On Generalized PF – Rings

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ABSTRACT

The aim of this paper is to extend several known results on GPF –rings. π-regular rings, PF-rings and GP-ideals are also considered. Among other results we prove that: If R is a uniform ring, then R is a GPF-ring if and only if every element of R is either non-zero divisor or nilpotent.

Key word: Generalized PF-ring, uniform ring, pure ideal.

حول الحلقات من النمط –GP

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الملخص

الهدف من هذا البحث هو توسيع بعض النتائج المعروفة في الحلقات من النمط GPF. فضـلاً عن ذلك درسنا العلاقات بين الحلقات من النمط GPF والحلقات المنتظمة من النمط π، والحلقات من النمط PF وكذلك المثاليات من النمط GP. ومن النتائج الأخرى التي حصلنا عليها هي " لتكن R حلقة موحدة ، فان R حلقة من النمط GPF إذا وإذا فقط فإنَ كل عنصر في R إما إلا يكون من قواسم الصفر وإما أن يكون عنصراً معدوم القوى " .

1. INTRODUCTION

Throughout this paper R denotes a commutative ring with identity and module means unitary R-module. Recall that (1) an ideal I of the ring R is said to be pure if for every $a \in I$, there exists $b \in I$ such that a = ab [2];(2) R is called π-regular if for every $a \in R$, there exists a positive integer n such that $a^n \in a^n R a^n$; (3) An R-model M is called general right principally injective (briefly GP-injective)if,for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any R-homomorphism of $a^n R$ into M extends to one of R into M [4]; (4) For any $a \in R$, ann(a) will denote the annihilator of a; (5) For any ideal I of R, R/I is a flat R-module if and only if for every $a \in I$, there exists $b \in I$ such that a=ab [6] ;(6) R is called quniform ring if every non-zero ideal of R is an essential ideal;(7) Z(R) and J(R) will stand respectively for the singular ideal of R, and the Jacobson radical of R.

2.GPF-RIGS (BASIC PROPERTIES)

Recall that a ring R is said to be a PF-ring if for every $a \in R$, the principal ideal aR is flat R-module .

A ring R is called generalized PF-ring (GPF-ring) if for any $a \in R$, there exists a positive integer n such that $a^n R$ is a flat R-module. Clearly every PF-ring is a GPF-ring[2] .

Recall the following result of Naoum [5].

Lemma 2.1: A ring R is a PF-ring if and only if for every $a \in R$ ann(a) is pure ideal of R.

In [2] AL – Ezeh proved that:

Lemma 2.2: A ring R is a GPF-ring if and only if for every a $\in R$, there exists a positive integer n such that ann(a^n) is a pure ideal.

In [1]Aritico and Marconi proved that:

Theorem2.3.A ring R is a PF-ring if and only if

ann(a) +ann(b)=R, whenever ab = 0

The next results generalize Theorem 2.3

Theorem 2.4: A ring R is a GPF-ring if and only if there exists a positive integer n, whenever $a^n b = 0$, ann(a^n)+ann(b)=R.

Proof. Let R be a GPF - ring, and Let $0 \neq a \in R$, then there exists a positive integer n such that ann(a^n) is pure . For any b $\in R$, if $a^n b=0$, we claim that ann(a^n) + ann(b) =R. Suppose that ann(a^n) + ann (b) $\neq$ R, then there exists a maximal ideal M containing ann(a^n) +ann(b), since $a^n b=0$, then b$\in$ ann(a^n) by purity of ann(a^n), there exists c$\in$ ann(a^n) such that b=bc. This implies that 1-c$\in$ ann(b)$\subseteq$ M, but c$\in$ ann (a^n) $\subseteq$ M, whence $1 \in M$ which contradicts M$\neq$ R. Consequently ann(a^n)+ann(b)=R. Conversely, assume that ann(a^n)+ann(b)=R where $a^n b=0$.

In particular $c_1+c_2=1$ for some $c_1 \in$ ann(a^n) and $c_2 \in$ ann (b) . Multiplying by b, we get $bc_1=b$. Hence ann(a^n) is pure ideal. Therefore R is GPF-ring.

Following [3], an ideal I of the ring R is said to be generalized pure ideal (GP-ideal) if for every a$\in$I there exists b $\in$I and a positive integer n such that $a^n=a^n b$.

The following theorem characterizes GPF-ring in terms of GP-ideal.

Theorem 2.5: Let R be a GPF-ring then for every a $\in$R, ann(a) is a GP-ideal.

Proof. Let $0 \neq a \in R$, and let b$\in$ann(a).Since R is a GPF-ring, then there exists a positive integer n such that ann(b^n) is pure(Lemma2.2). Applying Theorm2.5, we get ann(b^n)+ann(a) =R. In particular there exists $c_1 \in$ ann(b^n) and

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$c_2 \in \text{ann}(a)$ such that $c_1+c_2=1$ multiplying by b^n we have $b^n c_2=b^n$. Therefore ann(a) is a GP-ideal of R.

Next we consider the singular ideal of GPF-ring.

Proposition 2.6 Let R be a GPF-ring. Then Z(R) is a nilideal.

Proof. Let a be a non –zero element of Z(R) then $\text{ann}(a^n)$ is pure, for some positive integer n. We claim that $\text{ann}(a^n) \cap a^n R=(0)$.Let $x \in \text{ann}(a^n) \cap a^n R$. Then $x\, a^n = 0$ and $x = a^n r$ for some $r \in R$. On the other hand, since $\text{ann}(a^n)$ is pure then there exists $y \in \text{ann}(a^n)$ such that x=xy. Whence $x=a^n ry=ra^n y=0$, yielding ann $(a^n) \cap a^n R=0$. Since $a^n \in Z(R)$,then $\text{ann}(a^n)$ is an essential ideal of R. Whence it follows that $a^n R= 0$, thus $a^n = 0$. Z(R), therefore, is a nilideal.

3. THE CONNECTION BETWEEN GPF – RINGS AND OTHER RINGS

In this section we give further properties GPF-ring and the link between GPF-rings and other rings.

We shall begin this section with the following result, which gives the connection between GPF-ring and PF-rings.

Theorem 3.1: Let R be a GPF-ring, and let, $J^2=0$. Then R/J is PF-ring.

Proof. Let $a+J \in R/J$, $a \in R$. Since R is a GPF-ring, then there exists a positive integer n such that $\text{ann}(a^n)$ is pure. In order to prove R/J is a PF-ring we need to prove that ann(a+J) is pure. Let $x+J \in \text{ann}(a+J)$, then $ax \in J$. Since $J^2=0$, we have $(ax)^2=a^2 x^2=0$. Whence $x^2 \in \text{ann}(a^2)$. But $\text{ann}(a^2) \subseteq \text{ann}(a^n)$ for $n \geq 1$. On the other hand, since $\text{ann}(a^n)$ is pure, there exists $y \in \text{ann}(a^n)$ such that $x^2 y=x^2$, and this implies that $x^2(y-1)=0$. But $[x(y-1)]^2= x^2(y-1)^2=0$. Thus $x(y-1) \in N \subseteq J$. Whence it follows that $(x+J)(y+J)= x+J$ Since R/J is reduced,then $\text{ann}(a^n+J)=\text{ann}(a+J)$, whence it follows that $y+J \in \text{ann}(a^n+J)=\text{ann}(a+J)$ hence ann(a+J) is pure. R/J is, therefore, a PF-ring.

We now consider a necessary and sufficient condition for GPF-ring to be π-regular.

Before stating this result, the following lemma is needed.

Lemma 3.2: For any $a \in R$, if ann(a^n) pure,
then ann (a^n) =ann (a^{n+1}).

Proof. Obviously ann (a^n) $\subseteq$ ann (a^{n+1}).
Let $x \in$ ann(a^{n+1}), then $xa^{n+1}=0$, which implies that $xa \in$ ann (a^n). Since ann(a^n) is pure, there exists $y \in$ ann(a^n) such that $xa=yxa$, and this implies that $xa^n=yxa^n=0$. Hence $x \in$ ann(a^n). Therefore ann(a^n)=ann(a^{n+1}).

Theorem 3.3: A ring R is π -regular if and only if R is a GPF-ring with every maximal ideal is GP-ideal.

Proof. Let R be π -regular ring. Then for any $a \in R$, there exists a position integer n and $b \in R$ such that $a^n=a^{2n}b$ so $1-a^nb \in$ ann(a^n). Now for any $x \in$ ann(a^n), $x(1-a^n b)=x$, thus ann(a^n) is pure. Therefore R is a GPF-ring.
Conversely, assume that R is a GPF-ring , and let $0 \neq a \in R$. then there exists a positive integer n such that ann(a^n) is pure. We claim that $a^n R+$ann(a^n)=R. In fact, if not, there exists a maximal ideal M containing $a^n R+$ann(a^n). On the other hand, since M is GP-ideal , and $a^n \in M$, there exists $b \in M$, and a positive integer m such that $(a^n)=(a^n)^m b$. Whence it follows that $a^{nm}(1-b)=0$, and then we have $1-b \in$ ann(a^{nm}). Applying Lemma 3.2, we get ann(a^n)=ann(a^{nm}). Whence $1-b \in$ ann(a^n), yielding $1 \in M$ which contradicts M≠R. Hence $a^n R +$ ann(a^n)=R, and R is, therefore, π -regular.
Our next result characterizes a uniform GPF-ring in terms of nilpotent and non-zero divisor elements.

Theorem 3.4: Let R be a uniform ring. Then R is a GPF-ring if and only if every element of R is either non –zero divisor or nilpotent.

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Proof. Let R be a uniform GPF-ring, and let $0 \neq a \in R$, then there exists a positive integer n such that $ann(a^n)$ is pure. Now, for any $b \in ann(a^n)$, there exists $c \in ann(a^n)$ such that $b = bc$. First we claim that $bR \cap ann(c) = 0$. In fact, if not, let $0 \neq x \in bR \cap ann(c)$ then $x = br$ for some $r \in R$ and $cx = 0$. But $cx = cbr = br = x = 0$, a contradiction. On the other hand, since R is a uniform then either $ann(c) = 0$ or $bR = 0$. If $ann(c) = 0$, then $a^n c = 0$ gives $a^n = 0$. Thus a is a nilpotent element. Now, if $bR = 0$ then $b = 0$, and hence $ann(a^n) = 0$. Therefor, a is a non – zero divisor.

Conversely, for any $a \in R$, if a is a non-zero divisor, then $ann(a^n) = 0$, which is pure. On the other hand, if a is nipoltent then $a^m = 0$, for some positive integer m. So $ann(a^m) = R$ which is also pure. Therefore, R is a GPF-ring

Finally we introduce the following result.

Proposition 3.5: Let R be a ring with every simple singular R-module is GP-Injective. Then R is a GPF-ring.

Proof. Let $0 \neq a \in R$, and let $a^n b = 0$ for some positive integer n and $b \in R$. Suppose that $ann(a^n) + ann(b) \neq R$, then there exists a maximal ideal M containing $ann(a^n) + ann(b)$. First suppose that M is not essential ideal of R, then M must be a direct summand, and hence there exists an idempotent element $0 \neq e$ in R such that $M = ann(e)$. But $a^n b = 0$ gives $a^n \in ann(b) \subseteq ann(e)$. Whence it follows that $a^n e = 0$, yielding $e \in ann(a^n) \subseteq ann(e)$ which implies that $e = e^2 = 0$. Therefore M must be essential. Define $f : a^n R \to R/M$ by $f(a^n r) = r + M$, for all $r \in R$. Now since R/M is GP-Injective, then there exists $c \in R$ such that $f(a^n r) = c \, f(a^n r)$, which implies that $1 + M = f(a^n) = ca^n + M$. Whence $1 - ca^n \in M$ and hence $1 \in M$ which contradicts $M \neq R$. Therefore $ann(a^n) + ann(b) = R$, where $a^n b = 0$. Therefore R is a GPF-ring.

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