On sπ-Weakly Regular Rings

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ABSTRACT

A ring R is said to be right(left) sπ-weakly regular if for each $a \in R$ and a positive integer n, $a^n \in a^n R a^{2n} R$ ($a^n \in R a^{2n} R a^n$). In this paper, we continue to study sπ-weakly regular rings due to R. D. Mahmood and A. M. Abdul-Jabbar [8]. We first consider properties and basic extensions of sπ-weakly regular rings, and we give the connection of sπ-weakly regular, semi π-regular and π-biregular rings.

Key words: weakly regular rings , reduced rings , π-biregular rings.

حول الحلقات النتظمة بضعف من النمط sπ

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الملخص

يقال للحلقة R أنها يمنى (يسرى) منتظمة بضعف من النمط -sπ إذا كان لكل $a \in R$ و عدد صحيح موجب n و $a^n \in a^n Ra^{2n} R$. هذا البحث هو استمرار لدراسة الحلقات المنتظمة بضعف من النمط -sπ في المصدر [8] R. D. Mahmood و A. M. Abdul-Jabbar كما تم توسيع الخواص الأساسية للحلقات المنتظمة بضعف من النمط -sπ، وتم أعطاء العلاقة بين الحلقات المنتظمة بضعف من النمط -sπ و الحلقات شبه المنتظمة النمط -π و الحلقات المنتظمة من النمط bi -π .

الكلمات المفتاحية: حلقة منتظمة بضعف , حلقة مختزلة , الحلقات المنتظمة من النمط bi -π .

1. Introduction

Throughout this paper rings are associative with identity. For a subset X of a ring R, the right annihilator of X in R is $r(X) = \{ r \in R: xr = 0$, for all $x \in X\}$. J(R), Y(R) and N will stand respectively for the Jacobson radical, right singular ideal of R and the set of all nilpotent elements. A ring

R is called right (left) s-weakly regular if for each a $\in$ R, a $\in$ aRa^2R (a $\in$ Ra^2Ra). This concept was introduced by V. Gupta [4] and W. B. Vasantha Kandasamy [10]. As a generalization of this concept the authors in [8] defined sπ-weakly regular ring that is a ring such that for each a $\in$ R and a positive integer n, $a^n \in a^n R a^{2n} R$ ($a^n \in R a^{2n} R a^n$). In the present work we develop further properties of sπ-weakly regular rings, and we give the connection of sπ-weakly regular rings with other rings. Recall that:

(1) R is called reduced if it has no nonzero nilpotent elements.
(2) According to Cohn [3], a ring R is called reversible if ab = 0 implies ba = 0 for a, b $\in$ R. Obviously every reduced ring is reversible. It is easy to see that R is reversible if and only if right (left) annihilator in R is a two-sided ideal.

2. sπWR-Rings

In this section we continue to study sπWR-rings and we establish relation between such rings, semi π-regular and π-biregular rings.

Definition 2.1[8]:

R is called right(left) sπWR-ring if for each a $\in$ R, there exists a positive integer n = n(a), depending on a such that $a^n \in a^n R a^{2n} R$ ($a^n \in R a^{2n} R a^n$). R is called sπWR-ring if it is both right and left sπWR-ring.

Recall that, a right ideal P of a ring R is said to be completely prime if for all a, b $\in$ R such that ab $\in$ P, then either a $\in$ P or b $\in$ P.

Proposition 2.2:

If R is a right sπWR-ring and $r(a^n) \subset r(a)$, for every a $\in$ R and a positive integer n. Then every completely prime ideal of R is maximal.

Proof:

Let P be a completely prime ideal of R and J any ideal of R such that P $\subset$ J. Then, there exists x $\in$ J and x $\notin$ P. Since R is right sπWR-ring, there exist b, c $\in$ R and a positive integer n such that $x^n = x^n b x^{2n} c$ and $x(1-b x^{2n}c) = 0$ (since $r(a^n) \subset r(a)$). So, $x(1-b x^{2n}c) = 0 \in P$. Since x $\notin$ P and P is completely prime ideal of R, so $(1- b x^{2n} c) \in P \subset J$, and hence $(1- b x^{2n} c) \in J$. Hence $1- b x^{2n} c + b x^{2n} c = 1 \in J$ and therefore, J = R. Consequently P is a maximal ideal of R. ♦

Theorem 2.3:

If R is a right sπWR-ring and I is an ideal of R, then R/I is sπWR-ring.

Proof:

Let R be a right sπWR-ring. Then, there exist b,c $\in$ R such that $a^n = a^n b a^{2n} c$, for some positive integer n. Now,

$(a + I)^n (b + I) (a + I)^{2n} (c + I) = (a^n + I) (b + I) (a^{2n} + I) (c + I)$

$= a^n b a^{2n} c + I$

$= a^n + I.$

Therefore, R/I is a right sπWR-ring. ♦

Now, the following result is given in [7] and [1].

Lemma 2.4:

Let R be a reduced ring. Then, for every a $\in$ R and every positive integer n,

(1) $r(a^n) = \ell(a^n)$.

(2) $a^n R \cap r(a^n) = 0$.

(3) $r(a) = r(a^n)$.

Theorem 2.5:

Let R be a reduced ring. Then, R is a right sπWR-ring if and only if $R / r(a^m)$ is sπWR-ring, for every a $\in$ R and a positive integer m.

Proof:

Assume that $R / r(a^m)$ is a right sπWR-ring, for every a $\in$ R and a positive integer m. Now,

$a^n + r(a^m) = (a^n + r(a^m)) (b + r(a^m)) (a^{2n} + r(a^m)) (c + r(a^m))$.

$= a^n b a^{2n} c + r(a^m)$. Then, $a^n - a^n b a^{2n} c \in r(a^m)$, this implies that $a^m (a^n - a^n b a^{2n} c) = 0$ and hence $1 - b a^{2n} c \in r(a^{m+n}) = r(a^n)$ since R is reduced. So, $a^n (1 - b a^{2n} c) = 0$ and $a^n = a^n b a^{2n} c$. Therefore, R is a right sπWR-ring.

Conversely, assume that R is a right sπWR-ring. Then, by Theorem 2.3, $R / r(a^m)$ is an sπWR-ring. ♦

Next we consider the Jacobson radical, the right singular ideal of a and the set of all nilpotent elements N of right sπWR-ring.

Theorem 2.6:

Let R be a right sπWR-ring. Then,

(1) J(R) = N
(2) If R is a reduced, then Y(R) is a nilideal of R.

Proof:

(1) Let $0 \neq x \in J(R)$. Then, there exist $b, c \in R$ and a positive integer n such that $x^n = x^n b x^{2n} c$. Then, $1- b x^{2n} c$ is invertible. Therefore, there exists an invertible element u such that $(1- b x^{2n} c) u = 1$. Multiply from the left by x^n, we obtain $(x^n - x^n b x^{2n} c) u = x^n$. Whence it follows that $x^n = 0$, so $x \in N$ and $J(R) \subseteq N$. Since $N \subseteq J(R)$. Thus, $J(R) = N$.

(2) Let a be a nonzero element in $Y(R)$. Then, r(a) is an essential right ideal of R. Since R is right sπWR-ring, there exist $b, c \in R$ and a positive integer n such that $a^n = a^n b a^{2n} c$. Consider $r(a^n) \cap b a^{2n} R$, let $x \in r(a^n) \cap b a^{2n} R$. Then, $a^n x = 0$ and $x = b a^{2n} t$, for some $t \in R$. So, $a^n b a^{2n} t = 0$, then $a^n = 0$. Thus, $a^{2n} = 0$ and hence $b a^{2n} t = 0$, yields $x = 0$. Therefore, $r(a^n) \cap b a^{2n} R = 0$. Since $r(a^n)$ is a nonzero essential right ideal of R, then $b a^{2n} = 0$ and hence $a^{2n} = 0$. Thus, $a^n = 0$. So, $Y(R)$ is a nilideal of R. ♦

Recall that, a ring R is called right(left) semi π-regular [1] if for every $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^n b$ and $r(a^n) = r(b)$ ($a^n = b a^n$ and $\ell(a^n) = \ell(b)$).

The following theorem gives the conditions of sπWR-ring to be semi π-regular.

Theorem 2.7:

If R is a reversible, sπWR-ring and $a^{2n} R = R a^{2n}$, for every $a \in R$ and a positive integer n, then R is semi π-regular.

Proof:

Let R be sπWR-ring. Then, for every $a \in R$, there exist $b, c \in R$ and a positive integer n such that $a^n = a^n b a^{2n} c$. Since $b a^{2n} \in R a^{2n} = a^{2n} R$. So, $a^n = a^{3n} h c$, for some $h \in R$. This implies that $a^n = a^n a^{2n} t$ (set $t = hc$). If we set $y = a^{2n} t$, then $a^n = a^n y$. It only remains to show that $r(a^n) = r(y)$. Let $x \in r(a^n)$, then $a^n x = 0$ and $x a^n a^n t = 0$, so $x a^{2n} t = 0$ and $x y = 0$. Hence $x \in \ell(y) = r(y)$ and $r(a^n) \subseteq r(y)$. On the other hand, let $z \in r(y)$, then $y z = 0$ and $a^{2n} r z = 0$, this implies that $a^n a^{2n} t z = 0$, so $a^n z = 0$, thus $z \in r(a^n)$ and $r(y) \subseteq r(a^n)$. Therefore, $r(a^n) = r(y)$ and R is a right semi π-regular ring in the same method we can easily show that R is a left semi π-regular. ♦

A ring R is called π-biregular [9] if for any $a \in R$, $R a^n R$ is generated by a central idempotent, for some positive integer n.

We begin by stating following lemma, which will be used in proof of our main result.

Lemma 2.8:

A ring R is π-biregular if and only if for every $a \in R$, $R a^n R \oplus r(a^n) = R$, for some positive integer n.

Proof:

See [9]. ♦

Proposition 2.9:

Let R be a reduced ring. Then, R is an sπWR-ring if and only if R is a π-biregular ring.

Proof:

Assume that R is π-biregular ring. Then, by Lemma 2.8, $R = r(a^n) \oplus R a^n R$. Since R is reduced, then $r(a^n) = r(a^{2n})$ for every positive integer n and hence $R = r(a^{2n}) \oplus R a^n R$. In particular, $1 = b + t_1 a^{2n} t_2$, for some $t_1, t_2 \in R$ and $b \in r(a^n)$. So, $a^n = a^n t_1 a^{2n} t_2$. Therefore, R is sπWR-ring.

Conversely, assume that R is sπWR-ring, then for every $a \in R$, there exist $t_1, t_2 \in R$ such that $a^n = a^n t_1 a^{2n} t_2$, for some positive integer n. This implies that $a^n = a^n t_1 a^n a^n t_2$. So, $a^n = a^n t_1 a^n t$ ($t = a^n t_2$), for some $t \in R$.Thus, $a^n R = a^n R a^n R$. Also, by [8, Theorem 2.7], $R = r(a^n) \oplus R a^{2n} R$. Thus, $R = r(a^n) \oplus R a^n R$. So, by Lemma 2.8, R is π-biregular ring. ♦

3. Commutative sπ-Weakly Regular Rings

In this section, all our rings are commutative. We discuss some properties of sπ-weakly regular rings.

We now introduce the following lemma, which may be used frequently in the sequel.

Lemma 3.1:

A ring R is local if and only if for any elements $r, s \in R$ such that $r + s = 1$ implies that either r or s is a unit.

Proof:

See [6]. ♦

Theorem 3.2:

Let R be a local, sπ-weakly regular ring. Then, every element in R is either a unit or nilpotent.

Proof:

Assume that R is a local ring and $a \in R$. Since R is sπ-weakly regular, there exist $x, y \in R$ such that $a^n = a^n x a^{2n} y$, for some positive integer n. So, $a^n (1- x a^{2n} y) = 0$ and hence $a^n (1- a^{2n} x y) = 0$. If $a^n = 0$, then a is nilpotent. If $1- a^{2n} x y \neq 0$, and $a^n \neq 0$, then $1- a^{2n} x y$ is a zero divisor, that is $1- a^{2n} x y$ is non unit. Since $(1- a^{2n} x y) + a^{2n} x y = 1$ by Lemma 3.1,

a^{2n} x y is a unit. This implies that a is a unit. If 1- a^{2n} x y = 0, then a is a unit. Which completes the proof. ♦

Proposition 3.3:

Let R be sπ-weakly regular ring. Then, each element of R is either a unit or zero divisor.

Proof:

Let a be a non zero divisor in R and R is sπ-weakly regular, there exist elements x, y ∈ R and a positive integer n such that $a^n = a^n x a^{2n} y$. So, $a^n (1- x a^{2n} y) = 0$ and hence $a^n (1- a^{2n} x y) = 0$. Since a is a non zero divisor, then a^n is nonzero divisor. Therefore, 1- a^{2n} x y = 0, which implies that a^{2n} (x y) = 1. Hence a^{2n} x y = a (a^{2n-1} (x y)) = 1. Thus, a is a unit. ♦

Theorem 3.4:

If P is a prime ideal of a ring R and R / P is sπ-weakly regular ring. Then, P is a maximal ideal.

Proof:

Let a ∈ R. Then, a + P ∈ R / P, since R / P is sπ-weakly regular, there exist b + P, c + P ∈ R / P and a positive integer n such that

$a^n + P = (a^n + P) (b + P) (a^{2n} + P)(c + P)$

$a^n + P = a^n b a^{2n} c + P$. So, $a^n - a^n b a^{2n} c \in P$ and hence $a^n (1- b a^{2n} c) \in P$. Since $a^n \notin P$, then $(1- b a^{2n} c) \in P$, gives

$1 + P = b c a^{2n-1} a + P$

$= (b c a^{2n-1} + P) (a + P)$, this shows that a + P has an inverse. Thus, R/P is a division ring. Whence P is a maximal ideal. ♦

Corollary 3.5:

If R is an sπ-weakly regular ring, then every prime ideal of R is maximal.

Proof:

Let P be a prime ideal of R. Since R is sπ-weakly regular ring, then by Theorem 2.3, R/P is sπ-weakly regular. Thus, by Theorem 3.4, P is a maximal. ♦

Lemma 3.6:

If I is a primary ideal, then $\sqrt{I}$ is prime.

Proof:

See [5]. ♦

Lemma 3.7:

Let I be an ideal of R. If $\sqrt{I}$ is a maximal ideal of R, then I is primary ideal.

Proof:

See [2]. ♦

Theorem 3.8:

If R is sπ-weakly regular ring and I is an ideal of R, then I is primary if and only if $\sqrt{I}$ is prime.

Proof:

If I is primary, then by Lemma 3.6, $\sqrt{I}$ is prime.

Conversely, suppose that $\sqrt{I}$ is prime. Since R is sπ-weakly regular ring, then by Corollary 3.5, $\sqrt{I}$ is maximal. Therefore, by Lemma 3.7, I is primary. ♦

The following result is the relationship between sπ-weakly regular ring with it is ideals by adding the condition that every ideal of R is completely semi-prime.

Theorem 3.9:

If every ideal of R is completely semi-prime, then R is sπ-weakly regular if and only if for each ideal I of R, I = $\sqrt{I}$ holds.

Proof:

Let R be sπ-weakly regular. It is obvious that I $\subseteq \sqrt{I}$. Now, let b $\in \sqrt{I}$, then $b^n \in$ I and hence $b^{2n} \in$ I, for some positive integer n. Now, $b^n = b^n t_1 b^{2n} t_2$, for some $t_1, t_2 \in$ R. Since $b^n \in$ I and every ideal of R is completely semi-prime, then b $\in$ I. Therefore, $\sqrt{I}$ = I.

Conversely, assume that I = $\sqrt{I}$, for each ideal I of R. If we set I = $a^n R a^{2n} R$, for some positive integer n. Therefore, $a^n R a^{2n} R = \sqrt{a^n R a^{2n} R}$. Now, $a^{3n} \in a^n R a^{2n} R$, then $a^n \in \sqrt{a^n R a^{2n} R} = a^n R a^{2n} R$. So, $a^n \in a^n R a^{2n} R$. Whence R is sπ-weakly regular. ♦

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