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First and Second Zagreb Coindices for Chains of Cycles

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Abstract

Abstract—The graphs which are used in this paper are simple, finite and undirected. The first and second Zagreb indices for every non-adjacent vertices (also called first and second Zagreb coindices) are dependent only on the non-adjacent vertices degrees which interspersed the operations of addition and multiplication, respectively, for the degrees of non-adjacent vertices. The number of the edges incident on vertex v in a graph G is called the degree of a vertex v and the two vertices u and v are non-adjacent if it's no common edge between them. In this paper, we found the first and second Zagreb coindices of chains of even cycles and also, gave some examples.

Keywords:
First Zagreb coindex, Second Zagreb conidex, Identical, Even cycle.

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I. INTRODUCTION

The simple graph G is consist of pairs order of non-empty finite set is called set vertex V and a finite set of unordered pairs of distinct elements of vertex set is called set edge E and denoted by $G=(V,E)$. The number of elements the set vertex and set edge are called the order and the size of a graph G respectively which denoted by $p=p(G)$ and $q=q(G)$ respectively. The degree of vertex v in a graph G is the number of edges incident with vertex v denoted by $deg_G(v)$. The cycle in a graph G is alternate sequence of vertices and edges $u_1, e_1, u_2, e_2, \dots, u_h, e_h, u_1$ such that $u_i \neq u_j$ for all $1 \leq i, j \leq h$, $h \geq 3$, the cycle graph is a graph contain only a cycle, denoted to the cycle graph of order p by C_p. It is clear that every vertex of C_p has two degree and we say that the cycle graph is even if $p=2m,\ m \in N-\{1\}$ ($q=2m,\ m \in N-\{1\}$). For a lot of concepts in statement theory, see the references [1,2].

An invariant of any connected graph is the number related to graphs. The topological indices are one of invariant important in knowing many chemical and physical properties, one of the oldest invariants of graph is Zagreb indices (first and second) which introduced by Gutman and Trinajstić in 1972, [3]. In 2008, [4], Dǒslíc find the complement of first and second Zagreb indices which is called first and second Zagreb coindices. There are many researchers who have an interest in finding first and second Zagreb indices for a lot of graphs, such as: [5-9]. Also, there are many topological indices that are interested in many researchers such as: omega, degree deviation and Wiener indices [10-13].

In this paper, we found the first and second Zagreb coindices of special types of chains graphs "chains for identical of n −even cycles", the complement of first Zagreb index and the second of Zagreb index are defined respectively by:

$$Z_1^c(G) = \sum_{vu \notin E}[deg_G v + deg_G u],$$

and

$$Z_2^c(G) = \sum_{vu \notin E}[deg_G v \times deg_G u] .$$

II. The first and second Zagreb coindices for chains of n −even cycles:

In this section, we take two types of chains for identical of $n -$ even cycle as the following:

Type 1: A chain of $n -$ even cycles $C_{2m}, n, m \geq 2$ by identical symmetric vertices $I_V(CC_{2m})$, see Fig 1.1.

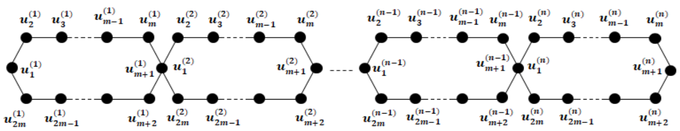

FIGURE 1.1. The graph $I_V(CC_{2m})$

Type 2: A chain of $n -$ even cycles $C_{2m},\ n, m \geq 2$ by introduced path $P_r,\ r \geq 2$, for symmetric vertices $I_P(CC_{2m})$, see Fig1.2.

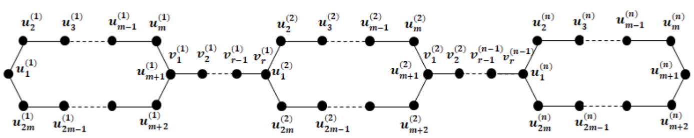

FIGURE 1.2. The graph $I_P(CC_{2m})$,

Now, we find general formula of the complement of first and second Zagreb indices for both types of chains.

Theorem 2.1.1: For all $n, m \geq 2$, then:

1. $Z_1^c\left(I_V(CC_{2m}^n)\right) = Z_1^c\left(I_V(CC_{2m}^{n-1})\right) + 8nm(2m-1) - 4(2m^2 + m + 2).$

2. $Z_2^c\left(I_V(CC_{2m}^n)\right) = Z_2^c\left(I_V(CC_{2m}^{n-1})\right) + 16nm^2 - 4(2m^2 + 3m + 5).$

Proof:

We add an even cycle C_{2m} to the chain $I_V(CC_{2m}^{n-1})$by identical the vertex u_1^n in C_{2m}^n with the vertex u_{m+1}^{n-1}in $I_V(CC_{2m}^{n-1})$, we obtain:

1.
$$Z_1^c\left(I_V(CC_{2m}^n)\right) = Z_1^c\left(I_V(CC_{2m}^{n-1})\right) + D^+_{I_V(CC_{2m}^n)}(2,2) + D^+_{I_V(CC_{2m}^n)}(2,4) + D^+_{I_V(CC_{2m}^n)}(4,4) - D^+_{I_V(CC_{2m}^{n-1}, u_{m+1}^{n-1})}(2,2) - D^+_{I_V(CC_{2m}^{n-1}, u_{m+1}^{n-1})}(2,4).$$

$$D^+_{I_V(CC_{2m}^n)}(2,2) = \sum_{i=2}^{2m-2}\sum_{j=i+2}^{2m}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n)} + deg_{I_V(CC_{2m}^n)}u_j^{(n)}\right] + \sum_{i=2}^{2m}\sum_{j=1}^{n-1}\sum_{h=2}^{m}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n)} + deg_{I_V(CC_{2m}^n)}u_h^{(j)}\right] + \sum_{i=2}^{2m}\sum_{j=1}^{n-1}\sum_{h=m+2}^{2m}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n)} + deg_{I_V(CC_{2m}^n)}u_h^{(j)}\right] + \sum_{i=2}^{2m}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n)} + deg_{I_V(CC_{2m}^n)}u_1^{(1)}\right] = 16nm^2 - 24nm - 8m^2 + 12m + 8n\,.$$

$$D^+_{I_V(CC_{2m}^n)}(2,4) = \sum_{i=2}^{2m}\sum_{j=2}^{n-1}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n)} + deg_{I_V(CC_{2m}^n)}u_1^{(j)}\right] + \sum_{i=3}^{2m-1}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n)} + deg_{I_V(CC_{2m}^n)}u_1^{(n)}\right] + \sum_{i=2}^{m-1}\left[deg_{I_V(CC_{2m}^n)}u_i^{(n-1)} + deg_{I_V(CC_{2m}^n)}u_{m+1}^{(n-1)}\right] + \sum_{i=3}^{m}\left[deg_{I_V(CC_{2m}^n)}u_{m+i}^{(n-1)} + deg_{I_V(CC_{2m}^n)}u_{m+1}^{(n-1)}\right] + \sum_{j=1}^{n-2}\sum_{i=2}^{m}\left[deg_{I_V(CC_{2m}^n)}u_i^{(j)} + deg_{I_V(CC_{2m}^n)}u_{m+1}^{(n-1)}\right] + \sum_{j=1}^{n-2}\sum_{i=2}^{m}\left[deg_{I_V(CC_{2m}^n)}u_{m+i}^{(j)} + deg_{I_V(CC_{2m}^n)}u_{m+1}^{(n-1)}\right] + \left[deg_{I_V(CC_{2m}^n)}u_1^{(1)} + deg_{I_V(CC_{2m}^n)}u_{m+1}^{(n-1)}\right] = 24nm - 24m - 18n\,.$$

$$D^+_{I_V(CC_{2m}^n)}(4,4) = \sum_{j=2}^{n-1}\left[deg_{I_V(CC_{2m}^n)}u_1^{(n)} + deg_{I_V(CC_{2m}^n)}u_1^{(j)}\right] = 8n - 16$$

$$D^+_{I_V(CC_{2m}^{n-1}, u_{m+1}^{n-1})}(2,2) = \left[deg_{I_V(CC_{2m}^{n-1})}u_1^{(1)} + deg_{I_V(CC_{2m}^{n-1})}u_{m+1}^{(n-1)}\right] + \sum_{i=2}^{m-1}\left[deg_{I_V(CC_{2m}^{n-1})}u_i^{(n-1)} + deg_{I_V(CC_{2m}^{n-1})}u_{m+1}^{(n-1)}\right] + \sum_{i=3}^{m}\left[deg_{I_V(CC_{2m}^{n-1})}u_{m+i}^{(n-1)} + deg_{I_V(CC_{2m}^{n-1})}u_{m+1}^{(n-1)}\right] + \sum_{j=1}^{n-2}\sum_{i=2}^{m}\left[deg_{I_V(CC_{2m}^{n-1})}u_i^{(j)} + deg_{I_V(CC_{2m}^{n-1})}u_{m+1}^{(n-1)}\right] + \sum_{j=1}^{n-2}\sum_{i=2}^{m}\left[deg_{I_V(CC_{2m}^{n-1})}u_{m+i}^{(j)} + deg_{I_V(CC_{2m}^{n-1})}u_{m+1}^{(n-1)}\right] = 8nm - 8m - 8n + 4\,.$$

$$D^+_{I_V(CC_{2m}^{n-1}, u_{m+1}^{n-1})}(2,4) = \sum_{j=2}^{n-1}\left[deg_{I_V(CC_{2m}^{n-1})}u_1^{(j)} + deg_{I_V(CC_{2m}^{n-1})}u_{m+1}^{(n-1)}\right] = 6n - 12\,.$$

There fore

$$Z_1^c\left(I_V(RC_{2m}^n)\right) = Z_1^c\left(I_V(RC_{2m}^{n-1})\right) + 16nm^2 - 24nm - 8m^2 + 12m + 8n + 24mn - 24m - 18n + 8n - 16 - 8nm + 8n + 8m - 4 - 6n + 12.$$

$$Z_1^c\left(I_V(RC_{2m}^n)\right) = Z_1^c\left(I_V(RC_{2m}^{n-1})\right) + 8nm(2m-1) - 4(2m^2 + m + 2).$$

2.
$$Z_2^c\left(I_V(CC_{2m}^n)\right) = Z_2^c\left(I_V(CC_{2m}^{n-1})\right) + D^\times_{I_V(CC_{2m}^n)}(2,2) + D^\times_{I_V(CC_{2m}^n)}(2,4) + D^\times_{I_V(CC_{2m}^n)}(4,4) - D^\times_{I_V(CC_{2m}^{n-1}, u_{m+1}^{n-1})}(2,2) - D^\times_{I_V(CC_{2m}^{n-1}, u_{m+1}^{n-1})}(2,4).$$

Since, $D^+_{I_V(CC_{2m}^n)}(2,2) = D^\times_{I_V(CC_{2m}^n)}(2,2)$
$= 16nm^2 - 24nm - 8m^2 + 12m + 8n\,.$
$D^\times_{I_V(CC_{2m}^n)}(2,4) = 32nm - 32m - 24n.$

$D^{\times}_{I_V(CC^n_{2m})}(4,4) = 16n - 32.$
$D^{\times}_{I_V(CC^{n-1}_{2m}, u^{n-1}_{m+1})}(2,2) = 8mn - 8m - 8n + 4.$
$D^{\times}_{I_V(CC^{n-1}_{2m}, u^{n-1}_{m+1})}(2,4) = 8n - 16.$

Then

$$Z^c_2\left(I_V(CC^n_{2m})\right) = Z^c_2\left(I_V(CC^{n-1}_{2m})\right) + 16nm^2 - 4(2m^2 + 3m + 5).$$

Corollary 2.1.2: For all $n, m \geq 2$, then:

1. $Z^c_1\left(I_V(CC^n_{2m})\right) = 4(2n^2m^2 - n^2m - 2nm - 2n + 2).$
2. $Z^c_2\left(I_V(CC^n_{2m})\right) = 4(2n^2m^2 - 3nm - 5n + 5).$

Proof:

From Theorem (2.1.1) and using recurrence relations, we get:

1.
$$\begin{aligned} Z^c_1\left(I_V(CC^n_{2m})\right) &= Z^c_1\left(I_V(CC^{n-1}_{2m})\right) + 8nm(2m-1) - 4(2m^2 + m + 2) \\ &= Z^c_1\left(I_V(CC^{n-2}_{2m})\right) + 8m(n-1)(2m-1) + 8nm(2m-1) - 8(2m^2 + m + 2) \\ &= Z^c_1\left(I_V(CC^{n-3}_{2m})\right) + 8m(n-2)(2m-1) + 8m(n-1)(2m-1) + 8nm(2m-1) - 12(2m^2 + m + 2) \\ &\vdots \\ &= Z^c_1\left(I_V\left(CC^{n-(n-1)}_{2m}\right)\right) + \sum_{i=0}^{n-2} 8m(n-i)(2m-1) - 4(n-1)(2m^2 + m + 2) \end{aligned}$$

Since $Z^c_1\left(I_V(CC^1_{2m})\right) = 8m^2 - 12m$, then

$$Z^c_1\left(I_V(CC^n_{2m})\right) = 4(2n^2m^2 - n^2m - 2nm - 2n + 2).$$

2.
$$\begin{aligned} Z^c_2\left(I_V(CC^n_{2m})\right) &= Z^c_2\left(I_V(CC^{n-1}_{2m})\right) + 16nm^2 - 4(2m^2 + 3m + 5) \\ &= Z^c_2\left(I_V(CC^{n-2}_{2m})\right) + 16m^2(n-1) + 16nm^2 - 8(2m^2 + 3m + 5) \\ &= Z^c_2\left(I_V(CC^{n-3}_{2m})\right) + 16m^2(n-2) + 16m^2(n-1) + 16nm^2 - 12(2m^2 + 3m + 5) \\ &\vdots \\ &= Z^c_2\left(I_V\left(CC^{n-(n-1)}_{2m}\right)\right) + \sum_{i=0}^{n-2} 16m^2(n-i) - 4(n-1)(2m^2 + 3m + 5) \end{aligned}$$

Since $Z^c_2\left(I_V(CC^1_{2m})\right) = 8m^2 - 12m$, then

$$Z^c_2\left(I_V(CC^n_{2m})\right) = 4(2n^2m^2 - 3nm - 5n + 5).$$

Example 2.1.3: The first Zagreb and the second Zagreb of $I_V(CC^n_6)$ for all $n \in N$ are:

1. $Z^c_1\left(I_V(CC^n_6)\right) = 60n^2 - 32n + 8.$
2. $Z^c_2\left(I_V(CC^n_6)\right) = 72n^2 - 56n + 20.$

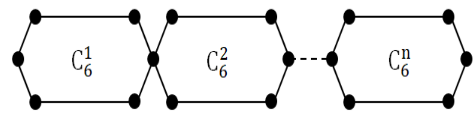

FIGURE 1.3. The graph $I_v(CC_6)$

Theorem 2.2.1: For all $n, m \geq 2$, $r \geq 3$, then:

1. $$Z^c_1\left(I_P(CC^n_{2m})\right) = Z^c_1\left(I_P(CC^{n-1}_{2m})\right) + 4n(4m^2 + 4mr + r^2 - 6m - 3r + 2) - 2(4m^2 - 6m - 6r + 3r^2 + 8mr + 6).$$
2. $$Z^c_2\left(I_P(CC^n_{2m})\right) = Z^c_2\left(I_P(CC^{n-1}_{2m})\right) + 4n(4m^2 + 4mr + r^2 - 4m - 2r + 1) - 2(4m^2 - 2m - 3r + 3r^2 + 8mr + 7.5).$$

Proof:

We add an even cycle C_{2m} to the chain $I_P(CC^{n-1}_{2m})$ by identical the vertex u^n_1 in C^n_{2m} and the vertex u^{n-1}_{m+1} in $I_P(CC^{n-1}_{2m})$ with both ends of the pathP_r, $v^{n-1}_r$and v^{n-1}_1 respectively, we obtain:

1.
$$\begin{aligned} Z^c_1\left(I_P(CC^n_{2m})\right) &= Z^c_1\left(I_P(CC^{n-1}_{2m})\right) + D^+_{I_P(CC^n_{2m})}(2,2) \\ &\quad + D^+_{I_P(CC^n_{2m})}(2,3) + D^+_{I_P(CC^n_{2m})}(3,3) \\ &\quad - D^+_{I_P(CC^{n-1}_{2m}, u^{n-1}_{m+1})}(2,2) - D^+_{I_P(CC^{n-1}_{2m}, u^{n-1}_{m+1})}(2,3). \end{aligned}$$

$$\begin{aligned} &D^+_{I_P(CC^n_{2m})}(2,2) \\ &= \sum_{i=2}^{2m-2}\sum_{j=i+2}^{2m}\left[deg_{I_P(CC^n_{2m})}u^{(n)}_i + deg_{I_P(CC^n_{2m})}u^{(n)}_j\right] \\ &+ \sum_{i=2}^{2m}\sum_{j=1}^{n-1}\sum_{h=2}^{m}\left[deg_{I_P(CC^n_{2m})}u^{(n)}_i + deg_{I_P(CC^n_{2m})}u^{(j)}_h\right] \\ &+ \sum_{i=2}^{2m}\sum_{j=1}^{n-1}\sum_{h=m+2}^{2m}\left[deg_{I_P(CC^n_{2m})}u^{(n)}_i + deg_{I_P(CC^n_{2m})}u^{(j)}_h\right] \\ &+ \sum_{i=2}^{2m}\left[deg_{I_P(CC^n_{2m})}u^{(n)}_i + deg_{I_P(CC^n_{2m})}u^{(1)}_1\right] \\ &+ \sum_{i=2}^{2m}\sum_{j=1}^{n-2}\sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}u^{(n)}_i + deg_{I_P(CC^n_{2m})}v^{(j)}_h\right] \\ &+ \sum_{i=2}^{2m}\sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}u^{(n)}_i + deg_{I_P(CC^n_{2m})}v^{(n-1)}_h\right] \\ &+ \sum_{h=2}^{r-3}\sum_{j=h+2}^{r-1}\left[deg_{I_P(CC^n_{2m})}v^{(n-1)}_h + deg_{I_P(CC^n_{2m})}v^{(n-1)}_j\right] \\ &+ \sum_{i=2}^{r-1}\sum_{j=1}^{n-2}\sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}v^{(n-1)}_i + deg_{I_P(CC^n_{2m})}v^{(j)}_h\right] \\ &+ \sum_{i=2}^{m}\sum_{j=1}^{n-1}\sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}u^{(j)}_i + deg_{I_P(CC^n_{2m})}v^{(n-1)}_h\right] \\ &+ \sum_{i=2}^{m}\sum_{j=1}^{n-1}\sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}u^{(j)}_{m+i} + deg_{I_P(CC^n_{2m})}v^{(n-1)}_h\right] \\ &+ \sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}v^{(n-1)}_h + deg_{I_P(CC^n_{2m})}u^{(n-1)}_{m+1}\right] \\ &= 16nm^2 + 16mnr + 4r^2n - 56nm + 48n - 8m^2 + 44m \\ &\quad - 6r^2 + 34r - 28rn - 16mr - 40\,. \end{aligned}$$

$D^{+}_{I_P(CC^n_{2m})}(2,3)$
$= \sum_{i=3}^{2m-1}\sum_{j=2}^{n}\left[deg_{I_P(CC^n_{2m})}u_i^{(n)} + deg_{I_P(CC^n_{2m})}u_1^{(j)}\right]$
$+ \sum_{j=2}^{n-1}\left[deg_{I_P(CC^n_{2m})}u_2^{(n)} + deg_{I_P(CC^n_{2m})}u_1^{(j)}\right]$
$+ \sum_{j=2}^{n-1}\left[deg_{I_P(CC^n_{2m})}u_{2m}^{(n)} + deg_{I_P(CC^n_{2m})}u_1^{(j)}\right]$
$+ \sum_{i=2}^{2m}\sum_{j=1}^{n-1}\left[deg_{I_P(CC^n_{2m})}u_i^{(n)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(j)}\right]$
$+ \sum_{j=1}^{n-1}\sum_{i=2}^{m}\left[deg_{I_P(CC^n_{2m})}u_i^{(j)} + deg_{I_P(CC^n_{2m})}u_1^{(n)}\right]$
$+ \sum_{j=1}^{n-1}\sum_{i=2}^{m}\left[deg_{I_P(CC^n_{2m})}u_{m+i}^{(j)} + deg_{I_P(CC^n_{2m})}u_1^{(n)}\right]$
$+ \left[deg_{I_P(CC^n_{2m})}u_1^{(1)} + deg_{I_P(CC^n_{2m})}u_1^{(n)}\right]$
$+ \sum_{j=1}^{n-2}\sum_{h=2}^{r-1}\left[deg_{I_P(CC^n_{2m})}v_h^{(j)} + deg_{I_P(CC^n_{2m})}u_1^{(n)}\right]$
$+ \sum_{h=2}^{r-2}\left[deg_{I_P(CC^n_{2m})}v_h^{(n-1)} + deg_{I_P(CC^n_{2m})}u_1^{(n)}\right]$
$+ \sum_{h=2}^{r-1}\sum_{j=2}^{n-1}\left[deg_{I_P(CC^n_{2m})}v_h^{(n-1)} + deg_{I_P(CC^n_{2m})}u_1^{(j)}\right]$
$+ \sum_{h=3}^{r-1}\sum_{j=1}^{n-1}\left[deg_{I_P(CC^n_{2m})}v_h^{(n-1)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(j)}\right]$
$+ \sum_{j=1}^{n-2}\left[deg_{I_P(CC^n_{2m})}v_2^{(n-1)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(j)}\right]$
$+ \sum_{i=2}^{m-1}\left[deg_{I_P(CC^n_{2m})}u_i^{(n-1)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{i=3}^{m}\left[deg_{I_P(CC^n_{2m})}u_{m+i}^{(n-1)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{i=2}^{m}\sum_{j=1}^{n-2}\left[deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)} + deg_{I_P(CC^n_{2m})}u_i^{(j)}\right]$
$+ \sum_{i=2}^{m}\sum_{j=1}^{n-2}\left[deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)} + deg_{I_P(CC^n_{2m})}u_{m+i}^{(j)}\right]$
$+ \sum_{h=2}^{r-1}\sum_{j=1}^{n-2}\left[deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)} + deg_{I_P(CC^n_{2m})}v_h^{(j)}\right]$
$+ \left[deg_{I_P(CC^n_{2m})}u_1^{(1)} + deg_{I_P(CC^n_{2m})}v_1^{(n)}\right]$
$= 40nm - 40m - 70n + 20rn - 30r + 70$.

$D^{+}_{I_P(CC^n_{2m})}(3,3)$
$= \sum_{j=2}^{n-1}[deg_{I_P(CC^n_{2m})}u_1^{(n)} + deg_{I_P(CC^n_{2m})}u_1^{(j)}]$
$+ \sum_{j=1}^{n-1}[deg_{I_P(CC^n_{2m})}u_1^{(n)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(j)}]$
$+ \sum_{j=2}^{n-1}[deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)} + deg_{I_P(CC^n_{2m})}u_1^{(j)}]$
$+ \sum_{j=1}^{n-2}[deg_{I_P(CC^n_{2m})}u_{m+1}^{(n-1)} + deg_{I_P(CC^n_{2m})}u_{m+1}^{(j)}]$
$= 24n - 42$

$D^{+}_{I_P(CC^{n-1}_{2m},u^{n-1}_{m+1})}(2,2)$
$= \left[deg_{I_P(CC^{n-1}_{2m})}u_1^{(1)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{j=1}^{n-2}\sum_{i=2}^{r-1}\left[deg_{I_P(CC^{n-1}_{2m})}v_h^{(j)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{i=2}^{m-1}\left[deg_{I_P(CC^{n-1}_{2m})}u_i^{(n-1)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{i=3}^{m}\left[deg_{I_P(CC^{n-1}_{2m})}u_{m+i}^{(n-1)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{j=1}^{n-2}\sum_{i=2}^{m}\left[deg_{I_P(CC^{n-1}_{2m})}u_i^{(j)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{j=1}^{n-2}\sum_{i=2}^{m}\left[deg_{I_P(CC^{n-1}_{2m})}u_{m+i}^{(j)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$= 4rn - 8r - 16n + 8nm - 8m + 20$.

$D^{+}_{I_P(CC^{n-1}_{2m},u^{n-1}_{m+1})}(2,3)$
$= \sum_{j=2}^{n-1}\left[deg_{I_P(CC^{n-1}_{2m})}u_1^{(j)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$+ \sum_{j=1}^{n-2}\left[deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(j)} + deg_{I_P(CC^{n-1}_{2m})}u_{m+1}^{(n-1)}\right]$
$= 10n - 20$.

There fore,

$$Z_1^c\left(I_P(RC^n_{2m})\right) = Z_1^c\left(I_P(RC^{n-1}_{2m})\right) + 16nm^2 + 16mnr + 4r^2n$$
$$-56nm + 48n - 8m^2 + 44m - 6r^2 + 34r - 28rn - 16mr$$
$$-40 + 40mn - 40m - 70n + 20nr - 30r + 70 + 24n - 42$$
$$-4rn + 8r + 16n - 8nm + 8m - 20 - 10n + 20 .$$
$$= Z_1^c\left(I_P(CC^{n-1}_{2m})\right) + 4n(4m^2 + 4mr + r^2 - 6m - 3r + 2)$$
$$-2(4m^2 - 6m - 6r + 3r^2 + 8mr + 6).$$

2. $Z_2^c\left(I_P(CC^n_{2m})\right) = Z_2^c\left(I_P(CC^{n-1}_{2m})\right) + D^{\times}_{I_P(CC^n_{2m})}(2,2)$
$+D^{\times}_{I_P(CC^n_{2m})}(2,3) + D^{\times}_{I_P(CC^n_{2m})}(3,3)$
$-D^{\times}_{I_P(CC^{n-1}_{2m},u^{n-1}_{m+1})}(2,2) - D^{\times}_{I_P(CC^{n-1}_{2m},u^{n-1}_{m+1})}(2,3).$

Since, $D^{+}_{I_P(CC^n_{2m})}(2,2) = D^{\times}_{I_P(CC^n_{2m})}(2,2)$
$= 16nm^2 + 16mnr + 4r^2n - 56nm + 48n - 8m^2 + 44m$
$-6r^2 + 34r - 28rn - 16mr - 40$.
$D^{\times}_{I_P(CC^n_{2m})}(2,3) = 48nm - 48m + 24nr - 84n - 36r + 84.$
$D^{\times}_{I_P(CC^n_{2m})}(3,3) = 36n - 63.$
$D^{\times}_{I_P(CC^{n-1}_{2m},u^{n-1}_{m+1})}(2,2) = 4rn - 8r - 16n + 8nm - 8m + 20.$
$D^{\times}_{I_P(CC^{n-1}_{2m},u^{n-1}_{m+1})}(2,3) = 12n - 24.$

Then, $Z_2^c\left(I_P(CC^n_{2m})\right) = Z_2^c\left(I_P(CC^{n-1}_{2m})\right)$
$+4n(4m^2 + 4mr + r^2 - 4m - 2r + 1)$
$-2(4m^2 - 2m - 3r + 3r^2 + 8mr + 7.5).$

Corollary 2.2.2 : For all $n, m \geq 2$ and $r \geq 3$, then:

1. $Z_1^c\left(I_P(CC^n_{2m})\right) = 8n^2m^2 + 8mn^2r + 2n^2r^2 - 12mn^2$
$-6rn^2 + 4n^2 - 8mnr - 4r^2n + 6rn - 8n$
$-32m^2 - 16mr + 200m + 46r - 284.$

2. $Z_2^c\left(I_P(CC^n_{2m})\right) = 8n^2m^2 + 8mn^2r + 2n^2r^2 - 8mn^2$
$-4rn^2 + 2n^2 - 8mnr - 4r^2n - 4nm + 2rn$
$-13n - 32m^2 - 16mr + 200m + 48r - 277.$

Proof:

From Theorem (2.2.1) and using recurrence relations, we get:

1. $Z_1^c\left(I_P(CC^n_{2m})\right) = Z_1^c\left(I_P(CC^{n-1}_{2m})\right)$
$+4n(4m^2 + 4mr + r^2 - 6m - 3r + 2)$
$-2(4m^2 - 6m - 6r + 3r^2 + 8mr + 6).$

$$= Z_1^c\left(I_P(CC_{2m}^{n-2})\right) + 4(n-1)(4m^2+4mr+r^2-6m-3r+2) + 4n(4m^2+4mr+r^2-6m-3r+2) - 4(4m^2-6m-6r+3r^2+8mr+6).$$

$$= Z_1^c\left(I_P(CC_{2m}^{n-3})\right) + 4(n-2)(4m^2+4mr+r^2-6m-3r+2) + 4(n-1)(4m^2+4mr+r^2-6m-3r+2) + 4n(4m^2+4mr+r^2-6m-3r+2) - 6(4m^2-6m-6r+3r^2+8mr+6).$$

$$\vdots$$

$$= Z_1^c\left(I_P\left(CC_{2m}^{n-(n-2)}\right)\right) + \sum_{i=0}^{n-3} 4(n-i)(4m^2+4mr+r^2-6m-3r+2) - 2(n-2)(4m^2-6m-6r+3r^2+8mr+6)$$

Since, $Z_1^c\left(I_P(CC_{2m}^{2})\right) = 152m - 34r - 284$ then

$$Z_1^c\left(I_P(CC_{2m}^{n})\right) = 8n^2m^2 + 8mn^2r + 2n^2r^2 - 12mn^2 - 6rn^2 + 4n^2 - 8mnr - 4r^2n + 6rn - 8n - 32m^2 - 16mr + 200m + 46r - 284.$$

2. $Z_2^c\left(I_P(CC_{2m}^{n})\right) = Z_2^c\left(I_P(CC_{2m}^{n-1})\right) + 4n(4m^2+4mr+r^2-4m-2r+1)$

$$-2(4m^2-2m-3r+3r^2+8mr+7.5)$$

$$= Z_2^c\left(I_P(CC_{2m}^{n-2})\right) + 4(n-1)(4m^2+4mr+r^2-4m-2r+1) + 4n(4m^2+4mr+r^2-4m-2r+1) - 2(4m^2-2m-3r+3r^2+8mr+7.5)$$

$$= Z_2^c\left(I_P(CC_{2m}^{n-3})\right) + 4(n-2)(4m^2+4mr+r^2-4m-2r+1) + 4(n-1)(4m^2+4mr+r^2-4m-2r+1) + 4n(4m^2+4mr+r^2-4m-2r+1) - 2(4m^2-2m-3r+3r^2+8mr+7.5)$$

$$\vdots$$

$$= Z_2^c\left(I_P\left(CC_{2m}^{n-(n-2)}\right)\right) + \sum_{i=0}^{n-3} 4(n-i)(4m^2+4mr+r^2-4m-2r+1) - 2(n-2)(4m^2-2m-3r+3r^2+8mr+7.5)$$

Since, $Z_2^c\left(I_P(CC_{2m}^{2})\right) = 160m + 36r - 295$ then

$$Z_2^c\left(I_P(CC_{2m}^{n})\right) = 8n^2m^2 + 8mn^2r + 2n^2r^2 - 8mn^2 - 4rn^2 + 2n^2 - 8mnr - 4r^2n - 4nm + 2rn - 13n - 32m^2 - 16mr + 200m + 48r - 277.$$

Example 2.2.3: The first Zagreb and second Zagreb of $(I_P(CC_6^n))$ for all $n \in N$, where P is a path between any two cycles C_6 are:

1. $Z_1^c\left(I_P(CC_6^n)\right) = 112n^2 - 98n + 22.$

2. $Z_2^c\left(I_P(CC_6^n)\right) = 128n^2 - 127n + 35.$

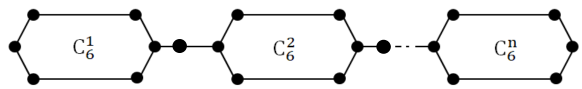

FIGURE 1.4. The graph $I_P(CC_6^n)$

Conclusion

We note from the results that we obtained in the previous, it is also possible to obtain the same results when the cycles are odd, but with the condition that two identical vertices of any cycle are not adjacent. Also can be generalized to cycles that are not of the same length.

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دليلي زغرب لحساب الرؤوس الغير متجاورة لسلاسل من الدارات الزوجية

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الملخص
البيان المستخدمة في هذا البحث هو بيان البسيط المنتهي . يعتمد دليلي زغرب الأول والثاني لكل رؤوس الغير متجاورة (تسمى أيضًا متمم زغرب الأول والثاني) فقط على درجات الرؤوس الغير متجاورة التي تتخللها عمليات الجمع والضرب على التوالي . يُطلق على عدد الحافات الواقعة على الرأس v في البيان G بدرجة الرأس v، في هذا البحث، وجدنا متمم دليلي زغرب الاول والثاني لسلاسل ذات الدارات الزوجية وأيضًا قدمنا بإعطاء بعض الأمثلة.

الكلمات المفتاحية: دليل زغرب الاول المتمم، دليل زغرب الثاني المتمم، التطابق ، الدارة الزوجية.