Detour Hosoya Polynomials of Some Compound Graphs

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ABSTRACT

In this paper we will introduce a new graph distance based polynomial; Detour Hosoya polynomials of graphs $H^*(G;x)$. The Detour Hosoya polynomials $H^*(G;x)$ for some special graphs such as paths and cycles are obtained. Moreover the Detour Hosoya polynomials $H^*(G_1 \bullet G_2;x)$, $H^*(G_1 : G_2;x)$ and $H^*(G_1 \odot G_2;x)$ are obtained.

***Keywords*:** Detour distance, compound graphs, Hosoya polynomials.

متعددات حدود Detour لبعض البيانات المركبة

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الملخص

في هذا البحث قمنا بتعريف متعددة حدود هوسويا نسبة الى مسافة اطول بعد $H^*(G;x)$. تم ايجاد متعددة حدود هوسويا نسبة الى مسافة أطول بعد لبعض البيانات ألخاصة مثل بيان ألدرب و بيان ألدارة. و كذلك تم ألحصول على كل من $H^*(G_1 \bullet G_2;x)$ ، $H^*(G_1 : G_2;x)$ و $H^*(G_1 \odot G_2;x)$.

الكلمات المفتاحية : مسافة Detour ، بيانات مركبة ، متعددة حدود هوسويا.

1. Introduction

The concept of Hosoya polynomial was first put forward in 1988 by Hosoya [**1**]. Several authors, such as [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**13**] and [**15**] had obtained Hosoya polynomials for special graphs, graphs having some kind of regularity and for compound graphs obtained by using some well-known binary operations in graph theory.

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see [**9**] and [**12**].

Ordinarily, when we wish to proceed from a point ***A*** to a point ***B*** we take a route which involves the least distance. We have all been faced with detour sign which require us to take a route from ***A*** to ***B*** that involves a greater distance. In any such detour route from ***A*** to ***B*** we assume that there is no possible shortcut along the route, for otherwise this should have been part of the route initially. When one is driving along such a detour, it sometimes seems that we are using the longest route possible from ***A*** to ***B*** (again subject to the "no shortcut" condition). In this paper we investigate longest detour routes in graphs.

The distance $d(u,v)$ between two vertices ***u*** and ***v*** in a connected graph ***G*** is the length of a shortest ***u-v*** path in ***G***. For a nonempty set ***S*** of vertices of ***G***, the subgraph $<S>$ of ***G*** induced by ***S*** as its vertex set while an edge of ***G*** belongs to $<S>$ if it joins two vertices of ***S***. If ***P*** is a ***u-v*** path of length $d(u,v)$, then the subgraph $<V(P)>$ induced by the vertices of ***P*** is ***P*** itself. This observation suggests the following concept. The detour distance $d^*(u,v)$ between ***u*** and ***v*** in ***G*** is the length of a longest induced ***u-v*** path, that is a longest ***u-v*** path ***P*** for which $<V(P)>=P$. An induced ***u-v*** path of length $d^*(u,v)$ is called a ***detour path*** [**10**].

Observe that $d^*(u,v) \geq d(u,v)$ for all vertices ***u*** and ***v*** of ***G*** and that $d^*(u,v)=d(u,v)=1$ if and only if ***u*** and ***v*** are adjacent. Also, note that $d^*(u,v)=d^*(v,u)$ for all vertices ***u*** and ***v*** of ***G***. Therefore the detour distance is symmetric. However, the triangle inequality does not hold in general. Consider the wheel W_p of order $p \geq 6$ with center at the vertex w; then: $d^*(u,v)=p-3>2=d^*(u,w)+d^*(w,v)$, for every two vertices ***u*** and ***v*** of W_p, $u,v \neq w$, that are both adjacent to a common vertex $x \neq w$.
Therefore, in general, the detour distance is not a metric on the vertex set of ***G***[**10**].

The ***detour eccentricity*** $e^*(v)$ of a vertex ***v*** is defined by $e^*(v)=\max\{d^*(v,w):w \in V(G)\}$. The ***detour eccentricity*** set $e^*(G)$ of a connected graph ***G*** is the set consisting of all detour

eccentricities of G, that is $e^*(G) = \{e^*(v) : v \in V(G)\}$. The ***detour radius*** $rad^*(G)$ of G is the minimum detour eccentricity, while the ***detour diameter*** $diam^*(G)$ of G is the maximum detour eccentricity.
For completeness we define $d^*(u,v) = 0$ if and only $u = v$.

A connected graph G is called a ***detour graph*** if $d^*(u,v) = d(u,v)$ for all vertices u and v of G. No cycle of length **5** or more is a detour graph. On the other hand, all trees and all complete graphs are detour graphs. If u and v are distinct vertices of a graph G such that $d^*(u,v) = 1$ or **2**, then $d^*(u,v) = d(u,v)$[**10**], the converse is not true in general, that is if $d(u,v) = 2$, then $d^*(u,v) \geq 2$, as for the wheel W_p, $p \geq 6$.

The concept of Hosoya polynomial $H(G;x)$ of a graph G was put forward by Hosoya[**13**], and defined as

$H(G;x) = \sum_{k=0}^{\delta(G)} C(G,k)x^k$; where $C(G,k)$ is the number of pairs of vertices in G that are distance k apart, and $\delta(G)$ is the diameter of the graph G.

In this paper, the concept of ***Hosoya polynomials of detour distance*** of a connected graph G (or simply ***detour Hosoya polynomial of a graph G***) has been defined by

$$H^*(G;x) = \sum_{k=0}^{\delta^*(G)} C^*(G,k)x^k = \sum_{\{u,v\} \subseteq V(G)} x^{d^*(u,v)} \qquad \ldots(1)$$

in which $C^*(G,k)$ is the number of pairs of vertices in G with detour distance k, and $\delta^*(G)$ is the detour diameter of G.
It is clear that if G is a detour graph, then $H^*(G;x) = H(G;x)$.

The sum $W^*(G)$ of detour distances between all pairs of vertices of the graph G is known as the ***Wiener index of detour distance*** of the graph G (or simply ***detour Wiener index*** of the graph G), that is

$$W^*(G) = \sum_{u,v} d^*(u,v),$$

where the sum is taken over all unordered pairs $\{u,v\}$ of distinct vertices in G.

It is clear that

$$W^*(G) = \frac{d}{dx} H^*(G;x)\Big|_{x=1}.$$

We illustrate these ideas in the following example.

***<u>Example</u>* 1.1.** Let G be a graph of order $p = 9$,depicted in figure **1.1(*a*)**. It is clear that

$e^*(v_1) = 5$, $e^*(v_2) = 4$, $e^*(v_3) = 4$, $e^*(v_4) = 3$, $e^*(v_5) = 4$, $e^*(v_6) = 3$, $e^*(v_7) = 4$, $e^*(v_8) = 5$ and $e^*(v_9) = 5$.

Hence

$$e^*(G) = \{5,4,4,3,4,3,4,5,5\},\ diam^*(G) = 5 \text{ and } rad^*(G) = 3.$$

A detour $v_1 - v_9$ path is given in Figure **1.1(b)**.Therefore $d^*(v_1, v_9) = 5$, and this gives us the maximum detour distance among all detour distances of pairs of vertices of $V(G)$.

The path P' is not a detour $v_1 - v_9$ path, because $\langle V(P') \rangle \neq P'$(see figures **1.1(c)** and **1.1(d)**).

By direct calculations, we get that

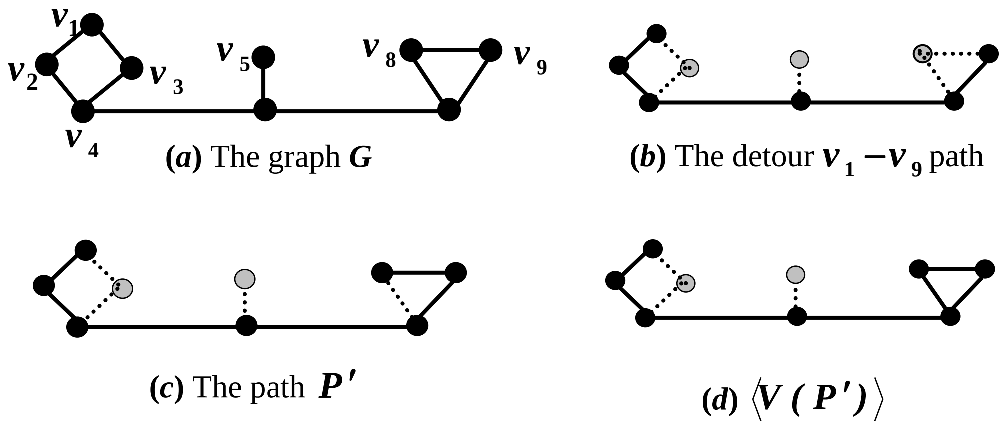
$C^*(G,0) = p = 9$, $C^*(G,1) = 10$, $C^*(G,2) = 9$, $C^*(G,3) = 9$, $C^*(G,4) = 6$ and $C^*(G,5) = 2$.

Hence, the detour Hosoya polynomial of G is

$$H^*(G;x) = 9 + 10x + 9x^2 + 9x^3 + 6x^4 + 2x^5,$$

and

$$W^*(G) = \frac{d}{dx} H^*(G;x)\Big|_{x=1} = 89.$$



(*a*) The graph G **(*b*)** The detour $v_1 - v_9$ path

(*c*) The path P' **(*d*)** $\langle V(P') \rangle$

Figure **1.1.**

In **1993**, Gutman [**8**], established few additional properties of the respective graph polynomials. He obtained Hosoya polynomials of some special graphs and obtained formula for the Hosoya polynomials of some compound graphs, namely $G_1 \bullet G_2$ and $G_1 : G_2$ which are defined in the following: Let G_1 and G_2 be vertex-disjoint connected graphs, and let $u \in V(G_1)$ and $v \in V(G_2)$. Then, the graph $G_1 \bullet G_2$ is obtained from G_1 and G_2 by identifying the two vertices u and v. This means that G_1 and G_2 have exactly one vertex in common in the compound graph $G_1 \bullet G_2$. The graph $G_1 : G_2$ is obtained from G_1 and G_2 by introducing a new edge joining the two vertices u and v. In this paper, formulas for $H^*(G_1 \bullet G_2; x)$ and $H^*(G_1 : G_2; x)$ in terms of the detour Hosoya polynomials of G_1 and G_2 will be obtained.

2. Detour Hosoya Polynomials of Some Special Graphs

Let P_n, K_n and S_n denotes the path, complete and star graphs of n vertices respectively. It is known that [**10**] all trees and complete graphs are detour graphs. This leads us to the following result.

<u>Proposition</u> **2.1**

(a) $$H^*(P_n; x) = \sum_{k=0}^{n-1} (n-k) x^k .$$

(b) $$H^*(K_n; x) = n + \frac{1}{2} n(n-1) x .$$

(c) $$H^*(S_n; x) = n + (n-1)x + \binom{n-1}{2} x^2 . \ \blacksquare$$

<u>Proposition</u> **2.2** Let C_p be a cycle of order $p \geq 5$, then

$$H^*(C_p; x) = \begin{cases} p(1 + x + \sum_{k=\frac{p+1}{2}}^{p-2} x^k) & \textit{if } p \textit{ is odd} \\ p(1 + x + \frac{1}{2} x^{\frac{p}{2}} + \sum_{k=\frac{p}{2}+1}^{p-2} x^k) & \textit{if } p \textit{ is even} \end{cases}$$

Proof. Let u,v be any two distinct vertices of C_p. We will consider the following cases:

(1) If $uv \in E(C_p)$ then $d^*(u,v)=1$ and $C^*(G,1)=p$.

(2) If $uv \notin E(C_p)$, then $d^*(u,v)=p-d(u,v)$,
where $d(u,v)$ denotes the ordinary distance.

We know that[**11**], for an odd p, the ordinary Hosoya polynomial of C_p is given by $H(C_p;x)=p+px+p\sum_{k=2}^{\frac{p-1}{2}} x^k$.

Hence

$$H^*(C_p;x)=p+px+p\sum_{k=p-2}^{p-\frac{p-1}{2}} x^k$$

or

$$H^*(C_p;x)=p+px+p\sum_{k=\frac{p+1}{2}}^{p-2} x^k .$$

Similarly, we prove the formula for the case when p is even.
This completes the proof. ■

Proposition **2.3** Let W_p be a wheel graph of $p \geq 6$ vertices, then

$$H^*(W_p;x)=p+2(p-1)x+(p-1)\begin{cases} \sum_{k=\frac{p}{2}}^{p-3} x^k, & \text{if } p \text{ is even} \\ \frac{1}{2}x^{\frac{p-1}{2}}+\sum_{k=\frac{p+1}{2}}^{p-3} x^k, & \text{if } p \text{ is odd} \end{cases}.$$

Proof. For $uv \notin E(W_p)$, $d^*_{W_p}(u,v)=d^*_{C_{p-1}}(u,v)$.

Hence, for $k \geq 2$

$$C^*(W_P,k)=C^*(C_{p-1},k).$$

Thus,

$$H^*(W_p;x)=1+(p-1)x+H^*(C_{p-1},x).$$

Now, using Proposition **2** we obtain the required result.■

Proposition **2.4** Let $K_{t,s}$ be a complete bipartite graph with partite subsets of sizes ***t*** and ***s***, then

$$H^*(K_{t,s};x)=(t+s)+(ts)x+\left[\binom{t}{2}+\binom{s}{2}\right]x^2.$$

Proof. Obvious ■

The following result gives us the Wiener index of the detour distance of the special graphs P_n, K_n, S_n, C_p, W_p and $K_{t,s}$.

Proposition **2.5**

(1) $W^*(P_n)=\frac{1}{6}n(n^2-1)$.

(2) $W^*(K_n)=\frac{1}{2}n(n-1)$.

(3) $W^*(S_n)=(n-1)^2$.

(4) For $p\geq 5$, $W^*(C_p)=\begin{cases}\frac{1}{8}p(3p^2-12p+17), & \text{if } p \text{ is odd}\\ \frac{1}{8}p(3p^2-12p+16), & \text{if } p \text{ is even}\end{cases}$.

(5) For $p\geq 6$, $W^*(W_p)=\begin{cases}\frac{1}{8}(p-1)(3p^2-18p+39), & \text{if } p \text{ is odd}\\ \frac{1}{8}(p-1)(3p^2-18p+40), & \text{if } p \text{ is even}\end{cases}$.

(6) $W^*(K_{t,s})=ts+t(t-1)+s(s-1)$.

3. Detour Hosoya Polynomials of Some Compound Graphs

Let u be a vertex of a connected graph G of order p. The number of pairs of vertices of G containing the vertex u such that $d_G^*(u,v)=k$, $\forall\ v\in V(G)$, will be denoted by $C^*(u,G;k)$.

We define the polynomial

$$H^*(u,G;x)=\sum_{k=0}^{e^*(u)}C^*(u,G;k)x^k \qquad \ldots(2)$$

It is clear that

$$\mathrm{H}^*(G;x)=\frac{1}{2}\sum_{u\in V(G)}H^*(u,G;x)+\tfrac{1}{2}p \qquad \ldots(3)$$

Let G_1 and G_2 be two disjoint connected graphs of orders p_1 and p_2 respectively. Moreover, let w be the vertex obtained by identifying the vertex u of G_1 with the vertex v of G_2 in order to construct the compound graph $G_1 \bullet G_2$. The compound graph $G_1 : G_2$ is obtained by introducing a new edge joining the vertex u of G_1 with the vertex v of G_2.

Now, we are ready to present formulas for $H^*(G_1 \bullet G_2; x)$ and $H^*(G_1 : G_2; x)$ in terms of $H^*(G_1; x)$ and $H^*(G_2; x)$.

Theorem **3.1** If G_1 and G_2 are disjoint connected graphs, then

$$H^*(G_1 \bullet G_2; x) = H^*(G_1; x) + H^*(G_2; x) + H^*(u, G_1; x).\, H^*(v, G_2; x) - H^*(u, G_1; x) - H^*(v, G_2; x).$$

Proof**:** Let s, t be any two vertices of $G_1 \bullet G_2$ such that $d^*_{G_1 \bullet G_2}(s,t) = k$. We will consider the following cases:

(1) If $s, t \in V(G_1)$, then $C^*(G_1 \bullet G_2; k) = C^*(G_1, k)$, which produces the polynomial $H^*(G_1; x)$.

(2) If $s, t \in V(G_2)$, then $C^*(G_1 \bullet G_2; k) = C^*(G_2, k)$, which produces the polynomial $H^*(G_2; x)$.

(3) $s \in V(G_1)$ and $t \in V(G_2)$: In this case, any longest induced (s,t)-path P will contain the vertex w. If P' is a longest (s,w)-path and P'' is a longest (t,w)-path with $\langle V(P') \rangle = P'$ and $\langle V(P'') \rangle = P''$, then

$$V(P) = V(P') \cup V(P''), \text{ and } \langle V(P) \rangle = \langle V(P') \cup V(P'') \rangle,$$

because no vertex of P', other than w is adjacent with a vertex of P'', other than w.

Therefore $P' \bullet P'' = \langle V(P) \rangle = P$.

Hence, $d^*_{G_1 \bullet G_2}(s,t) = d^*_{G_1}(s,w) + d^*_{G_2}(t,w)$.

This produces the polynomial $H^*(u, G_1; x).\, H^*(v, G_2; x)$. Notice that the polynomial $H^*(u, G_1; x)$ is counted twice in the Cases **(1)** and **(3),** and also $H^*(v, G_2; x)$ is counted twice in the Cases **(2)** and **(3)**.

Now, adding the polynomials obtained from the cases **(1)**, **(2)** and **(3)**, we get the required result. ■

Theorem **3.2** If G_1 and G_2 are disjoint connected graphs, then

$H^*(G_1 : G_2, x) = H^*(G_1, x) + H^*(G_2, x) + x.H^*(u, G_1; x).\ H^*(v, G_2; x)$.

Proof. Let s, t be any two distinct vertices of the compound graph $G_1 : G_2$. We consider the following cases:

(1) If $s, t \in V(G_1)$, then we get the polynomial $H^*(G_1; x)$.

(2) If $s, t \in V(G_2)$, then we get the polynomial $H^*(G_2; x)$.

(3) $s \in V(G_1)$ and $t \in V(G_2)$: In this case, any longest (s,t)-path will contains the edge uv, and as in the proof of Theorem **6**(Case **3**), this produces the polynomial

$$x.\ H^*(u, G_1; x).\ H^*(v, G_2; x)$$

Now, adding the polynomials obtained from the cases **(1)**, **(2)** and **(3)**, we get the required result. ■

Definition **3.3** Let G_1 and G_2 be disjoint connected graphs of orders p_1 and p_2, respectively. Let $G_2^{(i)}$ be the i^{th} copy of G_2. The ***Corona*** $G_1 \odot G_2$, is the graph[**13**] constructed from $G_1 \cup p_1 G_2$ with additional edges $\bigcup_{i=1}^{p_1} \{v_i u : u \in V(G_2^{(i)})\}$,
as depicted in Fig. **3.1**, in which $V(G_1) = \{v_1, v_2, ..., v_{p_1}\}$.

It is clear that

$$p(G_1 \odot G_2) = p_1(1 + p_2) = p,$$

and

$$q(G_1 \odot G_2) = q(G_1) + p_1(p_2 + q(G_2)) = q.$$

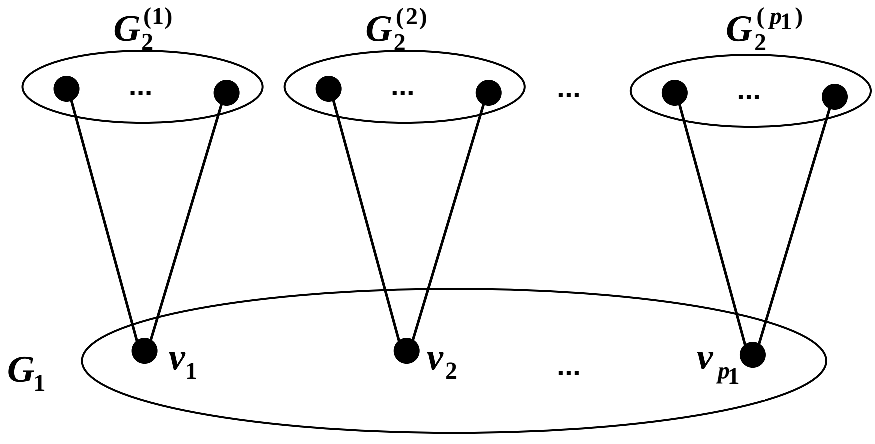


Fig. **3.1** The Corona $G_1 \odot G_2$

The next theorem computes the detour Hosoya polynomial of the corona $G_1 \odot G_2$.

***Theorem* 3.4** Let G_1 and G_2 be two disjoint connected graphs, then

$$H^*(G_1 \odot G_2; x) = (1 + p_2 x)^2 H^*(G_1; x) + p_1 H^*(G_2, x) - p_1 p_2 x(1 + p_2 x).$$

***Proof*.** Let s, t be any two distinct vertices of $G_1 \odot G_2$. We will consider the following cases:

Case 1. If $s, t \in V(G_1)$, then we get the polynomial $H^*(G_1; x)$.

Case 2. If $s, t \in V(G_2^{(i)})$, for $i = 1,2,..., p_1$, then we get the polynomial $p_1 H^*(G_2; x)$.

Case 3. $s \in V_2^{(i)}$ and $t = v_j$ **(or** $s = v_i$ and $t \in V_2^{(j)}$**)** for $i, j = 1,2,..., p_1$, then

(i) If $i = j$, then we get the polynomial $p_1 p_2 x$.

(ii) If $i \neq j$, then we get the polynomial $2 p_2 x\left[H^*(G_1; x) - p_1\right]$.

Case 4. If $s \in V_2^{(i)}$ and $t \in V_2^{(j)}$ for $i, j = 1,2,..., p_1$, $i \neq j$, then we get the polynomial $p_2^2 x^2\left[H^*(G_1; x) - p_1\right]$.

Now, adding the polynomials obtained from the above cases and simplifying, we get the required result. ■

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