Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM)

www.csmj.mosuljournals.com

On GAN-injective Rings : A systematic review

Hazim.T. Hazim 1,* Raida D. Mahmood 2

Northern Technical University / Technical Institute of Mosul, Mosul, Iraq [1], Department of Computer Science, College of computer science and mathematics, Mosul University, Mosul, Iraq [2]
**Corresponding author. Email:mti.lec80.hazim@ntu.edu.iq*

Article information

Article history:
Received : 2 /5 /2023
Accepted :28/5/2023
Available online :

Abstract

If for any maximal right ideal P of B and $a \in N(B)$,aB/ aP is almost N-injective, then a ring B is said to be right generalized almost N-injective. In this article, we present some significant findings that are known for right almost N-injective rings and demonstrate that they hold for right generalized almost N-injective rings. At the same time, we study the case in which every S.S.Right B-module is generalized almost N-injective.

Keywords:
Nil-injective rings, n-regular rings, reduced rings, ANil-injective rings.

Correspondence:
Author : Hazim.T. Hazim
Email:mti.lec80.hazim@ntu.edu.iq

1. INTRODUCTION

A ring B will be an associative ring with identity throughout this work, and all modules will be unitary.We write P_B to indicate right B-modules. For $a \in B$, we write Y(B), N(B), J(B) for the right singular ideal, the collection of nilpotent elements, and the Jacobson radical of B, and $r(\alpha)$ ($l(\alpha)$) for the right (left) annihilator of α. The right nil-injective ring was first defined Wei,J.C.and Chen,J.H in [10] and provided many properties of its. If $\alpha \in$N(B), lr(α)=$B\,\alpha$, a ring B is said to be right nil-injective. In [9], introduced a module that is almost Nil-injective or (AN-injective). Let $S = End\ (P_B)$ and let P be a right B-module. If the module P has an δ-submodule X_α of P such that $l_P r_B(a) = Pa \oplus X_a$ as left δ-modules for any $a \in N(B)$, then the module is known to as AN-injective. If there is a positive number n such that $a^n \neq 0$ and any right B-homomorphism of $a^n B$ into P extends to one of B into P, then the right B-module P is said to be GP-injective [2]. B is a reduced ring,if N(B)=0. Further work on reduced and injectivity rings appears in [2,4,5,6, and 11]. If we have ab=0 for every $a, b \in B$ implies that ba=0, then the ring B is a ZC-ring [1]. B is a ZC-ring if and only if $l(\alpha)(r(\alpha))$is an ideal of B for each case where $\alpha \in B$. If there is a $b \in B$ such that $aba = a$ exists for any $a \in B$, then the ring is said to be regular [2]. In accordance with [9], a ring B is referred to as n-regular if for each $a \in N(B), a \in aBa$. Every reduced ring is n-regular, as is a regular ring,[9] and B is said to be NJ if $N(B) \subseteq J(B)$ [7].

2-GAN-injective Rings:

If for any maximal right ideal P of B and for any $a \in P$, $aB/aP\ (Ba/Pa)$ is AGP-injective, then a ring B is said to be right (left) WAGPI according to [4].

We now provide the description that follows.

Definition (2.1): If for any maximal right ideal P of B and for any $a \in N(B)$ aB/aP is almost N-injective, then a ring B is said to be right generalized almost N-injective (for short GAN-injective).

Lemma (2.2):[9] Suppose that P is a right B-module with $S = End\ (P_B)$. If $l_P r_B(\alpha) = P\alpha \oplus X_\alpha$, where X_α is a left S-submodule of P_B with f: $\alpha B \to P$,a right B-homomorphism,then $f(\alpha) = p\alpha + x$, with $p \in P, x \in X_\alpha$.

From now on we considr every simple singular right B-module is GAN-injective (for short S.S.GAN-injective) and essential (maximal)right ideal (for short E.(M.)R.I.)

The next lemma, which is due to [3], plays a central role in several of our proofs.

Lemma (2.3): If P is M.R.I. of B and $r(\alpha) \subseteq P$, $\alpha \in P$, then:

a) $\alpha B \neq \alpha P$

b) $B/P \simeq \alpha B/\alpha P$

If Bα(αB) is an ideal of B for all instances where $\alpha \in N(B)$, a ring B is left (right) N-duo [8].

Theorem (2.4): B is reduced, if B is N-duo and GAN-injective ring.

Proof: If B is not reduced. Consequently, there is $0 \neq \alpha \in B$ such that $\alpha^2 = 0$. Hence there exists M.R.I. P of B containing $r(\alpha)$ .If αB= αP, then α = αc for some $c \in P$, hence $(1-c) \in r(\alpha) \subseteq P$, therefore $1 \in P$, which is contradiction. Now if αB≠ αP, then $\alpha B/\alpha P \simeq B/P$ and hence B/P is AN-injective and $l_{B/P} r_B(\alpha) = (B/P)\alpha \oplus X_\alpha$, $X_\alpha \leq B/P$. Let f: $\alpha B \to B/P$,be defined by $f(\alpha b) = b + P, b \in B$ .Note that f is a well-defined B-homomorphism.Then $1 + P = f(\alpha) = c\alpha + P + x$, $c \in B, x \in X_\alpha$ and $1 - c\alpha + P = x \in B/P \cap X_\alpha = 0$, $(1 - c\alpha) \in P$. Since B is right N duo, αB is an ideal of B, so $d\alpha \in \alpha B \subseteq r(\alpha) \subseteq P$, whence $1 \in P$, acontradiction. Thus, B is reduced.■

Corollary (2.5): Let B be a right GAN-injective and right N duo ring. Then B is n-regular.

Proof: From Th.(2.4) then B is reduced. Therefore B is n-regular ring. ■

Proposition (2.6): There is no non-zero nilpotent element in $J(B) \cap Y(B)$, if every S.S.GAN-injective.

Proof: Suppose $0 \neq \alpha \in J(B) \cap Y(B)$ with $\alpha^2 = 0$. Then BαB+r(α) is an E.R.I. of B, if it does not the M.R.I. P of B containing BαB+r(α). If αB= αP, then α= αc for some $c \in P$, It follows that $(1-c) \in r(\alpha) \subseteq P$, when $1 \in P$ contradicting $P \neq B$. If $\alpha P \neq \alpha B$, then $\alpha B/\alpha P \simeq B/P$.Since B/P is AN-injective then $l_{B/P} r(\alpha) = (B/P)\alpha \oplus X_\alpha$. Thus B$\alpha$B+r($\alpha$)=B. By the proof of Theorem (2.4) hence α =dα for some $d \in B\alpha B \subseteq J(B)$, $(1-d)\alpha = 0$ Since $d \in J(B)$, $1-d$ is invertible. It follows from this α =0, which is contradiction. Therefore $J(B) \cap Y(B)$ contains no non-zero nilpotent element . ■

Proposition (2.7): Let B is a ring. If every S.S.GAN-injective, then $J(B) \cap Y(B) = (0)$.

Proof: If $J(B) \cap Y(B) \neq 0$, then there exists 0≠ $a \in J(B) \cap Y(B)$ such that $a^2 = 0$. We'll show that BaB+r(a)=B. If not, we obtain BaB+r(a)=B as shown by the argument in Proposition (2.6), and a=da for some $d \in BaB \subseteq J(B)$.This gives that a=0, which is contradiction that a is non-zero. Therefore $J(B) \cap Y(B) = 0$.

Lemma (2.8):[2] If B is ZC-ring, then BxB+r(x) is an E.R.I. of B for every $x \in B$.

Lemma (2.9): Let BαB+r(α) is an E.R.I of B for every $\alpha \in N(B)$. If B is a ZC-ring

Proof: Similar to the evidence for, (Lemma (2.8)) ■

Theorem (2.10): If B is an S.S.GAN-injective and ZC- ring. Then B is a reduced.

Proof: Let $\alpha^2 \neq 0$. Suppose $\alpha \neq 0$. Consequently, there is a M.R.I. P of B containing r(α) . By Lemma (2.9) P is an E.R.I. of B containing r(α).If αB= αP , then α = αc for some $c \in P$, hence $1 - c \in r(\alpha) \subseteq P$. Therefore $1 \in P$.Now, if $\alpha B \neq \alpha P$, then $\alpha B/\alpha P \simeq B/P$ and hence B/P is AN-injective. Similar to prove of Theorem (2.4) we get, $1 - c\alpha \in P$ for some $c \in B$. Since B is ZC-ring, $c\alpha \in r(\alpha)$. It follows that, $1 \in P$, both cases contradicts that P is a M.R.I. and does not contain the identity of the ring. Therefore α =0 and hence, B is reduced . ■

Proposition (2.11): If B is an S.S.GAN-injective and ZC-ring. Then, for each $b \in N(B)$, BbB+r(b)=B.

Proof: Suppose $BbB + r(b) \neq B$ for some $b \in N(B)$. Consequently, there is a M.R.I. P of B containing BbB+r(b) by Lemma (2.9) P is an E.R.I. of B. If bB=bP, then b=bc for some $c \in P$. Therefore $1 \in P$. If $bB \neq bP$, then $bB/bP \simeq B/P$. Since bB/bP is AN-injective, then B/P is AN-injective, and $lr(b) = (B/P)b \oplus X_b$, $X_b \leq \frac{B}{P}$. Let $f: bB \to B/P$, be defined by $f(br) = r + P$. Note that f is a well-defined B-homomrphism. Then $1 + P = f(b) = db + P + x, d \in B, x \in X_b$, $1 - db + P = x \in \frac{B}{P} \cap X_b = 0$, $1 - db \in P$, so $1 \in P$. Both cases contradicts that P is a M.R.I. and does not contain the identity of the ring. Therefore $BbB + r(b) = B$, for any $b \in N(B)$. ■

Proposition (2.12): Let B be a ZC-ring. Then the following conditions are equivalent:

1. B is reduced ring.
2. B is n-regular and NJ-ring.
3. B is an S.GAN-injective.
4. B is an S.S.GAN-injective.

Proof: Obviously $1 \to 2 \to 3 \to 4$ and 2→1[8, Theorem 2.24]
$4 \to 1$ by Theorem (2.10) ■

References

[1] Cohn,P.M.(1999),"Reversible rings", Boll. London Mat Soc.31,PP.641-648.

[2] Kim,N.Nam,S,B.and J.Y.Kim, J.Y. (1999), On simple singular GP-injective module , Comm. Algebra, 27(5), PP. 2087-2096.

[3] Mahmood ,R.D. and Mohammad, H.Q.(2012),"On generalized simple singular AP-injective rings", Rafidan J. of Computer Science Math,Vol.(9),NO.1,PP. 175-180

[4] Mahmood,R.D. and Hazem,T.H,(2013)," On weakly AGP-injective rings" Aust.J.of Basic and Appl. Sciences ,7(12), PP. 151-155.

[5] Mohmood ,R.D. and Khider,K. J.(2020), "On Pwπ-regular rings" J.Physics:Coference Series 1591.

[6] Mahmood,R.D. and Mohammed,R.S.(2021), "QR and JCPI-ring" J.Physics: Conference Series 1879.

[7] Lee,C.I. and park,S.Y.(2018), "When nilpotents are contained in Jacobson radicals" J.korean Math. Soc.Vol.(55).No.5, PP. 1193-1205.

[8] Nam,S.B.Lee,H.T and Lee,y.k.(2016), "N-duo and SRB-rings" Gold J. of Pure and APP.Math., Vol.12(4), PP. 3037-3042.

[9] Xianneng,D.and Zhao,Y. (2011), "On almost Nil-injective rings" Inter.Eles.J.of Algebra ,Vol.9,PP. 103-113.

[10] Wei,J.C. and Chen,J.H.(2007), "Nil-injective rings" Inter .Electron .J. of Algebra ,Vol.2,PP. 1-21.

[11] Wei,J.C. and Chen,J.H.(2008),"Npp-rings, reduced rings and SNF-rings" Inter .Electron .J. of Algebra , Vol.4, PP. 9-96.

حول الحلقات الغامرة من النمط - GAN
: ورقة مراجعة

رائدة داؤد محمود

قسم علوم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة الموصل، الموصل، العراق

raida.1961@uomosul.edu.iq

حازم طلعت حازم

الجامعة التقنية الشمالية، المعهد التقني الموصل، الموصل ،العراق

mti.lec80.hazim@ntu.edu.iq

تاريخ الاستلام: 2/5/2023 **تاريخ القبول: 25/5/2023**

الملخص

يقال للحلقة B بأنها غامرة يمنى من النمط -N تقريباً المعممة ، اذا كان كل مثالي اعظمي ايمن P في B و $a \in N(B)$ ،aB/ ap غامراً من النمط -N تقريبا . في هذا البحث, اعطيت خواص هذه الحلقات مع تعميم بعض من نتائج للحلقات الغامرة من النمط – N تقريباً وكذالك درسنا الحلقات التي يكون فيها كل مقياس ايمن بسيط منفرد غامر من النمط –N المعممة.

الكلمات المفتاحية : حلقه مختزلة ،حلقه منتظمة من النمط -n، حلقه غامر من النمط – nil، حلقه غامر من النمط – Anil .