The n-Hosoya Polynomials of Some Classes of Thorn Graphs

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Received on:30/12/2008 **Accepted on:17/3/2009**

ABSTRACT

The n-Hosoya Polynomials of cog-complete graphs , thorn cog-complete graphs , cog-stars , thorn cog-stars , cog-wheels , and thorn cog-wheels are obtained . The n-Wiener indices of these graphs are also determined .

Keywords: cog-graph, thorn graph , n-distance , n-Hosoya polynomial ,n-Wiener index.

متعددات حدود هوسويا –n لبعض البيانات الشوكية

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تاريخ قبول البحث:2009/3/17 **تاريخ استلام البحث:2009/12/30**

الملخص

تضمن البحث إيجاد متعددات حدود هوسويا–n للبيانات المسننة التامة ، المسننة الشوكية التامة ، النجمات المسننة ، النجمات المسننة الشوكية ، العجلات المسننة ، العجلات المسننة الشوكية . كما حصلنا على أدلة وينر–n لهذه البيانات.

الكلمات المفتاحية : بينات مسننة ، بينات شوكية ، متعددات حدود هوسويا –n

1. Introduction:

We follow the terminology of [3],[4]. Let v be a vertex of a connected graph G , and let S be an (n-1)-subset of V(G) , $n \geq 2$, then the n-distance $d_n(v,S)$ is defined by [1]

$$d_n(v,S) = min\{d(v,u) : u \in S\}. \qquad \ldots(1.1)$$

The n-diameter of G is defined by

$$diam_n G = max\{d_n(v,S) : v \in V(G), |S| = n-1, S \subseteq V(G)\}. \qquad \ldots(1.2)$$

The n-Wiener index of G is defined by

$$W_n(G)=\sum_{(v,S)} d_n(v,S)\,. \qquad \dots(1.3)$$

The n-Hosoya polynomial of connected graph G of order p is defined by

$$H_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k\,, \qquad \dots(1.4)$$

where $3 \le n \le p$, δ_n is the n-diameter of G , and $C_n(G,k)$ is the number of order pairs $(v,S), v \in V(G), S \subseteq V(G), |S| = n-1,$ such that $d_n(v,S)=k$.

One can easily show that [1].

$$C_n(G,0)=p\binom{p-1}{n-2}\,,\; C_n(G,1)=p\binom{p-1}{n-1}-\sum_{v\in V(G)}\binom{p-1-\deg v}{n-1}. \qquad \dots(1.5)$$

The n-Hosoya polynomial of a vertex v in G , denoted by $H_n(v,G;x)$, is defined [1] by

$$H_n(v,G;x)=\sum_{k\ge 0} C_n(v,G,k)x^k\,, \qquad \dots(1.6)$$

where $C_n(v,G,k)$ is the number of (n-1)-subsets of vertices S such that $d_n(v,S)=k$. It is clear that for each k , $0 \le k \le \delta_n$,

$$C_n(G,k)=\sum_{v\in V(G)} C_n(v,G,k)\,, \qquad ...(1.7)$$

and

$$H_n(G;x)=\sum_{v\in V(G)} H_n(v,G;x)\,, \qquad ...(1.8)$$

The following simple lemma is useful for obtaining $C_n(v,G,k)$ for every vertex v of a connected graph G .

Lemma 1.1: [2] Let t be the number of vertices of ordinary distance k from vertex v , and let s be the number of vertices of distance more than k from v in a connected graph G . Then

$$C_n(v,G,k)=\binom{s+t}{n-1}-\binom{s}{n-1}, \qquad ...(1.9)$$

for $v \in V(G), 2 \le n \le p, 0 \le k \le \delta_n$. ■

Let T be a non-empty subset of vertices of G . We define

$$C_n(T,G,k)=\sum_{v\in T} C_n(v,G,k)\,. \qquad ...(1.10)$$

We shall use this notation in our proofs.

In this paper , we obtain n-Hosoya polynomials and n-Wiener indices of some classes of graphs that are not considered in the published research papers up to date as far as we know.

2. Thorn Cog-Complete Graphs:

The n-Hosoya polynomial will be obtained first for a cog-complete graph that is defined as follows.

Definition 2.1: A cog-complete graph K_m^c is the graph constructed from a complete graph $K_m, m \ge 3$, of vertex set $\{v_1, v_2, \ldots, v_m\}$ with m additional vertices $u_1, u_2, \ldots, u_m$, and 2m edges $\{u_i v_i, u_i v_{i+1} : i = 1, 2, \ldots, m\}$, $(v_{m+1} \equiv v_1)$, as shown in Fig. 2.1.

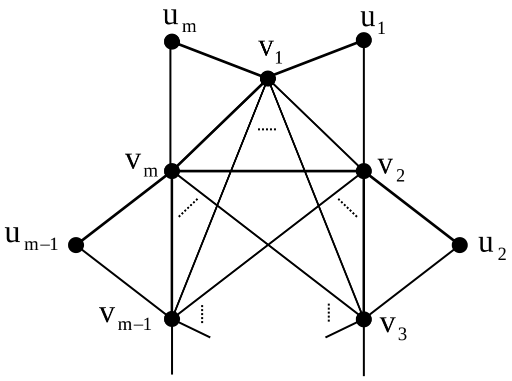

Fig. 2.1. K_m^c

It is clear that $p(K_m^c) = 2m$, $q(K_m^c) = \frac{1}{2}m(m+3)$, and for $m \ge 4$, diam K_m^c =3.

Moreover

$$diam_n K_m^c = \begin{cases} 3 \ , \ if \ \ 2 \le n \le m-2 \ \ , m \ge 4 \\ 2 \ , \ if \ \ m-1 \le n \le 2m-2, \\ 1 \ , \ if \ \ 2m-1 \le n \le 2m \ . \end{cases}$$

Proposition 2.2: For $m \ge 3$, $3 \le n \le 2m$, we have

$$H_n(K_m^c; x) = 2m\binom{2m-1}{n-2} + \sum_{k=1}^{3} C_n(K_m^c, k)x^k \quad ,$$

where

$$C_n(K_m^c, 1) = 2m\binom{2m-1}{n-1} - m\left[\binom{m-2}{n-1} + \binom{2m-3}{n-1}\right],$$

$$C_n(K_m^c, 2) = m\left[\binom{2m-3}{n-1} + \binom{m-3}{n-2}\right],$$

$C_n(K_m^c,3) = m\binom{m-3}{n-1}$.

Proof: The coefficients $C_n(K_m^c,0)$ and $C_n(K_m^c,1)$ are obtained from (1.5).
The coefficient $C_n(K_m^c,3)$ is obtained by taking $v = u_i, 1 \le i \le m$,and the (n-1)-subset S from the m-3 vertices $u_j, j \ne i-1, i, i+1$. Then , $C_n(K_m^c,2)$ is obtained from the fact $\sum_{k\ge 1} C_n(G,k) = p\binom{p-1}{n-1}$. ...(2.1)

■

The n-Wiener index of K_m^c is given by

$$W_n(K_m^c) = m\left[2\binom{2m-1}{n-1} + \binom{2m-3}{n-1} + \binom{m-2}{n-1} + \binom{m-3}{n-1}\right].$$

From Proposition 2.2 , we get the next corollary.

Corollary2.3: The Hosoya polynomial and the Wiener index of K_m^c, $m \ge 3$, are given by:

$$H(K_m^c;x) = 2m + \frac{1}{2}m(m+3)x + m(m-1)x^2 + \frac{1}{2}m(m-3)x^3 .$$

And

$$W(K_m^c) = m(4m-5) .$$

Now , we define a thorn cog- complete graph and find its n-Hosoya polynomial.

Definition 2.4: A thorn cog-complete graph , denoted by K_m^{c*}, is the cog-complete graph K_m^c, $m \ge 3$ constructed in Definition 2.1 , with 2m additional endvertices $w_1, w_2, ..., w_{2m}$, and edges $\{u_i w_{2i-1}, u_i w_{2i} : i = 1,2,...,m\}$, as shown in Fig. 2.2.

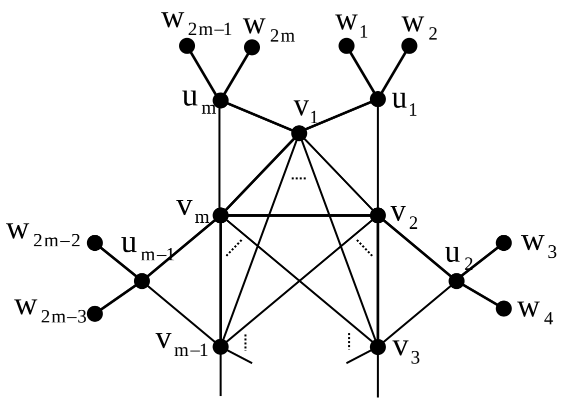

Fig.2.2. K_m^{c*}

It is clear that $p(K_m^{c*}) = 4m$, $q(K_m^{c*}) = \frac{1}{2}m(m+7)$, and, diam K_m^{c*} =5, for $m \geq 4$.

The n-diameter is as given below:

$$diam_n K_m^{c*} = \begin{cases} 5\ , \ if \ \ 2 \leq n \leq 2m-5 \ \ , \ m \geq 4 \\ 4\ , \ if \ \ 2m-4 \leq n \leq 3m-4 \ \ , \\ 3\ , \ if \ \ 3m-3 \leq n \leq 4m-4 \ \ , \\ 2\ , \ if \ \ 4m-3 \leq n \leq 4m-1 \ \ , \\ 1\ , \ if \ \ n = 4m \ \ . \end{cases}$$

The n-Hosoya polynomial of K_m^{c*} is given in the next theorem.

Theorem 2.4: The n-Hosoya polynomial of K_m^{c*}, $m \geq 3$, $3 \leq n \leq 4m$ is given by

$$H_n(K_m^{c*};x) = 4m\binom{4m-1}{n-2} + \sum_{k=1}^{5} C_n(K_m^{c*},k)x^k \ \ ,$$

where

$$C_n(K_m^{c*},1) = 4m\binom{4m-1}{n-1} - m\left[\binom{3m-2}{n-1} + \binom{4m-5}{n-1} + 2\binom{4m-2}{n-1}\right], \qquad ...(2.2)$$

$$C_n(K_m^{c*},2) = m\left[\binom{3m-2}{n-1} - \binom{3m-5}{n-1} - \binom{4m-5}{n-1} - \binom{2m-4}{n-1}\right] + 2m\binom{4m-2}{n-1}, \qquad ...(2.3)$$

$$C_n(K_m^{c*},3) = m\left[2\binom{4m-5}{n-1} + \binom{2m-4}{n-1} - \binom{3m-5}{n-1} - \binom{2m-6}{n-1}\right], \qquad ...(2.4)$$

$$C_n(K_m^{c*},4) = m\left[2\binom{3m-5}{n-1} - \binom{2m-6}{n-1}\right], \qquad ...(2.5)$$

$$C_n(K_m^{c*},5) = 2m\binom{2m-6}{n-1}. \qquad ...(2.6)$$

Proof: Using (1.5) we get $C_n(K_m^{c*},1)$ as given in (2.2).

We partition the vertex set of K_m^{c*} into three subsets W , U , and V, where $W = \{ w_1, w_2, \ ... \ , w_{2m} \}$, $U = \{ u_1, u_2, \ ... \ , u_m \}$, and $V = \{ v_1, v_2, \ ... \ , v_m \}$.

From Fig 2.2., for k = 2, 3, 4, 5 , one can easily notice that:

$C_n(W, K_m^{c*}, k) = 2mC_n(w_i, K_m^{c*}, k)$, for any vertex $w_i \in W$,

$C_n(U, K_m^{c*}, k) = mC_n(u_i, K_m^{c*}, k)$, for any vertex $u_i \in U$,

and

$C_n(V, K_m^{c*}, k) = mC_n(v_i, K_m^{c*}, k)$, for any vertex $v_i \in V$.

Thus , we have three cases for the values of k .

Case(1): k =3. There are m vertices , namely $u_2; v_3, v_4, \ ... \ , v_{m-1}, v_m; u_m$, of distance 3 from w_1; and there are (3m-5) vertices of distance more than 3 from w_1. Therefore , by Lemma 1.1 ,

$$C_n(w_1, K_m^{c*}, 3) = \binom{4m-5}{n-1} - \binom{3m-5}{n-1}. \qquad \ldots(2.7)$$

Also , there are (m+1) vertices , namely $w_3, w_4; u_3, u_4, \ldots, u_{m-2}, u_{m-1}; w_{2m-1}, w_{2m}$, of distance 3 from u_1 ; and there are (2m-6) vertices of distance more than 3 from u_1 . Therefore

$$C_n(u_1, K_m^{c*}, 3) = \binom{3m-5}{n-1} - \binom{2m-6}{n-1}. \qquad \ldots(2.8)$$

Finally , there are (2m-4) vertices , namely $w_3, w_4, \ldots, w_{2m-3}, w_{2m-2}$, of distance 3 from v_1; and there is no vertex of distance more than 3 from v_1. Hence

$$C_n(v_1, K_m^{c*}, 3) = \binom{2m-4}{n-1}. \qquad \ldots(2.9)$$

Thus from (2.7) , (2.8) , and (2.9) we get (2.4).

Case(2): k =4. There are (m+1) vertices , namely $w_3, w_4; u_3, u_4, \ldots, u_{m-2}, u_{m-1}; w_{2m-1}, w_{2m}$, of distance 4 from w_1 ; and there are (2m-6) vertices of distance more than 4 from w_1 . Therefore , by Lemma 1.1 ,

$$C_n(w_1, K_m^{c*}, 4) = \binom{3m-5}{n-1} - \binom{2m-6}{n-1}. \qquad \ldots(2.10)$$

Also , there are (2m-6) vertices , namely $w_5, w_6, \ldots, w_{2m-3}, w_{2m-2}$, of distance 4 from u_1; and no vertex of distance more than 4 from u_1. Therefore

$$C_n(u_1, K_m^{c*}, 4) = \binom{2m-6}{n-1}. \qquad \ldots(2.11)$$

Finally , there is no vertex of distance more than 4 from any vertex of V. Thus , from (2.10) and (2.11) we obtain (2.5).

Case(3): k =5. There are (2m-6) vertices , namely $w_5, w_6, \ldots, w_{2m-3}, w_{2m-2}$, of distance 5 from w_1 ; and no vertex of distance more than 5 from w_1 .
Also , there is no vertex of distance 5 from any vertex of $V \cup U$. Thus we get (2.6).

From (2.2), (2.4) , (2.5) , (2.6) and using (2.1) we get (2.3).
Hence , the proof is completed . ■

Corollary 2.5: The n-Wiener index of K_m^{c*} , $m \geq 3$, $3 \leq n \leq 4m$ is given by

$$W_n(K_m^{c*}) = 4m\binom{4m-1}{n-1} + 3m\left[\binom{4m-5}{n-1} + \binom{3m-5}{n-1} + \binom{2m-6}{n-1}\right]$$

$$+m\left[2\binom{4m-2}{n-1}+\binom{3m-2}{n-1}+\binom{2m-4}{n-1}\right]. \blacksquare$$

Corollary 2.6: The Hosoya polynomial of K_m^{c*}, $m \geq 3$ is given by

$$H(K_m^{c*};x) = 4m + \frac{1}{2}m(m+7)x + m(m+4)x^2 + \frac{1}{2}m(5m-3)x^3$$
$$+ 2m(m-1)x^4 + 2m(m-3)x^5.$$

Proof: When n = 2 , we have $d_2(u',\{v'\}) = d_2(v',\{u'\}) = d(u',v')$.
Thus $H(K_m^{c*};x)$ is obtained from Theorem 2.4 , by putting n=2 and dividing by 2. ■

Corollary 2.7: The Wiener index of K_m^{c*}, $m \geq 3$ is given by
$W(K_m^{c*}) = m(28m-31)$. ■

3. Thorn Cog-star Graphs:

Definition 3.1: A cog-star graph S_m^c is the graph constructed from a star [4] , S_m , $m \geq 4$, of vertex set $\{v_1, v_2, \ldots, v_{m-1}, v_m\}$ with (m-1) additional vertices $u_1, u_2, \ldots, u_{m-2}, u_{m-1}$, and edges $\{u_i v_{i+1}, u_i v_{i+2} : i = 1,2, \ldots, m-1\}$, $(v_{m+1} \equiv v_2)$, as shown in Fig. 3.1.

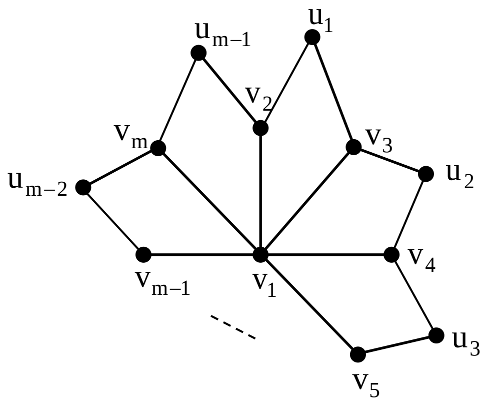

Fig. 3.1. S_m^c

It is clear that $p(S_m^c) = 2m-1$, $q(S_m^c) = 3(m-1)$, diam S_m^c =4 for $m \geq 5$, and $diam_n S_m^c \leq 4$.

From Fig.3.1. , we notice that there are (m-3) vertices of distance 3 from u_i; and there are (m-4) vertices of distance more than 3 from u_i, for $i = 1,2, \ldots, m-1$. Thus by Lemma 1.1 ,

$$C_n(u_i, S_m^c, 3) = \binom{2m-7}{n-1} - \binom{m-4}{n-1}, \quad i = 1,2, \ldots, m-1. \qquad \ldots(3.1)$$

Also , there are (m-3) vertices of distance 3 from v_i; and no vertex of distance more than 3 from v_i, $i = 2,3, \ ... \ ,m$, thus

$$C_n(v_i, S_m^c, 3) = \binom{m-3}{n-1}, \ i = 2,3, \ ... \ ,m . \qquad \ldots(3.2)$$

Thus , from (3.1) , and (3.2) we get

$$C_n(S_m^c, 3) = (m-1)\left[\binom{2m-7}{n-1} + \binom{m-4}{n-2}\right]. \qquad \ldots(3.3)$$

Moreover

$$C_n(S_m^c, 4) = (m-1)\binom{m-4}{n-1}. \qquad \ldots(3.4)$$

Using (1.5) , we get

$$C_n(S_m^c, 1) = (2m-1)\binom{2m-2}{n-1} - (m-1)\left[\binom{2m-4}{n-1} + \binom{2m-5}{n-1}\right] - \binom{m-1}{n-1}. \qquad \ldots(3.5)$$

Thus , from (1.10) , (3.3) , (3.4) , and (3.5) , we obtain

$$C_n(S_m^c, 2) = (m-1)\left[\binom{2m-4}{n-1} + \binom{2m-5}{n-1} - \binom{2m-7}{n-1} - \binom{m-3}{n-1}\right] + \binom{m-1}{n-1}. \qquad \ldots(3.6)$$

Hence , we have the following theorem :

Theorem 3.1: The n-Hosoya polynomial of S_m^c, $m \geq 4$, $3 \leq n \leq 2m-1$ is given by

$$H_n(S_m^c; x) = (2m-1)\binom{2m-2}{n-2} + \sum_{k=1}^{4} C_n(S_m^c, k)x^k \ ,$$

where $C_n(S_m^c, k)$, k = 1,2,3,4 are given in (3.5) , (3.6), (3.3) , and (3.4) , respectively. ■

Corollary 3.2: The n-Wiener index of S_m^c, $m \geq 4$, $3 \leq n \leq 2m-1$ is given by

$$W_n(S_m^c) = (m-1)\left[\binom{2m-7}{n-1} + \binom{2m-5}{n-1} + \binom{2m-4}{n-1} + \binom{m-3}{n-1} + \binom{m-4}{n-1}\right].$$
$$+ (2m-1)\binom{2m-2}{n-1} + \binom{m-1}{n-1}. \ ■$$

Definition 3.3: The thorn cog-star S_m^{c*} is the graph constructed from the cog-star S_m^c , $m \geq 4$, of vertex set $\{v_1, v_2, \ ... \ , v_m, u_1, u_2, \ ... \ , u_{m-1}\}$ (Fig.3.1) with 2(m-1) additional vertices $w_1, w_2, \ ... \ , w_{2m-3}, w_{2m-2}$, and edges $\{u_i w_{2i-1}, u_i w_{2i} : i = 1, 2, \ ... \ , m-1\}$, as shown in Fig. 3.2.

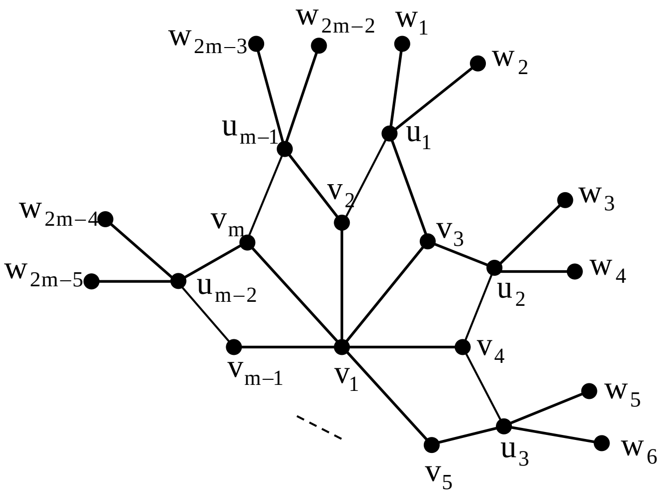

Fig. 3.2. S_m^{c*}

It is clear that $p(S_m^{c*}) = 4m-3$, $q(S_m^{c*}) = 5(m-1)$, diam S_m^{c*} =6 for $m \geq 5$, and $diam_n S_m^{c*} \leq 6$.

In the next theorem , we obtain the n-Hosoya polynomial of S_m^{c*} , $m \geq 4$.

Theorem 3.4: The n-Hosoya polynomial of S_m^{c*} , $m \geq 4$, $3 \leq n \leq 4m-3$ is given by

$$H_n(S_m^{c*};x) = (4m-3)\binom{4m-4}{n-2} + \sum_{k=1}^{6} C_n(S_m^{c*},k)x^k \quad , \qquad ...(3.7)$$

where

$$C_n(S_m^{c*},1) = (4m-3)\binom{4m-4}{n-1} - (m-1)\left[2\binom{4m-5}{n-1} + \binom{4m-7}{n-1} + \binom{4m-8}{n-1}\right] - \binom{3m-3}{n-1}, \qquad ...(3.8)$$

$$C_n(S_m^{c*},2) = 2(m-1)\binom{4m-5}{n-1} - (m-1)\left[\binom{3m-9}{n-1} + \binom{4m-11}{n-1} - \binom{4m-8}{n-2}\right] + \binom{3m-3}{n-1} - \binom{2m-2}{n-1}, \qquad ...(3.9)$$

$$C_n(S_m^{c*},3) = (m-1)\left[\binom{3m-9}{n-1} - \binom{4m-11}{n-1} - \binom{3m-12}{n-1} - \binom{2m-6}{n-1}\right] + \binom{2m-2}{n-1} + 2(m-1)\binom{4m-8}{n-1}, \qquad ...(3.10)$$

$$C_n(S_m^{c*},4)=(m-1)\left[\binom{2m-6}{n-1}-\binom{2m-8}{n-1}-\binom{3m-12}{n-1}\right]+2(m-1)\binom{4m-11}{n-1}, \qquad ...(3.11)$$

$$C_n(S_m^{c*},5)=(m-1)\left[2\binom{3m-12}{n-1}-\binom{2m-8}{n-1}\right], \qquad ...(3.12)$$

and

$$C_n(S_m^{c*},6)=2(m-1)\binom{2m-8}{n-1}. \qquad ...(3.13)$$

Proof: $C_n(S_m^{c*},1)$ follows from (1.5). Now we find $C_n(S_m^{c*},k)$, for k = 3,4,5,6. From Fig.3.2 , we notice that

$$C_n(v_1,S_m^{c*},k)=\begin{cases}\binom{2m-2}{n-1} & , \; if \;\; k=3 \, , \\ 0 & , \; otherwise.\end{cases} \qquad ...(3.14)$$

Let $V=\{v_2,v_3, \ ... \ ,v_m\}$, $U=\{u_1,u_2, \ ... \ ,u_{m-1}\}$,and $W=\{w_1,w_2, \ ... \ ,w_{2m-2}\}$. There are (m-3) vertices of distance 3 from any vertex $v\in V$; for example ; each of $u_2,u_3, \ ... \ ,u_{m-2}$, is of distance 3 from v_2 ;and there are (2m-6) vertices of distance more than 3 from v_2. Hence , by Lemma 1.1 ,

$$C_n(v_i,S_m^{c*},3)=\binom{3m-9}{n-1}-\binom{2m-6}{n-1} \, , \; 2\le i\le m \, . \qquad …(3.15)$$

Moreover , there are (m+1) vertices of distance 3 from any vertex $u\in U$; for example , each of $w_3,w_4;v_4,v_5, \ ... \ ,v_m;w_{2m-3},w_{2m-2}$ is of distance 3 from u_1; and there are (3m-12) vertices of distance more than 3 from u_1. Hence , by Lemma 1.1 ,

$$C_n(u_i,S_m^{c*},3)=\binom{4m-11}{n-1}-\binom{3m-12}{n-1} \, , \; 1\le i\le m-1 \, . \qquad …(3.16)$$

Finally , there are three vertices of distance 3 from any vertex $w\in W$;
and there are (4m-11) vertices of distance more than 3 from w. Hence , by Lemma 1.1 ,

$$C_n(w_i,S_m^{c*},3)=\binom{4m-8}{n-1}-\binom{4m-11}{n-1} \, , \; 1\le i\le 2m-2 \, . \qquad …(3.17)$$

From (3.14) - (3.17) , we get (3.10).
When k =4, there are (2m-6) vertices , of distance 4 from any vertex $v\in V$; and there is no vertex of graph S_m^{c*} of distance more than 4 from v. Hence

$$C_n(v_i,S_m^{c*},4)=\binom{2m-6}{n-1} \, , \; 2\le i\le m \, . \qquad …(3.18)$$

And there are (m-4) vertices of distance 4 from any vertex $u\in U$; for example , each of $u_3,u_4,u_5 \, , \ ... \ ,u_{m-3},u_{m-2}$ is of distance 4 from u_1; and there are (2m-8) vertices of distance more than 4 from u_1. Hence , by Lemma 1.1 ,

$$C_n(u_i,S_m^{c*},4)=\binom{3m-12}{n-1}-\binom{2m-8}{n-1}, 1\le i\le m-1 . \qquad \ldots(3.19)$$

Moreover , there are (m+1) vertices of distance 4 from any vertex $w\in W$; for example , each of $w_3, w_4 ; v_4, v_5 , \ldots , v_m ; w_{2m-3}, w_{2m-2}$ is of distance 4 from w_1 ; and there are (3m-12) vertices of distance more than 4 from w_1. Hence , by Lemma 1.1 ,

$$C_n(w_i,S_m^{c*},4)=\binom{4m-11}{n-1}-\binom{3m-12}{n-1} , 1\le i\le 2m-2 . \qquad \ldots(3.20)$$

Thus , from (3.18) , (3.19) , and (3.20) , we get (3.11).
Now , when k =5, then $C_n(v_i,S_m^{c*},5)=0$, $2\le i\le m$
Moreover , there are (2m-8) vertices of distance 5 from any vertex $u\in U$; and there is no vertex of graph S_m^{c*} of distance more than 5 from u . Hence

$$C_n(u_i,S_m^{c*},5)=\binom{2m-8}{n-1}, 1\le i\le m-1 . \qquad \ldots(3.21)$$

And there are (m-4) vertices of distance 5 from any vertex $w\in W$; for example , each of the vertices $u_3, u_4, u_5 , \ldots , u_{m-3}, u_{m-2}$ is of distance 5 from w_1; and there are (2m-8) vertices of distance more than 5 from w_1. Hence , by Lemma 1.1 ,

$$C_n(w_i,S_m^{c*},5)=\binom{3m-12}{n-1}-\binom{2m-8}{n-1} , 1\le i\le 2m-2 . \qquad \ldots(3.22)$$

Thus , from (3.21) , and (3.22), we get (3.12).
Finally , from Fig.3.2 , we notice that

$$C_n(v_i,S_m^{c*},6)=0 , 2\le i\le m ,$$

$$C_n(u_i,S_m^{c*},6)=0 , 1\le i\le m-1 ,$$

and

$$C_n(w_i,S_m^{c*},6)=\binom{2m-8}{n-1} , 1\le i\le 2m-2 .$$

Using (2.1) we obtain , from (3.8) and (3.10) – (3.13) , the value of $C_n(S_m^{c*},2)$ as given in (3.9).
This completes the proof . ■

Corollary 3.5: The n-Wiener index of S_m^{c*}, $m\ge 4 , 3\le n\le 4m-3$ is given by

$$W_n(S_m^{c*})=(4m-3)\binom{4m-4}{n-1}+2(m-1)\binom{4m-5}{n-1}+\binom{3m-3}{n-1}+\binom{2m-2}{n-1}$$
$$+3(m-1)\left[\binom{2m-8}{n-1}+\binom{4m-8}{n-1}+\binom{4m-11}{n-1}+\binom{3m-12}{n-1}\right]$$

$$+(m-1)\left[\binom{4m-7}{n-1}+\binom{3m-9}{n-1}+\binom{2m-6}{n-1}\right].$$ ■

Corollary 3.6: If S_m^{c*} is the thorn cog-star of order 4m-3 , then the Hosoya polynomial of S_m^{c*}, $m \ge 4$ is given by

$$H(S_m^{c*};x) = 4m-3+5(m-1)x+\frac{1}{2}(m-1)(m+12)x^2+(m-1)(m+3)x^3$$
$$+\frac{1}{2}(m-1)(5m-8)x^4+2(m-1)(m-4)x^5+2(m-1)(m-4)x^6 .$$

And, the Wiener index of S_m^{c*}, $m \ge 4$,is given by

$W(S_m^{c*}) = 36m^2 - 114m + 78$. ■

4. Thorn Cog-wheel Graphs:

Definition 4.1: A cog-wheel graph W_m^c is the graph constructed from a wheel [4] , W_m , $m \ge 4$, of order m , with vertex set $\{v_1, v_2, \ldots, v_{m-1}, v_m\}$, and with (m-1) additional vertices $u_1, u_2, \ldots, u_{m-2}, u_{m-1}$, and edges $\{u_i v_{i+1}, u_i v_{i+2} : i = 1,2, \ldots, m-1\}$, $(v_{m+1} \equiv v_2)$, as shown in Fig. 4.1.

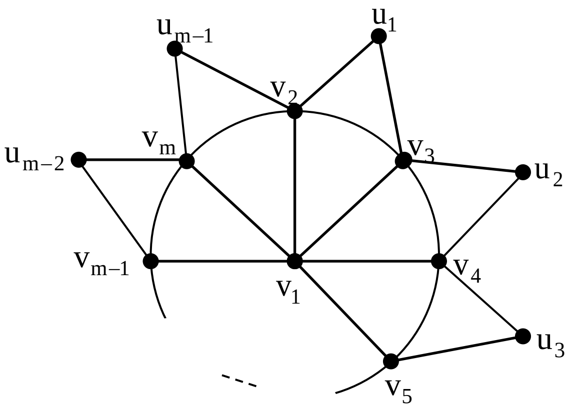

Fig. 4.1. W_m^c

It is clear that $p(W_m^c) = 2m-1$, $q(W_m^c) = 4(m-1)$, diamW_m^c =4 for $m \ge 7$, and $diam_n W_m^c \le 4$.

From Fig.4.1. , we notice that :

$$C_n(u_i, W_m^c, 3) = \sum_{j=1}^{n-1}\binom{m-3}{j}\binom{m-6}{n-j-1}$$
$$= \binom{2m-9}{n-1} - \binom{m-6}{n-1},\ i = 1,2, \ldots, m-1,\ m \ge 6, \qquad \ldots(4.1)$$

$$C_n(v_i, W_m^c, 3) = \binom{m-5}{n-1}, \; i = 2, 3, \; \ldots \; , m \, , \; m \geq 5 \, , \qquad \ldots(4.2)$$

Thus , from (4.1) , and (4.2) we get

$$C_n(W_m^c, 3) = (m-1)\left[\binom{2m-9}{n-1} + \binom{m-6}{n-2}\right], \; m \geq 6 \; . \qquad ...(4.3)$$

Moreover

$$C_n(W_m^c, 4) = (m-1)\binom{m-6}{n-1}, \; m \geq 6 \; . \qquad ...(4.4)$$

Using (1.5) , we obtain

$$C_n(W_m^c, 1) = (2m-1)\binom{2m-2}{n-1} - (m-1)\left[\binom{2m-4}{n-1} + \binom{2m-7}{n-1}\right] - \binom{m-1}{n-1}. \qquad ...(4.5)$$

Finally , from (4.3) , (4.4) , (4.5) , and (2.1) , we get

$$C_n(W_m^c, 2) = (m-1)\left[\binom{2m-4}{n-1} + \binom{2m-7}{n-1} - \binom{2m-9}{n-1} - \binom{m-5}{n-1}\right] + \binom{m-1}{n-1}. \qquad ...(4.6)$$

Hence , we have the following results :

Theorem 4.1: The n-Hosoya polynomial of W_m^c, $m \geq 6$, $3 \leq n \leq 2m-1$ is given by

$$H_n(W_m^c; x) = (2m-1)\binom{2m-2}{n-2} + \sum_{k=1}^{4} C_n(W_m^c, k)x^k \; ,$$

where $C_n(W_m^c, k)$, k = 1,2,3,4 are given in (4.5) , (4.6), (4.3) , and (4.4) , respectively. ■

Corollary 4.2: The n-Wiener index of W_m^c, $m \geq 6$, $3 \leq n \leq 2m-1$ is given by

$$W_n(W_m^c) = (m-1)\left[\binom{2m-4}{n-1} + \binom{2m-7}{n-1} + \binom{2m-9}{n-1} + \binom{m-5}{n-1} + \binom{m-6}{n-1}\right]$$

$$+ (2m-1)\binom{2m-2}{n-1} + \binom{m-1}{n-1}. \quad ■$$

Remarks (4.1):

(i) For $3 \leq n \leq 7$, we have

- $$H_n(W_4^c; x) = 7\binom{6}{n-2} + \left[7\binom{6}{n-1} - 3\binom{4}{n-1} - \binom{3}{n-1}\right]x + \left[3\binom{4}{n-1} + \binom{3}{n-1}\right]x^2 .$$

(ii) For $3 \leq n \leq 9$, we have

- $$H_n(W_5^c; x) = 9\binom{8}{n-2} + \left[9\binom{8}{n-1} - 4\binom{6}{n-1} - 4\binom{3}{n-1} - \binom{4}{n-1}\right]x$$

$$+\left[4\binom{6}{n-1}+4\binom{3}{n-1}+\binom{4}{n-1}\right]x^2 . \quad \#$$

Definition 4.3: The thorn cog-wheel graph W_m^{c*} is the graph constructed from the cog- wheel graph W_m^c , $m \ge 4$, of vertex set $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_{m-1}\}$ (Fig.4.1) with 2(m-1) additional vertices $w_1, w_2, \ldots, w_{2m-3}, w_{2m-2}$, and edges $\{u_i w_{2i-1}, u_i w_{2i} : i = 1,2, \ldots, m-1\}$, as shown in Fig. 4.2.

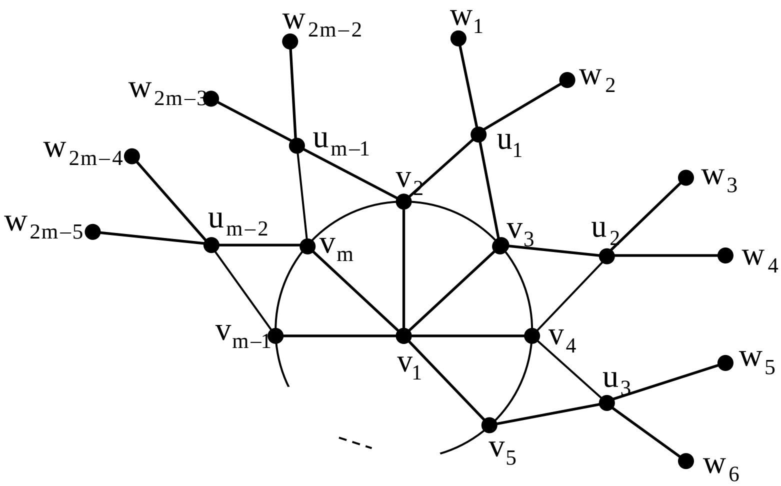

Fig. 4.2. W_m^{c*}

It is clear that $p(W_m^{c*}) = 4m-3$, $q(W_m^{c*}) = 6(m-1)$, diamW_m^{c*} =6 for $m \ge 7$, and $diam_n W_m^{c*} \le 6$.

Using the procedure followed for obtaining $H_n(S_m^{c*};x)$, we establish the next theorem , which determines $H_n(W_m^{c*};x)$.

Theorem 4.4: If W_m^{c*} is the thorn cog-wheel of order 4m-3 , $m \ge 6$, $3 \le n \le 4m-3$,then

$$H_n(W_m^{c*};x) = (4m-3)\binom{4m-4}{n-2} + \sum_{k=1}^{6} C_n(W_m^{c*},k)x^k \ , \qquad \ldots(3.7)$$

where

$$C_n(W_m^{c*},1) = (4m-3)\binom{4m-4}{n-1} - (m-1)\left[2\binom{4m-5}{n-1}+\binom{4m-8}{n-1}+\binom{4m-9}{n-1}\right] - \binom{3m-3}{n-1}, \qquad \ldots(4.7)$$

$$C_n(W_m^{c*},2)=(m-1)\left[2\binom{4m-5}{n-1}+\binom{4m-9}{n-1}-\binom{4m-8}{n-2}-\binom{4m-13}{n-1}-\binom{3m-11}{n-1}\right]$$
$$+\binom{3m-3}{n-1}-\binom{2m-2}{n-1}, \qquad ...(4.8)$$
$$C_n(W_m^{c*},3)=(m-1)\left[2\binom{4m-8}{n-1}+\binom{3m-11}{n-1}-\binom{4m-13}{n-1}-\binom{3m-14}{n-1}-\binom{2m-10}{n-1}\right]$$
$$+\binom{2m-2}{n-1}, \qquad ...(4.9)$$
$$C_n(W_m^{c*},4)=(m-1)\left[2\binom{4m-13}{n-1}+\binom{2m-10}{n-1}-\binom{3m-14}{n-1}-\binom{2m-12}{n-1}\right], \qquad ...(4.10)$$
$$C_n(W_m^{c*},5)=(m-1)\left[2\binom{3m-14}{n-1}-\binom{2m-12}{n-1}\right], \qquad ...(4.11)$$
and
$$C_n(W_m^{c*},6)=2(m-1)\binom{2m-12}{n-1}. \qquad ...(4.12)$$

■

Remarks (4.2):

(i) For $3\le n\le 13$, we have

- $$H_n(W_4^{c*};x)=13\binom{12}{n-2}+\left[13\binom{12}{n-1}-6\binom{11}{n-1}-3\binom{8}{n-1}-3\binom{7}{n-1}-\binom{9}{n-1}\right]x$$
$$+\left[6\binom{11}{n-1}+\binom{9}{n-1}-\binom{6}{n-1}+3\left[\binom{7}{n-1}-\binom{8}{n-1}-\binom{4}{n-1}-\binom{2}{n-1}\right]\right]x^2$$
$$+\left[6\binom{8}{n-1}+3\binom{2}{n-1}+\binom{6}{n-1}-3\binom{4}{n-1}\right]x^3+6\binom{4}{n-1}x^4 .$$

(ii) For $3\le n\le 17$, we have

- $$H_n(W_5^{c*};x)=17\binom{16}{n-2}+\left[17\binom{16}{n-1}-8\binom{15}{n-1}-5\binom{12}{n-1}-4\binom{11}{n-1}\right]x$$
$$+\left[8\binom{15}{n-1}+4\binom{11}{n-1}-3\binom{12}{n-1}-4\binom{7}{n-1}-4\binom{4}{n-1}-\binom{8}{n-1}\right]x^2$$
$$+\left[4\left[2\binom{12}{n-1}+\binom{4}{n-1}-\binom{7}{n-1}-\binom{2}{n-1}\right]+\binom{8}{n-1}\right]x^3$$
$$+\left[8\binom{7}{n-1}-4\binom{2}{n-1}\right]x^4+8\binom{2}{n-1}x^5 . \quad \#$$

Corollary 4.5: The n-Wiener index of W_m^{c*}, of order 4m-3 $m\ge 6$, $3\le n\le 4m-3$ is given by

$$W_n(W_m^{c*}) = (4m-3)\binom{4m-4}{n-1} + \binom{3m-3}{n-1} + \binom{2m-2}{n-1} + (m-1)\left[\binom{4m-9}{n-1} + \binom{3m-11}{n-1} + \binom{2m-10}{n-1} + 2\binom{4m-5}{n-1} + 3\binom{4m-8}{n-1} + 3\binom{4m-13}{n-1} + 3\binom{3m-14}{n-1} + 3\binom{2m-12}{n-1}\right]. \blacksquare$$

Corollary 4.6: If W_m^{c*} is the thorn cog-wheel of order 4m-3 , $m \geq 6$, then the Hosoya polynomial of W_m^{c*} is given by

$$H(W_m^{c*};x) = 4m-3+6(m-1)x+\frac{1}{2}(m-1)(m+14)x^2+(m-1)(m+6)x^3 + \frac{5}{2}(m-1)(m-2)x^4+2(m-1)(m-4)x^5+2(m-1)(m-6)x^6 .$$

Moreover

$$H(W_4^{c*};x) = 13+18x+24x^2+24x^3+12x^4 ,$$

and

$$H(W_5^{c*};x) = 17+24x+38x^2+42x^3+24x^4+8x^5 . \qquad \blacksquare$$

Corollary 4.7:

The Wiener index of W_m^{c*} of order 4m-3 , $m \geq 6$, is given by

$$W(W_m^{c*}) = 2(m-1)(18m-47) ,$$

Moreover

$W(W_4^{c*}) = 186$, and $W(W_5^{c*}) = 362$. $\blacksquare$

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