The Analytic solution for some non-linear stochastic differential equation by linearization (Linear-transform)

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Abstract

In this paper, we study a reducible method which is called linearization(Linear-transform) for some non-linear stochastic differential equations (SDEs) to linear by using the Ito-integrated formula. And then finding their analytic solution, we compare the obtained solution for the nonlinear SDEs with the approximate solution by using numerical (Euler-Maruyama and Milstein) Methods.

Keywords:
Stochastic differential equation, Ito formula, Reducible, Euler-Maruyama & Milstein.

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1. Introduction:

The stochastic differential equations are increasingly used in many scientific and industrial fields which demonstrate the importance of stochastic modeling, since they can be used to any address problem caused by noise, accidents, etc. [1]


In this paper, we study the linearization to the non-linear SDEs in order to find the analytic solution to the obtained linear stochastic differential equations, there are several examples with numerical solution are used to validate the results.

Let \( X(t) \) satisfied the following stochastic differential equation

\[
dX(t) = f(t, X(t))dt + g(t, X(t))dW(t)
\]  

\( \ldots \) (1)

Where \( W(t) \) denote wiener process , \( f(t, X(t)) \) and \( g(t, X(t)) \) are deterministic non-linear functions and \( dW(t) \) represent the differential form of \( W(t) \), equation(1) is called nonlinear stochastic differential equation otherwise it is called linear.

let the real-valued function \( F(t, x(t)) \) has continuous partial derivatives
\[
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2}, \quad t \in [0, T] \quad \text{and} \quad x \in \mathbb{R}, \quad \text{then} \quad F(t, x(t)) \quad \text{satisfies the following equation:}
\]
\[
dF(t, x(t)) = \left[ \frac{\partial F}{\partial t} + f \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 F}{\partial x^2} \right] (t, X(t)) dt + \left[ \frac{\partial F}{\partial x} \right] (t, X(t)) dW(t) \quad \text{...(2)}
\]
Equation (2) is called (Ito-formula). \[6\]

2. Preliminaries and Results:

We introduce some fundamental and necessary concept steps for the transformation of non-linear SDEs to reduce to linear and then find the analytic solution.

Definition 2.1. (Wiener Process) \[1][7\]
The continuous-time stochastic process \( \{W_t : t \geq 0\} \) is called Wiener process (Brownian motion) over the interval \([0, T]\) which satisfies the following conditions:
1. \( W(0) = 0 \)
2. If \( t, s \geq 0, then W(t) - W(s) \) is normally distributed with zero mean and variance \( |t - s| \).
3. For \( 0 \leq s < t < k < j \leq T, W(t) - W(s) \text{ and } W(j) - W(k) \) are independent increments.

Definition 2.2. (Stochastic integral) \[8\]
A stochastic integral is an integral which is defined as a sum more than integration and it is increased by the rise in time on the Wiener process trajectory. That is:
\[
\int_0^T c(t) dW(t) = \sum_{i=0}^{\infty} \delta_i (W_{t_{i+1}} - W_{t_i}) \quad \text{...(3)}
\]

Definition 2.3. (The analytic solution of linear SDE) \[9\]
Let we have the general linear stochastic differential equation given by
\[
dX_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t \quad \text{...(4)}
\]
Then the solution will be in the form:
\[
X_t = \Phi_{t,0}(X_0) + \int_0^t \Phi_{t,s}^{-1}(a_2(s) - b_1(s)b_2(s))ds + \int_0^t \Phi_{t,s}^{-1}b_2(s)dW_s \quad \text{...(5)}
\]
where
\[
\Phi_{t,s} = \exp(\int_{t_0}^t \left[ a_1(s) - \frac{1}{2}b_2(s)^2 \right] ds + \int_{t_0}^t b_1(s) dW_s) \quad \text{...(6)}
\]

3. Linearization method for non-linear (SDEs): \[10\]
The linearization method means a suitable function which reduce such nonlinear stochastic differential equation to linear in order to find their analytic solution, we explain it by the following main steps:

Step 1. Suppose we have the following nonlinear stochastic differential equation:
\[
dY(t) = f(t, Y(t)) dt + g(t, Y(t)) dW(t) ; \quad y(t_0) = y_0 \quad \text{...(7)}
\]
Here \( f(t, y) \) and \( g(t, y) \) are real non-linear functions, Suppose that the function \( U(t, y(t)) \) be smooth and has continuous derivatives \( \frac{\partial U}{\partial t}, \frac{\partial U}{\partial y}, \frac{\partial^2 U}{\partial y^2} \) that reduce equation (7) to some linear stochastic differential equation. Which we want to find it.

By using Ito formula to \( U(t, y(t)) \), we get
\[
dU(t, y(t)) = \left[ \frac{\partial U}{\partial t} + f \frac{\partial U}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial y^2} \right] (t, Y(t)) dt + \left[ \frac{\partial U}{\partial y} \right] (t, Y(t)) dW(t) \quad \text{...(8)}
\]
Step 2. suppose that equation (8) transformed to linear stochastic differential equation of the form:
\[
dx(t) = (a_1(t)X(t) + a_2(t))dt + (b_1(t)X(t) + b_2(t))dW(t) \quad \text{...(9)}
\]
Step 3. Determined the parameters \( a_1, a_2, b_1, b_2 \) for the linearity by using the comparison of eq. (8) and eq. (9), then we have
\[
\frac{\partial U}{\partial t} + f \frac{\partial U}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial y^2} = (a_1(t)U(t) + a_2) \quad \text{...(10.1)}
\]
\[
\frac{\partial U}{\partial y} = (b_1(t)U(t) + b_2) \quad \text{...(10.2)}
\]
By solving equation (10.1) and equation (10.2), we get the values of the parameter of the linear equation(9)

Step 4. The analytic solution for the obtained linear stochastic differential equation has the form:
\[
X_t = \Phi_t \left[ X_0 + (a_2 - b_1(b_2) \int_0^t \Phi_r^{-1}ds + b_2 \int_0^t \Phi_r^{-1}dW_r \right]
\]
where \( \Phi_t = \exp(\int_{t_0}^t (a_1(s) - \frac{1}{2}b_2(s)^2)ds + \int_{t_0}^t b_1(s) dW_s) \)
By using transformation
\[
Y_t = U^{-1}(t, X_t)
\]
we obtain the solution of eq. (7).

3.1. The method of finding the analytic solution: In this paragraph, we explain the main step of finding the suitable function which transformed the nonlinear stochastic differential equation to linear stochastic differential equation.

From eq(10.2), let \( g(Y) \neq 0 \) and \( b_1(t) = b_1 \neq 0 \), then we get
\[
\frac{\partial U}{\partial y} = \frac{b_2}{g(y)} \quad \text{...(10.3)}
\]
which is a first order differential equation, and the integrated factor is:
\[
M = \exp(\int_{y_0}^{y} \frac{b_1(y)}{g(y)} dy), \quad \text{let} B(y) = \int_{y_0}^{y} \frac{ds}{g(y)} \quad \text{then}
\]
\[
U(y) = e^{b_1B(y)} \int e^{-b_1B(y)} \frac{b_2}{g(B)} dy + k e^{b_1B(y)}
\]
\[
U(y) = \frac{-b_2}{b_1} e^{b_1B(y)} \int e^{-b_1B(y)} \frac{-b_2}{g(B)} dy + k e^{b_1B(y)}
\]
\[
U(y) = k e^{b_1B(y)} \quad \frac{-b_2}{b_1}
\]
Then equation (10.1) becomes:
\[
k e^{b_1B(y)} \left[ f(y) \right] B_1 + \frac{1}{2} b_2 = a_1 \left[ k e^{b_1B(y)} - \frac{b_2}{b_1} \right] + a_2
\]
\[
k e^{b_1B(y)} \left[ f(y) \right] B_1 + \frac{1}{2} b_2 = a_1 k e^{b_1B(y)} - a_2 \frac{b_2}{b_1} + a_2
\]
\[
\left[ f(y) \right] B_1 + \frac{1}{2} b_2 = \frac{a_1 b_2}{b_1} + \frac{a_2}{b_1}
\]
let \( A(y) = \frac{f(y)}{g(y)} - \frac{1}{2} \frac{dU}{dy} \), then

\[
[b_1 A(y) + \frac{1}{2} b_1^2 - a_1] ke^{b_1 B(y)} = a_2 - \frac{a_1 b_2}{b_1} \quad \ldots (11)
\]

The derivative of equation (11) is equal to zero, i.e.

\[
\left[b_1 A(u) + \frac{1}{2} b_1^2 - a_1\right] ke^{b_1 B(u)} = 0
\]

\[
b_1 \frac{dA}{dy} \left[k e^{b_1 B(u)}\right] + k b_1 A(u) \frac{d}{dy}\left(e^{b_1 B(u)}\right)
\]

\[
k b_1 e^{b_1 B(y)} \frac{dA}{dy} + k b_1 A(y) \frac{b_1}{g(y)} e^{b_1 B(y)}
\]

Multiplying by \( \frac{g(y)e^{-b_1 B(y)}}{b_1} \), we get

\[
k b_1 \left[\frac{dA}{dy} + \frac{b_1}{g(y)} A(y)\right] = g(y) \frac{dA}{dy} + b_1 A(y), \text{ then taking the derivative with respect to } y \text{ we get}
\]

\[
= b_1 A_y + \frac{d}{dy}(b_1 A_y) = 0
\]

\[
\text{then we have the relation}
\]

\[
b_1 \frac{dA}{dy} + \frac{d}{dy} \left( g \frac{dA}{dy} \right) = 0 \quad \ldots (12)
\]

Which is satisfied if

\[
\frac{dA}{dy} = 0 \text{ or } \frac{d}{dy} \left( \frac{g \frac{dA}{dy}}{dy} \right) = 0 \quad \ldots (13)
\]

Provided that \( b_1 = -\frac{\frac{d}{dy} \left( \frac{g \frac{dA}{dy}}{dy} \right)}{dy} \)

**Result.** The suitable linearization of a non-linear stochastic differential equation is:

1. If \( b_1 \neq 0 \) then the best transformation is
   \( U(y) = ke^{b_1 B(y)} - \frac{b_2}{b_1} \)
2. If \( b_1 = 0 \), then the transformation is \( U(y) = b_1 B(y) + k \)

by using a suitable value for \( b_1 \) and obtaining \( U_y \) and \( U_{yy} \). From equations (10.1) and (10.2) we obtain the parameters \( a_1, a_2, b_1, \) and \( b_2 \) for the linear stochastic differential equation (which is given in eq.(9)).

From equation (4) we obtain the general solution of the transformed linear stochastic differential equation by the transformed function rewrite \( y(t) = U(x)^{-1} \) \[11\]

### 4. Numerical methods for solving nonlinear SDEs:

Since many stochastic differential equations have unknown solution, so it is necessary to derive numerical methods to generate approximations to the exact solution.

we used (Euler-Maruyama and Milstein’s) method.

#### 4.1. Euler - Maruyama method: [12] [13]

Euler-Maruyama method is similar to Euler method for solving ordinary differential equations that are presented from the point of view of Taylor's algorithm which greatly simplifies accurate analysis.

Euler approximation is one of the simplest discrete time estimates for Ito-Taylor expansion .Let \( X_t \) be an ito process on \( t \in [t_0, T] \) satisfying the stochastic differential equation

\[
dX_t = f(t, X_t) dt + g(t, X_t) dW_t
\]

\[
X_{t_0} = X_0
\]

....... (14)

For a given estimate \( t_0 < t_1 < t_2 < \cdots < t_N = T \), Euler approximation is a random process in a continuous time \( X_t \). Now we write

\[
X_{tn+1} = X_{tn} + \int_{tn}^{tn+1} f(t, x) dt + \int_{tn}^{tn+1} g(t, x) dW_t
\]

...(15)

By replacing the integral of (15) with Taylor's chain expansion for functions \( f(t, x) \) and \( g(t, x) \) around the point \( (t_n, x_n) \) and \( X_n = X_{tn} \) terms containing \( (x - x_n) \) that arise in the expansion above ,we get

\[
f(t, x) = f(t, x_n) + \frac{\partial f}{\partial t} \delta t + \frac{\partial f}{\partial x} \delta x + \ldots
\]

\[
= f(t_n, x_n) + \frac{\partial f}{\partial t} \delta t + \frac{\partial f}{\partial x} \delta x + \ldots \quad \ldots (16)
\]

\[
g(t, x) = g(t_n, x_n) + \frac{\partial g}{\partial t} \delta t + \frac{\partial g}{\partial x} \delta x + \ldots
\]

\[
= g(t_n, x_n) + \frac{\partial g}{\partial t} \delta t + \frac{\partial g}{\partial x} \delta x + \ldots \quad \ldots (17)
\]

Substituting equations (16) and (17) into (15)

\[
X_{tn+1} = X_{tn} + \int_{tn}^{tn+1} \left[f(t_n, x_n) + \frac{\partial f}{\partial t} \delta t + \frac{\partial f}{\partial x} \delta x \right] dt
\]

\[
+ \int_{tn}^{tn+1} \left[\frac{\partial g}{\partial t} \delta t + \frac{\partial g}{\partial x} \delta x \right] dW_t
\]
\[ \int_{t_n}^{t_{n+1}} \left( f(t_n, x_n) + \frac{\partial g}{\partial t}(t_n - x_n) \right) dt + \frac{\partial g}{\partial x} ((t_n - x_n) dW_t \quad \ldots (18) \]

\[ X_{t_{n+1}} = X_{t_n} + f_n(t_n + 1 - t_n) + g_n(W_{t_{n+1}} - W_{t_n}) \quad \ldots (19) \]

This implies \( X_{t_{n+1}} = X_{t_n} + f_n(\Delta t) + g_n(\Delta W) \)

Which is the Euler-Maruyama formula.

4.2. Milstein’s method: [12] [13]

The Milstein’s method obtained by adding the following second order term for Ito integral to the Euler-Maruyama scheme

\[ g(X_{t_n})g'(X_{t_n}) \int_{t_n}^{t_{n+1}} g(W_{t_n})^2 \, dt = g(X_{t_n})g'(X_{t_n}) (\Delta W)^2 - \Delta t \]

From the Ito-Taylor expansion, we obtain (Milstein formula) given by

\[ X_{t_{n+1}} = X_{t_n} + f_n(\Delta t) + g_n(\Delta W) \]

This implies

\[ X_{t_{n}} = X_{t_n} + f(\Delta t) + g(\Delta W) \]

Eq. (21) is called Milstein’s formula.

5. Examples: In this paragraph we give some examples in order to explain the methods (analytically and numerically).

Example 5. 1. Consider the nonlinear SDEs given by

\[ dY_t = Y_t(\alpha + \beta Y_t^{n-1}) dt + \gamma Y_t \, dw_t ; Y(0) = y_0 \]

\[ \alpha, \beta, \gamma \text{ are constants.} \]

Solve it (analytically and numerically).

The analytic solution:

Here \( a(y) = Y_t(\alpha + \beta Y_t^{n-1}) \) and \( b(y) = \gamma Y_t \)

Since \( A = \frac{a(y)}{b(y)} = \frac{1}{2} \frac{\partial b}{\partial y} \)

Then \( A = \left( \frac{\alpha}{\gamma} + \frac{\beta }{\gamma} Y_t^{n-1} - \frac{\gamma}{2} \right) \neq 0 \)

\[ A_y = \frac{\beta}{\gamma} (n - 1) Y_t^{n-2} \quad \text{and} \quad b(y)A_y = \gamma Y_t^2 (n - 1) Y_t^{n-2} \]

\[ \frac{d}{dy} (b(y)A_y) = \beta (n - 1)^2 Y_t^{n-2} \]

then \( \frac{d}{dy} (b(y)A_y) = \gamma (n - 1) \text{ which is constant} \)

\[ \frac{d}{dy} \left( \frac{d}{dy} (b(y)A_y) \right) = 0 \quad \text{, we can take} \]

\[ b_1 = - \frac{d}{dy} (b(y)A_y) = \gamma (1 - n) \neq 0, b_2 = 0, k = 1 \]

We get \( U = k_b b_i B(t) \)

Where \( B(t) = \int_0^\infty \frac{ds}{B(s)} = \frac{1}{\gamma} \ln \left( \frac{y}{y_0} \right) ; y_0 > 0 \)

Then \( U = \frac{y}{y_0} \quad \ldots (22) \)

so we can assume

\[ X = U(y) = \frac{y}{y_0} \]

To find \( a_1, a_2 \), we need to find \( U_y, U_{yy} \) to use equation (10.1)

From equation (22), we get

\[ U_y(y) = (1 - n) \frac{y}{y_0} \quad \text{and} \quad U_{yy}(y) = -n(1 - n) \frac{y}{y_0} \]

So we have

\[ a(y) U_y(y) + \frac{1}{2} b^2(y) U_{yy} = a_1 U + a_2 \]

\[ y(\alpha + \beta y^{n-1}) (1 - n) \left( \frac{y}{y_0} \right)^{-n} \]

\[ \frac{1}{2} y^2 y^{n-2} \left( (1 - n) \left( \frac{y}{y_0} \right)^{-n-1} \right) \]

\[ a_1 \left( \frac{y}{y_0} \right)^{1-n} + a_2 \]

This implies

\[ y^{1-n} \left( (1 - n) a y_0 + \frac{1}{2} (n^2 - n) y^2 y_0^2 - a_1 \right) \]

\[ + [(1 - n)n y_0 x(t) + a_2] = 0, \]

since \( y^{n-1} \neq 0 \)

Then

\[ a_1 = (1 - n) a y_0 + \frac{1}{2} (n^2 - n) y^2 y_0^2 + a_2 = \beta (1 - n) y_0^{n} \]

Therefore, the linearization for the given equation is

\[ dx(t) = \left[ \left( (1 - n) a y_0 + \frac{1}{2} (n^2 - n) y^2 y_0^2 \right) x(t) + \beta (1 - n) y_0^{n} \right] dt + \gamma (1 - n) x(t) \, dw(t) \]

by using equation (4) , we get

\[ X_t = \Phi_t \exp \left( a(1 - n) y_0 + \frac{1}{2} (n^2 - n) y^2 y_0^2 \right) t - \frac{1}{2} y^2 (n - 1)^2 t + \gamma (1 - n) w_t \]

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From the transformation \( Y_t = U^{-1}(X_t) \),
we obtain the general solution of the original equation
\[
y(t) = \frac{\exp\left[\frac{1}{2}(n-1)^2 - (a_1^n - \nu_1^n) + \frac{1}{2}(n-1)^2 \nu_1^n\right] - \gamma y(1-n)w_t}{y_0^{-1} + \beta(1-n) \int_0^t \phi_t^{-1} ds}
\]

For special case, let \( n=2 \), \( y_{t_0} = 1 \)

\[
dy_t = y_t(\alpha + \beta y_t) dt + \gamma y_t dw_t ; y(y_{t_0}) = y_0
\]

Then
\[
b_t = -\gamma \neq 0 \quad \text{we can take}
\]
\[
U = c e^{b_1(t)}
\]
\[
B(t) = \int_0^t ds = \frac{1}{y} \ln(y)
\]
\[
U = c e^{\frac{-\gamma}{y} \ln(y)} = c \frac{1}{e^{\gamma y}} = \frac{c}{y}
\]

If we take \( c = 1, b_2 = 0 \), \( b_1 = -\gamma \)

Then \( \gamma = \frac{1}{y} ; U_y = -\frac{1}{y^2} ; U_{yy} = -\frac{2}{y^3} \)

From equation (10.1), we get

\[
a(y), U_y + \frac{1}{2} b^2(y) U_{yy} = a_1 U + a_2
\]

\[
y(\alpha + \beta y)^{-\frac{1}{2}} + \frac{1}{2} y^2 y^2 - \frac{2}{y^3} = a_1 \left[ 1 \right] + a_2
\]

\[a_1 = y^2 - \alpha ; \quad a_2 = -\beta, b_1 = -\gamma, b_2 = 0\]

From the linear equation

\[X_t = [a_1 x_t + a_2] dt + [b_1 x_t + b_2] dw_t\]

we obtain

\[X_t = [(y^2 - x)x_t - that is\beta] dt + [-y x_t + 0] dw_t\]

Then the solution is

\[X_t = \varphi_t(x_0 - \beta \int_0^t \varphi_t^{-1} ds)\]

\[\varphi_t = \exp\{y^2 - \alpha t - \frac{1}{2} \gamma^2 t - \gamma w_t\}\]

\[= \exp\{y^2 t - \alpha t + \frac{1}{2} \gamma^2 t - \gamma w_t\}\]

\[= \exp\{\frac{3}{2} y^2 t - \alpha t - \gamma w_t\}\]

\[= \exp\{-\left[\left(\frac{3}{2} y^2\right) t + \gamma w_t\right]\}\]

\[\int_0^t \varphi_t^{-1} ds = \int_0^t \exp \left[ -\left(\frac{3}{2} y^2\right) s + \gamma w_s\right] ds\]

\[= \exp\left[\left(\frac{3}{2} y^2\right) s + \gamma w_s\right] ds\]

\[\varphi_t = \varphi_t(x_0 - \beta \int_0^t \varphi_t^{-1} ds)\]

\[= \exp\left[\left(\frac{3}{2} y^2\right) t + \gamma w_t\right] ds\]

\[X_t = \exp\left[\left(\frac{3}{2} y^2\right) t + \gamma w_t\right] x_0 - \beta \int_0^t \varphi_t^{-1} ds\]

Then the solution for this example is

\[Y_t = \frac{\exp\left[\left(\frac{3}{2} y^2\right) t + \gamma w_t\right]}{y_0^{-1} - \beta \int_0^t \exp\left[\left(\frac{3}{2} y^2\right) s + \gamma w_s\right] ds}\]

The numerical solution:

Here \( f_n = y_{t_n}(\alpha + \beta y_{t_n}) \) and \( g_n = y y_{t_n} \)

\( \alpha = 1.3; \beta = 0.05; y = 0.1 \) : \( y_0 = 0.1 \)

The Euler-Maruyama method for this example take the form

\[Y_{t+1} = Y_t + f_t \Delta t + g_t \sqrt{\Delta t} \eta_j \quad (23)\]

With \( \Delta t \Rightarrow \Delta t = \frac{1}{N}, i-(0,1) , t \in (0,1) \]

And Milstein’s equation has the form:

\[y_{t_n+1} = y_{t_n} + f_t \Delta t + g_t \sqrt{\Delta t} \eta_j + \frac{1}{2} \beta g_t \frac{\partial g_{t_n}}{\partial y_{t_n}} (\eta_j^2 - 1) \Delta t\]

Where \( dW_t = W_t - W_{t-1} \), \( dW_t = \sqrt{\Delta t} \eta_j \)

The following figure shows the comparison between numerical (Euler -Maruyama & Milstein) with the analytic solution:

![Figure(1): comparison between the analytic and numerical solution.](image)

**Table (1): The values of the exact and numerical solutions.**

<table>
<thead>
<tr>
<th>No.</th>
<th>exact</th>
<th>Euler-Maruyama</th>
<th>Milstein</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1.214723305</td>
<td>1.214618882</td>
</tr>
</tbody>
</table>
Example 5.2. consider the following nonlinear SDEs:
\[ dy(t) = -e^{-4yt}dt + e^{-2yt}dw_t \]
Reduce it to a linear SDEs and find the solution (analytically and numerically)

The analytic solution: By compare the above equation with eq. (7), we get
\[ A = \frac{a}{b}, \quad b = g(t, Y(t)) = g(y) = e^{-2y} \]
\[ -e^{-4y} - \frac{1}{2}(-2e^{-2y}) = 0 \]
\[ -e^{-2y} + e^{-2y} = 0 \]
\[ b_1 = 0 \]
then we can take
\[ U = b_2B(s) + k; \quad b_2 = 1, k = 0 \]

Then \[ U = \frac{1}{2}(e^{2y_0} - 1), U_y = e^{2y}, U_{yy} = 2e^{2y} \], so from equation (10.1)
\[ [-e^{-4y}(e^{2y})] + \frac{1}{2}(e^{-4y})(2e^{2y}) = a_1 \frac{1}{2}(e^{2y_0} - 1) + a_2 \]
Then \[ -2e^{-2y} + 2(e^{-2y}) = a_1 e^{y} + a_2 \]
\[ a_1 \frac{1}{2}(e^{2y_0} - 1) + a_2 = 0 \] if and only if \[ a_1 = a_2 = 0 \], from eq. (9), we get
\[ dx_t = dw_t \]
Then solution is
\[ x_t = x_0 + wt \]
Where \[ x_0 = \frac{1}{2}(e^{2y_0} - 1) \]
This implies
\[ x_t = U = \frac{1}{2}(e^{2y_0} - 1). \]

Since \[ x_t = U = \frac{1}{2}(e^{2y_0} - 1). \]
we get \[ y_t = \frac{1}{2}ln[x_t + 1] \]
Then the analytic solution is.
\[ y_t = \frac{1}{2}ln[e^{2y_0} + 2wt] \]

The numerical solution:
since \[ f_n = -e^{-4yn} \quad \text{and} \quad g_n = e^{-2yn} \]
Then Euler- Maruyama and Milstein’ formulas take the form respectively:
\[ y_{tn+1} = y_{tn} + (-e^{-4yn})\Delta t + e^{-2yn}\sqrt{\Delta t}\eta_i \]
\[ y_{tn+1} = y_{tn} + (-e^{-4yn})\Delta t + e^{-2yn}\sqrt{\Delta t}\eta_j + \frac{1}{2}(-e^{-2yn})\theta(e^{-2yn})\Delta t \]

The following figure shows the comparison between numerical (Euler -Maruyama & Milstein) with the analytic(exact) solution.

Figure (2): comparison between the analytic and numerical solution.

The following figure shows the comparison between numerical (Euler -Maruyama & Milstein) with the analytic(exact) solution.

Table (2): The values of the exact and numerical solutions.

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5. CONCLUSION AND FUTURE WORKS:
Through our study we applied the linearization method (linear -transform) for the nonlinear stochastic differential equation(SDES) by applying Ito formula. After we obtained a suitable function \[ X_t = U(t, Y_t) \] for the reducible linear stochastic differential equation, we find the analytic
solution. Lastly we compare the exact solution with the numerical solution (Euler- Maruyama and Milestein) methods by several examples, in the numerical solution we see that the Milestein method better than Euler- Maruyama in convergence with the exact solution for the first example while in the square root example it seems that they are the same.

As a future studies one can study the linearization of some nonlinear(harmonic) stochastic differential equation by using stratonovich formula for their solution and compare it with Ito formula.

References: