E(G)|$. We say for any two vertices u, v in G adjacent in G if there exists edge between them, and we write $e = uv$, as well as we say the edge e incident on u and v. We called the degree of vertex u as the number of edges incident on it and denoted that by $degu$ as such that for vertex v in G [5].

Now, we define the distance between any two vertices u, v in G. The **distance** is the length of a shortest path that join between u and v in G which is denoted by $d_G(u, v)$ or $d(u, v)$. We called the maximum distance between any two vertices u and v in G the **diameter** and denoted that by $diamG$ [4]. In 2005, Gutman introduced the graph

polynomials related to the Schultz and modified Schultz indices [12], and in 2011, Behmaram, et al. found the Schultz polynomials of some graph operation [3]. Farahani [9], gave Schultz and modified Schultz polynomials of some Harary graphs in 2013. Ahmed and Haitham studied Schultz and modified Schultz polynomials, indices, and index average for two Gutman's operations [1]. Also they found general formulas for Schultz and modified Schultz polynomials, indices, and index average of cog-special graphs [2]. Also there are many studies about their applications ([6,7,8,10, 11]).
Schultz had introduced and studied in 1989 Schultz index (*molecular topological index*) [18]. Then, in 1997 Klavžar and Gutman introduced the modified Schultz index [17].

They have defined **Schultz** and **modified Schultz**, **indices**, respectively, as:
$Sc(G) = \sum_{\{u,v\}\subseteq V(G)}(degv + degu)\ d(u,v)$.
$Sc^*(G) = \sum_{\{u,v\}\subseteq V(G)}(degv \cdot degu)\ d(u,v)$.

Schultz and modified Schultz polynomials are considered very important polynomials through studying some properties of their coefficients. Schultz and modified Schultz polynomials are defined, respectively, as:
$Sc(G;x) = \sum_{\{u,v\}\subseteq V(G)}(degv + degu)\, x^{d(u,v)}$.
$Sc^*(G;x) = \sum_{\{u,v\}\subseteq V(G)}(degv \cdot degu)\, x^{d(u,v)}$.

We can obtain the indices of Schultz and modified Schultz by taking derivative of them with respect to x at $x = 1$, as explained below.
$Sc(G) = \frac{d}{dx}(Sc(G;x))|_{x=1}$ and $Sc^*(G) = \frac{d}{dx}(Sc^*(G;x))|_{x=1}$.

While we can obtain the average of the Schultz and modified Schultz indices for connected graph G with order $p(G)$ that are defined as:
$\overline{Sc}(G) = 2Sc(G)/p(G)\ (p(G)\ - 1)$ and $\overline{Sc^*}(G) = 2Sc^*(G)/p(G)\ (p(G)\ - 1)$.

In any connected graph G, we refer to the set of unordered pairs of vertices which are distance k apart by the symbol $D_k(G)$ and let $|D_k(G)| = D(G,k)$.
Now let that $D_k(r,h)$ be the set of all unordered pairs of vertices u, v in G, which are of distance k and of $degu = r, degv = h$.

It is obvious that $\sum_{k=1}^{diam(G)}|D_k(G)| = p(G)(p(G) - 1)/2$, where $D(G,k) = |D_k(G)|$.

Finally, Schultz indices are considered very interesting to determine some properties of chemical structures, see more ([13,14,15,16]).

2. Main Results:

2.1. The Vertex – Identification Chain (VIC) – Graphs:

Let $\{G_1, G_2, \ldots, G_n\}$ be a set of pairwise disjoint graphs with vertices $u_i, v_i \in V(G_i), i = 1,2,\ldots,n\ , n \geq 2$, then the vertex-identification chain graph $C_v(G_1, G_2, \ldots, G_n) \equiv C_v(G_1, G_2, \ldots, G_n\colon v_1 \cdot u_2\ ;\ v_2 \cdot u_3; \ldots; v_{n-1} \cdot u_n\ )$ of $\{G_i\}_{i=1}^{n}$ with respect to the vertices $\{v_i, u_{i+1}\}_{i=1}^{n-1}$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_n$ by identifying the vertex v_i with the vertex u_{i+1} for all $i = 1,2,\ldots,n-1$. (See Fig. 2-1) in which:

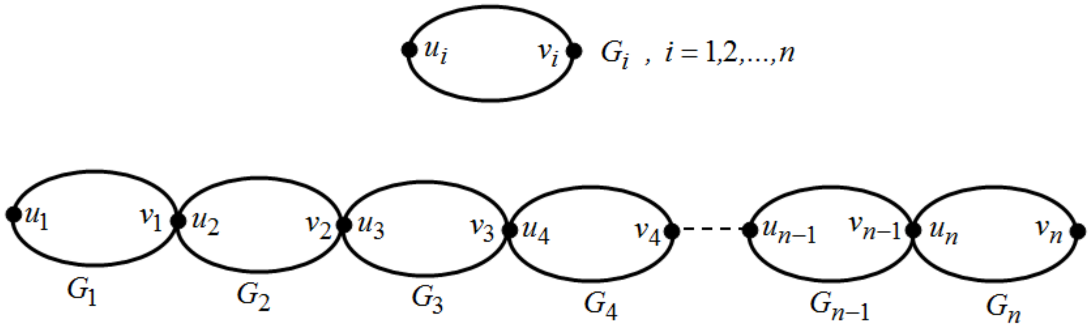

Fig. 2-1-1. $C_v(G_1, G_2, \dots, G_n)$.

Some Properties of Graph $C_v(G_1, G_2, \dots, G_n)$:

1. $p(C_v(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} p(G_i) - (n-1)$.
2. $q(C_v(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} q(G_i)$.
3. $n \le diam\big(C_v(G_1, G_2, \dots, G_n)\big) \le \sum_{i=1}^{n} diam(G_i)$.

The equality of both bounds are satisfied at complete graphs, but the upper bound is satisfied at path graphs in which v_i , u_i are end-vertices of G_i for $i = 1,2, \dots, n$.
If $G_i \equiv H_p$, for all $1 \le i \le n$, where H_p is a connected graph of order p, we denoted $C_v(H_p, H_p, \dots, H_p)$ by $C_v(H_p)_n$.

Schultz and modified Schultz of $C_v(C_6)_{p/2}$

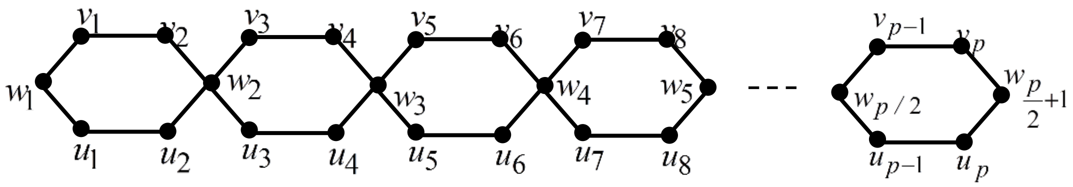

Fig. 2-1-2 The Graph $C_v(C_6)_{p/2}$

From Fig. 2-1-2, we note that $p\left(C_v(C_6)_{\frac{p}{2}}\right) = \frac{5p}{2} + 1$, $q(C_v(C_6)_{\frac{p}{2}}) = 3p$ and $diam\left(C_v(C_6)_{\frac{p}{2}}\right) = \frac{3p}{2}$. For all $1 \le i, j \le p$, $i \ne j$ and $2 \le m, h \le \frac{p}{2}$, $m \ne h$ we have:

Table 2.1

+ ×	$degu_i = 2$	$degv_i = 2$	$degw_1 = 2$	$degw_{\frac{p}{2}+1} = 2$	$degw_m = 4$
$degu_j = 2$	4 4	4 4	4 4	4 4	6 8
$degv_j = 2$	4 4	4 4	4 4	4 4	6 8
$degw_1 = 2$	4 4	4 4		4 4	6 8
$degw_{\frac{p}{2}+1} = 2$	4 4	4 4	4 4		6 8
$degw_h = 4$	6 8	6 8	6 8	6 8	8 8

Theorem 2.1.1: For $p \geq 4$, then:

1. $Sc\left(C_v(C_6)_{\frac{p}{2}}; x\right) = 8(2p-1)x + 24(p-1)x^2 + 12(2p-3)x^3 + \frac{20}{3}\sum_{k=4}^{\frac{3p}{2}}(3p-2k)x^k + \frac{4}{3}x(3x^2+2x+4)\sum_{k=1}^{\frac{p}{2}-1}x^{3k}.$

2. $Sc^*\left(C_v(C_6)_{\frac{p}{2}}; x\right) = 4(5p-4)x + 4(7p-8)x^2 + 4(7p-12)x^3 + \sum_{k=4}^{\frac{3p}{2}-1}(24p-16)x^k + 4x^{\frac{3p}{2}}.$

Proof: For all $p \geq 8$ and every two vertices $u, v \in V(C_v(C_6)_{\frac{p}{2}})$, there is $d(u,v) = k$, $1 \leq k \leq \frac{3p}{2}$, we will have ten partitions for proof:

P1. If $d(u,v) = 1$, then $|D_1| = 3p = q\left(C_v(C_6)_{\frac{p}{2}}\right)$ and we have two subsets of the edge set:

P1.1 $|D_1(2,2)| = |\left\{(u_{2i-1}, u_{2i}), (v_{2i-1}, v_{2i}): 1 \leq i \leq \frac{p}{2}\right\} \cup \{(w_1, u_1), (w_1, v_1), \left(w_{\frac{p}{2}+1}, u_p\right), \left(w_{\frac{p}{2}+1}, v_p\right)\}| = p + 4.$

P1.2 $|D_1(2,4)| = \left|\left\{(u_{2i}, w_{i+1}), (v_{2i}, w_{i+1}), (u_{2i+1}, w_{i+1}), (v_{2i+1}, w_{i+1}): 1 \leq i \leq \frac{p}{2} - 1\right\}\right| = 2p - 4.$

P2. If $d(u,v) = 2$, then, we have two subsets of D_2

P2.1 $|D_2(2,2)| = |\{(u_{2i}, u_{2i+1}), (v_{2i}, v_{2i+1}), (u_{2i}, v_{2i+1}), (v_{2i}, u_{2i+1}): 1 \leq i \leq \frac{p}{2} - 1\} \cup \{(w_1, u_2), (w_1, v_2), \left(w_{\frac{p}{2}+1}, u_{p-1}\right), \left(w_{\frac{p}{2}+1}, v_{p-1}\right)\} \cup \{(u_i, v_i): 1 \leq i \leq p\}| = 3p.$

P2.2 $|D_2(2,4)| = |\{(u_{2i-1}, w_{i+1}), (v_{2i-1}, w_{i+1}), (u_{2i+2}, w_{i+1}), (v_{2i+2}, w_{i+1}): 1 \leq i \leq \frac{p}{2} - 1\}| = 2p - 4.$

Therefor $|D_2| = 5p - 4$.

P3. If $d(u,v) = 3$, then, we have three subsets of D_3:

P3.1 $|D_3(2,2)| = |\{(u_i, u_{i+2}), (v_i, v_{i+2}), (u_i, v_{i+2}), (v_i, u_{i+2}): 1 \leq i \leq p-2\} \cup \{(u_{2i-1}, v_{2i}), (v_{2i-1}, u_{2i}): 1 \leq i \leq \frac{p}{2}\}| = 5p - 8.$

P3.2 $|D_3(2,4)| = \left|\left\{(w_1, w_2), \left(w_{\frac{p}{2}+1}, w_{\frac{p}{2}}\right)\right\}\right| = 2.$

P3.3 $|D_3(4,4)| = \left|\left\{(w_{i+1}, w_{i+2}): 1 \leq i \leq \frac{p}{2} - 2\right\}\right| = \frac{p}{2} - 2.$

Therefor $|D_3| = \frac{11p}{2} - 8.$

P4. If $d(u,v) = k$, when $k = 3j + 4$, j $= 0,1,\ldots,\frac{p}{2} - 3$, then, we have two subsets of D_k:

P4.1 $|D_k(2,2)| = |\{\left(u_{2i-1}, u_{2i+\frac{2(k-1)}{3}}\right), \left(v_{2i-1}, v_{2i+\frac{2(k-1)}{3}}\right), \left(u_{2i-1}, v_{2i+\frac{2(k-1)}{3}}\right), \left(v_{2i-1}, u_{2i+\frac{2(k-1)}{3}}\right): 1 \leq i \leq \frac{p}{2} - \frac{k-1}{3}\} \cup \{\left(w_1, u_{\frac{2k+1}{3}}\right), \left(w_1, v_{\frac{2k+1}{3}}\right), \left(w_{\frac{p}{2}+1}, u_{p-\frac{2(k-1)}{3}}\right), \left(w_{\frac{p}{2}+1}, v_{p-\frac{2(k-1)}{3}}\right)\}| = 2p - \frac{4(k-4)}{3}.$

P4.2 $|D_k(2,4)| = |\{\left(u_{2i}, w_{i+\frac{k+2}{3}}\right), \left(v_{2i}, w_{i+\frac{k+2}{3}}\right), \left(u_{2i+\frac{2k+1}{3}}, w_{i+1}\right), \left(v_{2i+\frac{2k+1}{3}}, w_{i+1}\right): 1 \leq i \leq \frac{p}{2} - \frac{k+2}{3}\}| = 2p - \frac{4(k+2)}{3}.$

Therefore $|D_k| = 4p - \frac{8}{3}(k-1)$, for $k = 3j+4$, j $= 0,1,\dots,\frac{p}{2}-3$

P5. If $d(u,v) = k$, when $k = 3j+5$, j $= 0,1,\dots,\frac{p}{2}-3$, then, we have two subset of D_k:

P5.1 $|D_k(2,2)| = |\{\left(u_{2i}, u_{2i+\frac{2k-1}{3}}\right), \left(v_{2i}, v_{2i+\frac{2k-1}{3}}\right), \left(u_{2i}, v_{2i+\frac{2k-1}{3}}\right), \left(v_{2i}, u_{2i+\frac{2k-1}{3}}\right):$
$1 \le i \le \frac{p}{2} - \frac{k+1}{3}\} \cup \{\left(w_1, u_{\frac{2(k+1)}{3}}\right), \left(w_1, v_{\frac{2(k+1)}{3}}\right), \left(w_{\frac{p}{2}+1}, u_{p-\frac{2k-1}{3}}\right),$
$\left(w_{\frac{p}{2}+1}, v_{p-\frac{2k-1}{3}}\right)\}| = 2p - \frac{4(k-2)}{3}.$

P5.2 $|D_k(2,4)| = |\{\left(u_{2i-1}, w_{i+\frac{k+1}{3}}\right), \left(v_{2i-1}, w_{i+\frac{k+1}{3}}\right), \left(u_{2i+\frac{2(k+1)}{3}}, w_{i+1}\right), \left(v_{2i+\frac{2(k+1)}{3}}, w_{i+1}\right)$
$: 1 \le i \le \frac{p}{2} - \frac{k+1}{3}\}| = 2p - \frac{4(k+1)}{3}.$

Thus $|D_k| = 4p - \frac{4}{3}(2k-1)$ for $k = 3j+5$, j $= 0,1,\dots,\frac{p}{2}-3$.

P6. If $d(u,v) = k$, when $k = 3j+6$, $j = 0,1, \dots ,\frac{p}{2}-4$, then, we have three subsets of D_k:

P6.1 $|D_k(2,2)| = |\{\left(u_i, u_{i+\frac{2k}{3}}\right), \left(v_i, v_{i+\frac{2k}{3}}\right), \left(u_i, v_{i+\frac{2k}{3}}\right), \left(v_i, u_{i+\frac{2k}{3}}\right):$
$1 \le i \le p - \frac{2k}{3}\}| = 4p - \frac{8k}{3}.$

P6.2 $|D_k(2,4)| = |\{\left(w_1, w_{\frac{k}{3}+1}\right), \left(w_{\frac{p}{2}+1}, w_{\frac{p}{2}-\frac{k}{3}+1}\right)\}| = 2.$

P6.3 $|D_k(4,4)| = \left|\left\{\left(w_{i+1}, w_{i+\frac{k}{3}+1}\right) : 1 \le i \le \frac{p}{2} - \frac{k}{3} - 1\right\}\right| = \frac{p}{2} - \frac{k}{3} - 1.$

Thus $|D_k| = \frac{9p}{2} - 3k + 1$ for $k = 3j+6$, $j = 0,1, \dots ,\frac{p}{2}-4$.

P7. If $d(u,v) = \frac{3p}{2} - 3$, then, we have two subsets of $D_{\frac{3p}{2}-3}$:

P7.1 $\left|D_{\frac{3p}{2}-3}(2,2)\right| = \left|\{(u_i, u_{p+i-2}), (v_i, v_{p+i-2}), (u_i, v_{p+i-2}), (v_i, u_{p+i-2}) : i = 1,2\}\right|$
$= 8.$

P7.2 $\left|D_{\frac{3p}{2}-3}(2,4)\right| = |\{\left(w_1, w_{\frac{p}{2}}\right), \left(w_{\frac{p}{2}+1}, w_2\right)\}| = 2.$

Therefore $\left|D_{\frac{3p}{2}-3}\right| = 10.$

P8. If $d(u,v) = \frac{3p}{2} - 2$, then $\left|D_{\frac{3p}{2}-2}\right| = 8$, because:

$\left|D_{\frac{3p}{2}-2}(2,2)\right| = |\{(u_1, u_p), (u_1, v_p), (v_1, v_p), (v_p, u_1), \left(u_2, w_{\frac{p}{2}+1}\right), (u_{p-1}, w_1),$
$\left(v_2, w_{\frac{p}{2}+1}\right), (v_{p-1}, w_1)\}| = 8.$

P9. If $d(u,v) = \frac{3p}{2} - 1$, then $\left|D_{\frac{3p}{2}-1}\right| = 4$, because:

$\left|D_{\frac{3p}{2}-1}(2,2)\right| = \left|\left\{\left(u_1, w_{\frac{p}{2}+1}\right), (u_p, w_1), \left(v_1, w_{\frac{p}{2}+1}\right), (v_p, w_1)\right\}\right| = 4.$

P10. If $d(u,v) = \frac{3p}{2}$, then $\left|D_{\frac{3p}{2}}\right| = 1$, since $\left|D_{\frac{3p}{2}}(2,2)\right| = \left|\left\{\left(w_1, w_{\frac{p}{2}+1}\right)\right\}\right| = 1.$

From P_1 to P_{10} and Table 2.1.1, we have:

$$Sc\left(C_e(C_6)_{\frac{p}{2}}; x\right) = \{4(p+4) + 6(2p-4)\}x + \{4(3p) + 6(2p-4)\}x^2$$
$$+ \left\{4(5p-8) + 6(2) + 8\left(\frac{p}{2}-2\right)\right\}x^3$$

$$+\sum_{k=4,7,10,\dots}^{\frac{3p}{2}-5}\{4\left(2p-\frac{4(k-4)}{3}\right)+6\left(2p-\frac{4(k+2)}{3}\right)\}x^k$$
$$+\sum_{k=5,8,11,\dots}^{\frac{3p}{2}-4}\{4\left(2p-\frac{4(k-2)}{3}\right)+6(2p-\frac{4(k+1)}{3})\}x^k$$
$$+\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-6}\{4\left(4p-\frac{8k}{3}\right)+6(2)+8(\frac{p}{2}-\frac{k}{3}-1)\}x^k$$
$$+\{4(8)+6(2)\}x^{\frac{3p}{2}-3}+\{4(8)\}x^{\frac{3p}{2}-2}+\{4(4)\}x^{\frac{3p}{2}-1}+\{4(1)\}x^{\frac{3p}{2}}.$$
$$=8(2p-1)x+24(p-1)x^2+12(2p-3)x^3$$
$$+4\sum_{k=4,7,10,\dots}^{\frac{3p}{2}-2}(5p-\frac{2(5k-2)}{3})x^k\;+4\sum_{k=5,8,11,\dots}^{\frac{3p}{2}-1}(5p-\frac{2(5k-1)}{3})x^k$$
$$+4\sum_{k=6,9,12,\dots}^{\frac{3p}{2}}(5p-\frac{10k-3}{3})\,x^k.$$
$$=8(2p-1)x+24(p-1)x^2+12(2p-3)x^3$$
$$+\frac{20}{3}\sum_{k=4}^{\frac{3p}{2}}(3p-2k)x^k+\frac{4}{3}x(3x^2+2x+4)\sum_{k=1}^{\frac{p}{2}-1}x^k.$$

Now, we find modified Shultz polynomial:

$$Sc^*\left(C_v(C_6)_{\frac{p}{2}};x\right)=\{4(p+4)+8(2p-4)\}x+\{4(3p)+8(2p-4)\}x^2$$
$$+\left\{4(5p-8)+8(2)+16\left(\frac{p}{2}-2\right)\right\}x^3$$
$$+\sum_{k=4,7,10,\dots}^{\frac{3p}{2}-5}\{4\left(2p-\frac{4(k-4)}{3}\right)+8(2p-\frac{4(k+2)}{3})\}x^k$$
$$+\sum_{k=5,8,11,\dots}^{\frac{3p}{2}-4}\{4\left(2p-\frac{4(k-2)}{3}\right)+8(2p-\frac{4(k+1)}{3})\}x^k$$
$$+\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-6}\{4\left(4p-\frac{8k}{3}\right)+8(2)+16(\frac{p}{2}-\frac{k}{3}-1)\}x^k$$
$$+\{4(8)+8(2)\}x^{\frac{3p}{2}-3}+\{4(8)\}x^{\frac{3p}{2}-2}+\{4(4)\}x^{\frac{3p}{2}-1}+\{4(1)\}x^{\frac{3p}{2}}.$$
$$=4(5p-4)x+4(7p-8)x^2+4(7p-12)x^3$$
$$+\sum_{k=4,7,10,\dots}^{\frac{3p}{2}-2}(24p-16k)x^k+\sum_{k=5,8,11,\dots}^{\frac{3p}{2}-1}(24p-16k)x^k$$
$$+\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-3}(24p-16k)\,x^k+4x^{\frac{3p}{2}}.$$
$$=4(5p-4)x+4(7p-8)x^2+4(7p-12)x^3$$
$$+\sum_{k=4}^{\frac{3p}{2}-1}(24p-16k)\,x^k+4x^{\frac{3p}{2}}.\quad\blacksquare$$

Remark:

1. $Sc(C_v(C_6)_2;x)=56x+72x^2+60x^3+32x^4+16x^5+4x^6.$
   $Sc^*(C_v(C_6)_2;x)=64x+80x^2+64x^3+32x^4+16x^5+4x^6.$
2. $Sc(C_v(C_6)_3;x)=88x+120x^2+108x^3+72x^4+56x^5+44x^6+32x^7+16x^8$ $+4x^9.$
3. $Sc^*(C_v(C_6)_3;x)=104x+136x^2+120x^3+80x^4+64x^5+48x^6+32x^7+16x^8$ $+4x^9.$

Corollary 2.1.2: For $p\geq 4$, then we have:

1. $Sc\left(C_v(C_6)_{p/2}\right)=\frac{3p}{2}(5p^2+3p+10).$
2. $Sc^*\left(C_v(C_6)_{p/2}\right)=9p(p^2+2).$ ■

Corollary 2.1.3: If n is the number of cycles C_6 in the graph $C_v(C_6)_n$, $n\geq 2$, then

1. $Sc(C_v(C_6)_n)=6n(10n^2+3n+5).$

2. $Sc^*(C_v(C_6)_n) = 36n(2n^2 + 1)$ ■

Corollary 2.1.4: For $p \geq 4$, then we have:

1. $\overline{Sc}(C_v(C_6)_{\frac{p}{2}}) = \frac{12}{25}(5p + 1 + \frac{48}{5p+2})$.

2. $\overline{Sc^*}(C_v(C_6)_{p-1}) = \frac{72}{125}(5p - 2 + \frac{54}{5p+2})$. ■

2.2. The Vertex – Identification Ring (VIR) – Graph:

Let $\{G_1, G_2, \dots, G_n\}$ be a set of pairwise disjoint graphs with vertices $u_i, v_i \in V(G_i), i = 1,2,\dots,n, \geq 3$, then the vertex-identification Ring graph $R_v(G_1, G_2, \dots, G_n) \equiv R_v(G_1, G_2, \dots, G_n: v_1 \cdot u_2;\ v_2 \cdot u_3; \dots; v_{n-1} \cdot u_n; v_n \cdot u_1)$ of $\{G_i\}_{i=1}^{n}$ with respect to the vertices $\{v_i, u_i\}_{i=1}^{n}$ is the graph obtained from the graphs $G_1, G_2, \dots, G_n$ by identifying the vertex v_i with the vertex u_{i+1} for all $i = 1,2,\dots,n$. (See Fig. 2-2) where $u_{n+1} \equiv u_1$.

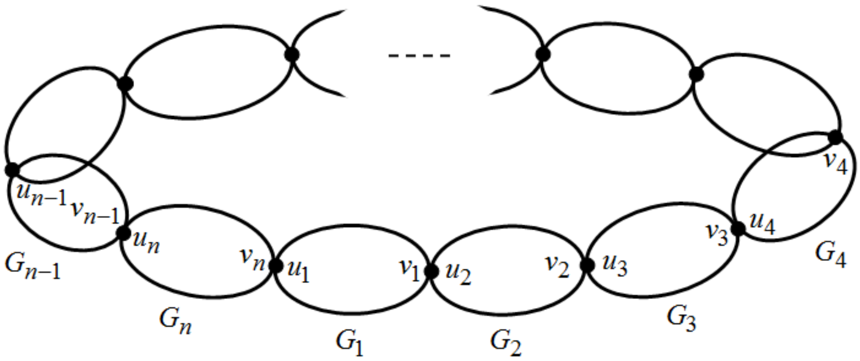

Fig. 2-2-1. $R_v(G_1, G_2, \dots, G_n)$

Some Properties of the graph $R_v(G_1, G_2, \dots, G_n)$:

1. $p(R_v(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} p(G_i) - n$.
2. $q(R_v(G_1, G_2, \dots, G_n)) = \sum_{i=1}^{n} q(G_i)$.
3. $\left\lceil \frac{n-1}{2} \right\rceil \leq diam(R_v(G_1, G_2, \dots, G_n)) \leq \left\lceil \frac{\sum_{i=1}^{n} diam(G_i)}{2} \right\rceil$.

The equality of both bounds are satisfied at complete graphs but the upper bound is satisfied at path graphs in which v_i, u_i are end-vertices of G_i for $i = 1,2,\dots,n$.

If $G_i \equiv H_p$, for all $1 \leq i \leq n$, where H_p is a connected graph of order p, we denoted $R_v(H_p, H_p, \dots, H_p)$ by $R_v(H_p)_n$.

Schultz and modified Schultz of $R_v(C_6)_{p/2}$:

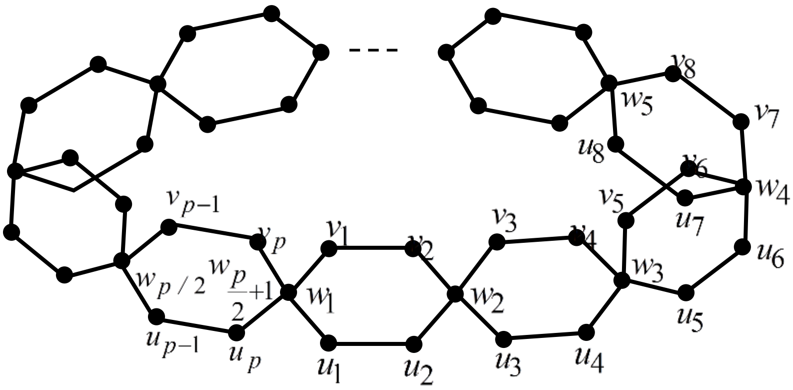

Fig. 2-2-2. The Graph $R_v(C_6)_{p/2}$, $p \geq 6$, even p.

From Fig. 2-2-2, we note that $p\left(R_v(C_6)_{\frac{p}{2}}\right)=\frac{5p}{2}$, $q(R_v(C_6)_{\frac{p}{2}})=3p$ and $diam\left(R_v(C_6)_{\frac{p}{2}}\right)=\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil$. For all $1\le i,j\le p$, $i\neq j$, then we have:

Table 2.1.1

+ \ ×	$degu_i=2$	$degv_i=2$	$degw_i=4$
$degu_j=2$	4 4	4 4	6 8
$degv_j=2$	4 4	4 4	6 8
$degw_j=4$	6 8	6 8	8 16

Theorem 2.1.2: For $p\ge 8$, then we have:

1. $Sc\left(R_v(C_6)_{\frac{p}{2}};x\right)=16px+24px^2+24px^3$

$$+\begin{cases}20p\sum_{k=4,5,6,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1}x^k+10px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p=12,16,20,\dots\\ 20p\sum_{k=4,5,6,,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}x^k, when\ p=14,18,22,\dots\end{cases}$$

2. $Sc^*\left(R_v(C_6)_{\frac{p}{2}};x\right)=20px+28px^2+28px^3$

$$+\begin{cases}24p\sum_{k=4,5,6,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1}x^k+12px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p=12,16,20,\dots\\ 24p\sum_{k=4,5,6,,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}x^k, when\ p=14,18,22,\dots\end{cases}$$

Proof: For all $p\ge 12$, and every two vertices $u,v\in V(R_v(C_6)_{\frac{p}{2}})$, there is $d(u,v)=k$, $1\le k\le\frac{3p}{2}$, we will have seven partitions for proof:

P1. If $d(u,v)=1$, then $|D_1|=3p=q\left(R_v(C_6)_{\frac{p}{2}}\right)$ and we have two subsets of the edge set:

P1.1 $|D_1(2,2)|=\left|\left\{(u_{2i-1},u_{2i}),(v_{2i-1},v_{2i}):1\le i\le\frac{p}{2}\right\}\right|=p.$

P1.2 $|D_1(2,4)|=\left|\left\{(u_{2i-1},w_i),(v_{2i-1},w_i),(u_{2i},w_{i+1}),(v_{2i},w_{i+1}):1\le i\le\frac{p}{2}\right\}\right|=2p,$ where $w_{\frac{p}{2}+1}\equiv w_1$.

P2. If $d(u,v)=2$, then, we have two subsets of D_2:

P2.1 $|D_2(2,2)|=|\{(u_{2i},u_{2i+1}),(v_{2i},v_{2i+1}),(u_{2i+1},v_{2i}),(v_{2i+1},u_{2i}):1\le i\le\frac{p}{2}\}\cup$
$\{(u_i,v_i):1\le i\le p\}|=3p$, where $u_{p+1}\equiv u_1\ and\ v_{p+1}\equiv v_1$.

P2.2 $|D_2(2,4)|=\left|\left\{(u_{2i-1},w_{i+1}),(v_{2i-1},w_{i+1}),(u_{2i},w_i),(v_{2i},w_i):1\le i\le\frac{p}{2}\right\}\right|=2p,$ where $w_{p+1}\equiv w_1$.

Thus $|D_2|=5p$.

P3. If $d(u,v)=3$, then,we have three subsets of D_3:

P3.1 $|D_3(2,2)|=|\{(u_i,u_{i+2}),(v_i,v_{i+2}),(u_i,v_{i+2}),(v_i,u_{i+2}):1\le i\le p\}\cup$

$\{(u_{2i},v_{2i-1}),(v_{2i},u_{2i-1}):1\le i\le \frac{p}{2}\}| = 5p,$

where $u_{p+a}\equiv u_a$ *and* $v_{p+a}\equiv v_a, a=1,2.$

P3.2 $|D_3(4,4)| = \left|\left\{(w_i,w_{i+1}):1\le i\le \frac{p}{2}\right\}\right| = \frac{p}{2}$, where $w_{\frac{p}{2}+1}\equiv w_1$.

Thus $|D_3| = \frac{11p}{2}$.

P4. If $d(u,v)=k$, when $k=3j+4$, and $p=12,16,20,\dots$, $j=0,1,2,\dots,\frac{p}{4}-2$, and when $p=14,18,22,\dots$, $j=0,1,\dots,\frac{p-2}{4}-2$, then, we have two subsets of such (u,v) pairs of D_k:

P4.1 $|D_k(2,2)| = |\{\left(u_{2i-1},u_{2i+\frac{2(k-1)}{3}}\right),\left(v_{2i-1},v_{2i+\frac{2(k-1)}{3}}\right),\left(u_{2i-1},v_{2i+\frac{2(k-1)}{3}}\right),$
$\left(v_{2i-1},u_{2i+\frac{2(k-1)}{3}}\right):1\le i\le \frac{p}{2}\}| = 2p,$

where $u_{p+a}\equiv u_a$ *and* $v_{p+a}\equiv v_a, a=2,4,6,\dots,\frac{2(k-1)}{3}$

P4.2 $|D_k(2,4)| = |\{\left(u_{2i},w_{i+\frac{k+2}{3}}\right),\left(v_{2i},w_{i+\frac{k+2}{3}}\right),\left(u_{2i+\frac{2k-5}{3}},w_i\right),\left(v_{2i+\frac{2k-5}{3}},w_i\right):$
$1\le i\le \frac{p}{2}\}| = 2p$, where $u_{p+a}\equiv u_a$ *and* $v_{p+a}\equiv v_a, a=1,2,3,\dots,\frac{2k-5}{3}$,
where $w_{\frac{p}{2}+b}\equiv w_b, b=1,2,3,\dots,\frac{k+2}{3}$.

Thus $|D_k| = 4p$, $k=3j+4$,for $j=0,1,2,\dots,\frac{p}{4}-2$.

P5. If $d(u,v)=k$, when $k=3j+5$, and $p=12,16,20,\dots$, $j=0,1,2,\dots,\frac{p}{4}-2$, and when $p=14,18,22,\dots$, $j=0,1,2,\dots,\frac{p-2}{2}-2$, then, we have two subsets of such (u,v) pairs of D_k:

P5.1 $|D_k(2,2)| = |\{\left(u_{2i},u_{2i+\frac{2k-1}{3}}\right),\left(v_{2i},v_{2i+\frac{2k-1}{3}}\right),\left(u_{2i},v_{2i+\frac{2k-1}{3}}\right),\left(v_{2i},u_{2i+\frac{2k-1}{3}}\right):$
$1\le i\le \frac{p}{2}\}| = 2p,$

where $u_{p+a}\equiv u_a$ *and* $v_{p+a}\equiv v_a, a=1,2,3,\dots,\frac{2k-1}{3}$.

P5.2 $|D_k(2,4)| = |\{\left(u_{2i-1},w_{i+\frac{k+1}{3}}\right),\left(v_{2i-1},w_{i+\frac{k+1}{3}}\right),\left(u_{2i+\frac{2(k-2)}{3}},w_i\right),\left(v_{2i+\frac{2(k-2)}{3}},w_i\right):$
$1\le i\le \frac{p}{2}\}| = 2p,$

where $u_{p+a}\equiv u_a$, $v_{p+a}\equiv v_a, a=2,4,6,\dots,\frac{2(k-2)}{3}$ and $w_{\frac{p}{2}+b}\equiv w_b, b=1,2,3,\dots,\frac{k+1}{3}$.

Thus $|D_k| = 4p$, $k=3j+5$, for $j=0,1,2,\dots,\frac{p-2}{2}-2$.

P6. If $d(u,v)=k$, when $k=3j+6$, and when $p=12,16,20,\dots$, $j=0,1,2,\dots,\frac{p}{4}-3$, and when $p=14,18,22,\dots$, $j=0,1,\dots,\frac{p-2}{4}-2$, then, we have three subsets of such (u,v) pairs of D_k:

P6.1 $|D_k(2,2)| = \left|\left\{\left(u_i,u_{i+\frac{2k}{3}}\right),\left(v_i,v_{i+\frac{2k}{3}}\right),\left(u_i,v_{i+\frac{2k}{3}}\right),\left(v_i,u_{i+\frac{2k}{3}}\right):1\le i\le p\right\}\right| = 4p,$

where $u_{p+a}\equiv u_a$ *and* $v_{p+a}\equiv v_a$ $a=1,2,3,\dots,\frac{2k}{3}$.

P6.2 $|D_k(4,4)| = \left|\left\{\left(w_i,w_{i+\frac{k}{3}}\right):1\le i\le \frac{p}{2}\right\}\right| = \frac{p}{2},$

where $w_{\frac{p}{2}+b}\equiv w_b, b=1,2,3,\dots,\frac{2(2k-3)}{3}$.

Thus $|D_k| = \frac{9p}{2}$,, $k=3j+6$,for $j=0,1,2,\dots,\frac{p}{4}-3$.

P7. If $d(u,v)=\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil$, then we have:

a- If $p=12,16,20,\ldots$, then, we have two subsets of $D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}$:

P7.1 $\left|D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}(2,2)\right|=\left|\left\{\left(u_i,u_{i+\frac{p}{2}}\right),\left(v_i,v_{i+\frac{p}{2}}\right),\left(u_i,v_{i+\frac{p}{2}}\right),\left(v_i,u_{i+\frac{p}{2}}\right):1\le i\le\frac{p}{2}\right\}\right|=2p.$

P7.2 $\left|D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}(4,4)\right|=\left|\left\{\left(w_i,w_{i+\frac{p}{4}}\right):1\le i\le\frac{p}{4}\right\}\right|=\frac{p}{4}.$

Thus $\left|D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}\right|=\frac{9}{4}p$, for even $\frac{p}{2}$.

b- If $p=14,18,22,\ldots$ then , we have two subsets of $D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}$:

Ṕ7.1 $\left|D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}(2,2)\right|=\left|\left\{\left(u_i,u_{i+\frac{p}{2}}\right),\left(v_i,v_{i+\frac{p}{2}}\right),\left(u_i,v_{i+\frac{p}{2}}\right),\left(v_i,u_{i+\frac{p}{2}}\right):1\le i\le\frac{p}{2}\right\}\right|$
$=2p.$

Ṕ7.2 $\left|D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}(2,4)\right|=|\{\left(u_{2i},w_{i+\frac{p+2}{4}}\right),\left(u_{2i-1},w_{i+\frac{p+2}{4}}\right),\left(v_{2i},w_{i+\frac{p+2}{4}}\right),\left(v_{2i-1},w_{i+\frac{p+2}{4}}\right):$
$1\le i\le\frac{p}{2}\}|=2p,$

where $w_{\frac{p}{2}+b}\equiv w_b, b=1,2,3,\ldots,\frac{p+2}{4}$.

Thus $\left|D_{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}\right|=4p$, for odd $\frac{p}{2}$.

From P_1 to P_7 and Table 2.1.2, we have:

$$Sc\left(R_e(C_6)_{\frac{p}{2}};x\right)=\{4(p)+6(2p)\}x+\{4(3p)+6(2p-4)\}x^2+\left\{4(5p)+8\left(\frac{p}{2}\right)\right\}x^3$$
$$+\begin{cases}\sum_{k=4,7,10,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2}\{4(2p)+6(2p)\}x^k+\sum_{k=5,8,11,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1}\{4(2p)+6(2p)\}x^k\\ +\sum_{k=6,9,12,\ldots}^{\frac{3p}{2}-3}\{4(4p)+8(\frac{p}{2})\}x^k+\{4(2p)+8(\frac{p}{4})\}x^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}\\ \qquad when\ p=12,16,20,\ldots\\ \sum_{k=4,7,10,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-3}\{4(2p)+6(2p)\}x^k+\sum_{k=5,8,11,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2}\{4(2p)+6(2p)\}x^k\\ +\sum_{k=6,9,12,\ldots}^{\frac{3p}{2}-1}\{4(4p)+8(\frac{p}{2})\}x^k+\{4(2p)+6(2p)\}x^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}\\ \qquad when\ p=14,18,22,\ldots\end{cases}$$
$$=16px+24px^2+24px^3$$
$$+\begin{cases}20p\sum_{k=4,7,10,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2}x^k+20p\sum_{k=5,8,11,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1}x^k+20p\sum_{k=6,9,12,\ldots}^{\frac{3p}{2}-3}x^k\\ +10px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p=12,16,20,\ldots\\ 20p\sum_{k=4,7,10,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-3}x^k+20p\sum_{k=5,8,11,\ldots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2}x^k+20p\sum_{k=6,9,12,\ldots}^{\frac{3p}{2}-1}x^k\\ +20px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p=14,18,22,\ldots\end{cases}$$
$$=16px+24px^2+24px^3$$
$$+\begin{cases}20p\sum_{k=4}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1}x^k+10px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p=12,16,20,\ldots\\ 20p\sum_{k=4}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}x^k, when\ p=14,18,22,\ldots\end{cases}$$

Now, we find modified Shultz polynomial:

$$Sc^*\left(R_v(C_6)_{\frac{p}{2}};x\right) = \{4(p)+8(2p)\}x + \{4(3p)+8(2p-4)\}x^2 + \left\{4(5p)+16\left(\frac{p}{2}\right)\right\}x^3$$

$$+\begin{cases} \sum_{k=4,7,10,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2}\{4(2p)+8(2p)\}\,x^k + \sum_{k=5,8,11,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1}\{4(2p)+8(2p)\}\,x^k \\ +\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-3}\{4(4p)+16(\frac{p}{2})\}\,x^k + \{4(2p)+16(\frac{p}{4})\}x^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil} \\ \qquad when\ p = 12,16,20,\dots \\ \sum_{k=4,7,10,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-3}\{4(2p)+8(2p)\}\,x^k + \sum_{k=5,8,11,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2}\{4(2p)+8(2p)\}\,x^k \\ +\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-1}\{4(4p)+16(\frac{p}{2})\}\,x^k + \{4(2p)+8(2p)\}x^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil} \\ \qquad when\ p = 14,18,22,\dots \end{cases}$$

$$= 20px + 28px^2 + 28px^3$$

$$+\begin{cases} 24p\sum_{k=4,7,10,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2} x^k + 24p\sum_{k=5,8,11,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1} x^k + 24p\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-3} x^k \\ +12px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p = 12,16,20,\dots \\ 24p\sum_{k=4,7,10,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-3} x^k + 24p\sum_{k=5,8,11,\dots}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-2} x^k + 24p\sum_{k=6,9,12,\dots}^{\frac{3p}{2}-1} x^k \\ +24px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p = 14,18,22,\dots \end{cases}$$

$$= 20px + 28px^2 + 28px^3$$

$$+\begin{cases} 24p\sum_{k=4}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil-1} x^k + 12px^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil}, when\ p = 12,16,20,\dots \\ 24p\sum_{k=4}^{\frac{p}{2}+\left\lceil\frac{p-2}{4}\right\rceil} x^k, when\ p = 14,18,22,\dots \end{cases}$$

By simply, we can calculate:

1. $Sc(R_v(C_6)_4;x) = 128x + 192x^2 + 192x^3 + 160x^4 + 160x^5 + 80x^6$.
   $Sc^*(R_v(C_6)_4;x) = 160x + 224x^2 + 224x^3 + 192x^4 + 192x^5 + 96x^6$.
2. $Sc(R_v(C_6)_5;x) = 160x + 240x^2 + 240x^3 + 200x^4 + 200x^5 + 200x^6 + 200x^7$.
   $Sc^*(R_v(C_6)_5;x) = 200x + 280x^2 + 280x^3 + 240x^4 + 240x^5 + 240x^6 + 240x^7$. ■

Remark:

1. $Sc(R_v(C_6)_2;x) = 64x + 96x^2 + 56x^3$.
   $Sc^*(R_v(C_6)_2;x) = 80x + 112x^2 + 64x^3$.
2. $Sc(R_v(C_6)_3;x) = 96x + 144x^2 + 144x^3 + 120x^4$.
   $Sc^*(R_v(C_6)_3;x) = 120x + 168x^2 + 168x^3 + 144x^4$.

Corollary 2.1.2: For $p \geq 4$, then we have:

1. $Sc\left(R_v(C_6)_{\frac{p}{2}}\right) = \begin{cases} \frac{p}{8}(45p^2+128), when\ p = 4,8,12,\dots \\ \frac{3p}{8}(15p^2+36), when\ p = 6,10,14\dots \end{cases}$
2. $Sc^*\left(R_v(C_6)_{\frac{p}{2}}\right) = \begin{cases} \frac{p}{4}(27p^2+64), when\ p = 4,8,12,\dots \\ \frac{p}{4}(27p^2+52), when\ p = 6,10,14,\dots \end{cases}$. ■

Corollary 2.1.3: If n is the number of cycles C_6 in the graph $R_v(C_6)_n$, $n \geq 2$, then we have:

1. $Sc(R_v(C_6)_n) = \begin{cases} n(45n^2+32), when\ n = 2,4,6,\dots \\ 9n(5n^2+3), when\ n = 3,5,7,\dots \end{cases}$

2. $Sc^*(R_v(C_6)_n) = \begin{cases} 2n(27n^2+16), when\ n = 2,4,6,\dots \\ 2n(27n^2+13), when\ n = 3,5,7,\dots \end{cases}$. ■

Corollary 2.1.4: For $p \geq 4$, then we have:

1. $\overline{Sc}(R_v(C_6)_{\frac{p}{2}}) = \begin{cases} \frac{1}{5}\left(9p+\frac{18}{5}+\frac{676}{5(5p-2)}\right), \ when\ p = 4,8,12,\dots \\ \frac{3}{5}\left(3p+\frac{6}{5}+\frac{162}{5(5p-2)}\right), when\ p = 6,10,14,\dots \end{cases}$

2. $\overline{Sc^*}(C_v(C_6)_{\frac{p}{2}}) = \begin{cases} \frac{2}{5}\left(27p+\frac{54}{5}+\frac{1708}{5(5p-2)}\right), \ when\ p = 4,8,12,\dots \\ \frac{2}{25}\left(27p+\frac{54}{5}+\frac{1408}{5(5p-2)}\right), when\ p = 6,10,14,\dots \end{cases}$. ■

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