Strongly Nil* Clean Ideals
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1. INTRODUCTION
In this paper, a ring $R$ is associative with unity $1 \neq 0$ unless otherwise expressed. $J(R)$, $U(R)$, $Id(R)$ and $N(R)$ are respectively Jacobson radical, the set of unit, idempotent and nilpotent elements of respectively. "An ideal $I$ of a unital ring $R$ is clean in case every element in $I$ is a sum of an idempotent and a unit of $R$"[4]. In [1] Sharma and Basnet defined the concept nil clean ideal (henceforth: NCI) as for each $a \in I$, there is a nilpotent element $n$ in $R$ and an idempotent element $e$ in $R$ then $a = e + n$. We call $I$ is strongly nil clean ideal (henceforth: SNCI) if for each $a \in I$ are written as $a = e + n$ where $e \in Id(R), n \in N(R)$ and $en = ne$ [1]. An element $t$ in a ring $R$ is called tripotent if $t^3 = t$ [6]. "An ideal $I$ is called strongly tri nil clean ideal (henceforth: STNCI) if for each element $a \in I$ can be expressed as $a = t + n$ where $t$ is tripotent and $n$ is nilpotent elements with $tn = nt^*$[5].

This paper introduces the concept of strongly nil* clean ideal (henceforth: SN*C). We give some of it's properties, and find it's relationships with NC ideals.

2. Strongly nil* clean ideals:
In this section we introduce the concept of the SN*C. Some of it's characteristics are discussed as well as some examples.

An element $a$ is known a strongly nil* clean element if $a = e_1 - e_2 + n$, where $e_1$, $e_2$ are idempotents and $n$ is nilpotent, that commute with one another. An ideal $I$ of a ring $R$ is called a strongly nil* clean ideal if each element of $I$ is strongly nil* clean element. We investigate some of its fundamental features, as well as its relationship to the nil clean ideal.

Keywords:
Nilpotent element, Jacobson radical, Strongly nil* clean ideal, Idempotent element.

Example 2.1:
An element $a$ of a ring $R$ is said to be strongly nil clean if $a = e + n$, where $e \in Id(R)$ and $n \in N(R)$ and $en = ne$. A ring $R$ is said to be strongly nil clean if every element of $R$ is strongly nil clean[2].

Example 2.2:
The ring of integers modulo 4, $Z_4$ is SNC ring.

Definition 2.3:
An element $a$ of a ring $R$ is known SN*C element if for each $a \in R$ there exist two idempotent elements $e_1$, $e_2$ in $R$ and a nilpotent element $n$ in $R$ that commute with one another, such that $a = e_1 - e_2 + n$. A ring $R$ is said to be SN*C ring if each element of $R$ is SN*C element.

Example 2.4:
The ring of integers modulo 8, $Z_8$ is SN*C ring.

Definition 2.5:
An ideal $I$ of a ring $R$ is known SN*C ideal if each element of $I$ is SN*C element.

Example 2.6:
Consider the ring of integers modulo 16, the ring of $Z_{16}$ contained three proper ideals namely:
$I_1 = \{0, 2, 4, 6, 8, 10, 12, 14 \}$, $I_2 = \{0, 4, 8, 12 \}$ and $I_3 = \{0, 8 \}$. The ideals $I_1$, $I_2$ and $I_3$ of a ring $Z_{16}$ are SN*C ideals.
Lemma 2.7:
Let \(e_1, e_2 \in Id(R)\), with \(e_1 (e_1 e_2) = (e_1 e_2)e_1\). Then \(e_1 - e_1 e_2\) is an idempotent.

Proof:
Since \(e_1 (e_1 e_2) = (e_1 e_2)e_1\), then \(e_1 e_2 = e_1 e_2 e_1\).

Note that:
\[(e_1 - e_1 e_2)^2 = e_1 - e_1 e_2 - e_1 e_2 + e_1 e_2 = e_1 - e_1 e_2.\]

Hence \(e_1 - e_1 e_2\) is an idempotent.

Proposition 2.8:
Let \(I\) be a SN*CI. Then \(I\) is a SNCI.

Proof:
Let \(I\) be a strongly nil* clean ideal. Then for all \(a \in I\), we have \(a = e_1 - e_1 e_2 + n\), where \(e_1\) and \(e_2\) are idempotent elements and \(n\) is a nilpotent that commute with one another. By (Lemma 2.7) \(e_1 - e_1 e_2\) is idempotent.

Then we get \((e_1 - e_1 e_2)^2 = e_1 - e_1 e_2\), idempotent, since \((e_1 - e_1 e_2)n = n(e_1 - e_1 e_2)\).

Hence \(a = e + n\) where \(e = e_1 - e_1 e_2\) and \(n = ne\).

Hence \(I\) is strongly nil – clean ideal.

Next, we give the following results:

Lemma 2.9:
If \(u \in U(R)\) and \(n \in N(R)\) with \(un = nu\), then \(u + n\) is a unit.

Proof:
Since \(n \in N(R)\), then \(n^r = 0\), for some positive integer \(r\). If we set
\[1 = 1 + n^r = (1 + n)(1 - n + \cdots + (-1)^{r-1}n^{r-1}),\]
showing that \(1 + n\) is a unit. Since \(u^{-1} n \in N(R)\), hence \(1 + u^{-1} n \in U(R)\). So \(u + n\) is also unit.

Proposition 2.10:
If \(I\) is SN*CI, and if 2 \(\in I\), then 2 is a nilpotent.

Proof:
Let \(I\) be SN*CI such that \(a \in I\), then
\[a = e_1 - e_1 e_2 + n\]
where \(e_1, e_2 \in Id(R)\) and \(n \in N(R)\) that commute with one another.

By (Lemma 2.7) \((e_1 - e_1 e_2)\) is an idempotent.

Then \(a = e + n\) where \(e = e_1 - e_1 e_2\).

Let \(2 = e + n\) this implies \(1 + 1 = e + n\).

We get \(1 - e = n - 1\), since \(n - 1\) is unit.

Let \(1 - e = u\). Suppose \(1 - e = 1\), hence \(e = 0\).

Then \(2 = 0 + n\). So \(2 \in N(R)\). Then 2 is nilpotent.

Proposition 2.11:
If \(R\) is SN*C ring, then \(J(R)\) is a nil ideal.

Proof:
Suppose \(a \in J(R)\), then \(1 - a\) is a unit. Since \(R\) is SN*C ring, then \(a = e_1 - e_1 e_2 + n\).

Now \(1 - a = 1 - (e_1 - e_1 e_2) - n\), this implies \(u = 1 - (e_1 - e_1 e_2) - n\).

Hence \(u + n = 1 - (e_1 - e_1 e_2)\).

This implies \(u_1 = 1 - (e_1 - e_1 e_2)\) where \(u_1\) is a unit.

Thus \(1 = 1 - (e_1 - e_1 e_2)\), but \(e_1 - e_1 e_2\) is an idempotent. Then \(e_1 - e_1 e_2 = 0\).

So \(e_1 = e_1 e_2\). Therefore \(a = n \in J(R)\).

Proposition 2.12:
Let \(I\) be an ideal of a ring \(R\) with every \(a \in I\), \(a = f - gf + u, fg = gf\) and if \(a^2 - a\) is nilpotent, then \(I\) is SNCI.

Proof:
Since \(fg = gf\) then \((f - gf)\) is an idempotent we get \(a = e + u\). Now \(a^2 = (e + u)^2\).

Then \(a^2 = e + 2eu + u^2\).

Now \(a^2 - a = e + 2eu + u^2 - e - u\).

Thus \(a^2 - a = (2e + u - 1)u \in N(R)\).

On the other hand, \(a = 1 - e + (2e - 1 + u)\).

Since \(2e - 1 + u \in N(R)\). Then \(I\) is SNCI.

Proposition 2.13:
If \(R\) is local ring, and \(I\) is SN*CI of \(R\), then \(I\) is a nil ideal.

Proof:
Let \(R\) be a local ring, then either \(a\) or \(1 - a\) is a unit. Let \(I\) be a SN*CI of \(R\), and let \(a \in I\), if \(a\) is a unit. Then \(I = R\). Let \(1 - a\) is a unit. Since \(I\) is a SN*CI, then \(a = e_1 - e_1 e_2 + n\), where \(e_1, e_2 \in Id(R)\) and \(n \in N(R)\), that commute with one another.

Now \(1 - a = 1 - (e_1 - e_1 e_2) - n\) then \((1 - a + n) = 1 - (e_1 - e_1 e_2)\).

Since \(1 - a\) is a unit we get \(u + n\) also is unit, say \(u_1\). Then \(u_1 = 1 - (e_1 - e_1 e_2)\).

By (Lemma 2.7) \((e_1 - e_1 e_2)\) is also idempotent. Hence \(1 - (e_1 - e_1 e_2) = 1\), this implies \(e_1 = e_1 e_2\). Therefore \(a = n \in I\). Hence \(I\) is a nil ideal.

Lemma 2.14:
Let \(R\) be a ring, with \(2 \in U(R)\), and if \(e\) is idempotent element, then \(1 + e\) is a unit.

Proof:
Let \(e = e^2 \in R\).

Then \((1 + e)(2 - e) = 2 - e + 2e - e = 2\).

Therefore \(1 + e\) is a unit.

Theorem 2.15:
Let \(R\) be a ring, with \(2 \in U(R)\), and \(I\) be a SN*CI, then each element of \(I\), can be written as a sum of two units.

Proof:
Let \(I\) be a SN*CI and \(a \in I\), then
\[a = e_1 - e_1 e_2 + n \in N(R)\] that commute with one another.

Consider \(a = 1 + (e_1 - e_1 e_2) + n - 1\). Since \(e_1 - e_1 e_2\) is an idempotent. Then by (Lemma 2.14) \(1 + (e_1 - e_1 e_2)\) is a unit, say \(u_2\) and \(n - 1\) is a unit, say \(u_1\), then \(a = u_1 + u_2\).

3. Tri nil clean ideal
In this section we give the definition of the tri nil clean ideal. We investigate some of its properties and provide some examples.

Definition 3.1:
An ideal \(I\) is known TNCI if for each element \(a \in I\) can be expressed as \(a = t + n\) where \(t = t^3\) and \(n \in N(R)\) if further \(tn = nt\), then \(I\) is called STNCI[5]. Clearly every NCI is TNCI.
Example 3.2:  
In the ring of integers modulo 6, the ideals of $\mathbb{Z}_6$ are $I_1=\{0, 2, 4\}$ and $I_2=\{0, 3\}$ are TNC ideals.

The next results shows the relation between TNC with strongly clean ideal and nil ideals.

**Proposition 3.3:**  
If $I$ is an ideal with every $a \in I$, $a = t + n$, $tn = nt$ and $t^2 = t$. Then $I$ is a strongly clean ideal.

**Proof:**  
Let $a \in I$, then $a = t + n$, $tn = nt$, $t^2 = t$.  
Consider $t^2 + t - 1$, then $(t^2 + t - 1)^2 = t^2 + t - 2 + t^2 - t - 1 = 1$. Hence $t^2 + t - 1$ is a unit.  
This implies $a = (1 - t^2) + (t^2 + t - 1)$ + $n$.  
Since $t^2 + t - 1$ is a unit, then $a = (1 - t^2) + u + n$ where $u = t^2 + t - 1$.  
by (Lemma2.9) $u + n \in U(R)$. We get $a = (1 - t^2) + u^*$, where $u^* = u + n$, since $(1 - t^2)$ is an idempotent.  
Then $a = e^* + u^*$ where $e^* = 1 - t^2$. Hence $I$ is strongly clean ideal.

**Proposition 3.4:**  
Let $I$ be an ideal of a ring $R$ and $2 \in N(R)$. If every element of $I$, $a = t - t^2 + n$ where $t^2 = t$ and $tn = nt$.  
Then $I$ is a nil ideal of $R$.

**Proof:**  
Let $a = t - t^2 + n$ where $t^2 = t$ and $tn = nt$.  
Now $(t - t^2)^2 = t^2 - 2t + t^2 = 2(t^2 - t)$. Since $2 \in N(R)$. Then $2(t^2 - t) \in N(R)$.  
Let $-t^2 = n_u$, hence $a = n_1 - n$. Then $a \in N(R)$.  
Thus $I$ is a nil ideal of $R$.

**Proposition 3.5:**  
Let $t = t^3$, and let $t \in J(R)$, then $t = 0$.

**Proof:**  
Let $t \in J(R)$ then $t^2 \in J(R)$ then $1 - t^2$ is unit.  
Let $1 - t^2 = u$. Since $(t = t^3)$.  
Then it follows $t^2 - t^2 = t^2u$, then $t^2u = 0$. So $t^2 = 0$. Hence $t = 0$.

**Proposition 3.6:**  
If $I$ is strongly tripotent ideal $a \in I$, $a = t + n$, $t^3 = t$, $n \in N(R)$, if $2 \in N(R)$. Then $I \cap J(R)$ is a nil ideal.

**Proof:**  
Let $a \in I$, $a = t + n$, $tn = nt$, and let $a \in I \cap J(R)$.  
Since $a \in J(R)$ then also $a^2 \in J(R)$, hence $1 - a^2$ is a unit, let $1 - a^2 = u$.  
Now $a^2 = (t + n)^2 = t^2 + 2tn + n^2$. Since $2tn + n^2$ is nilpotent.  
Then $a^2 = t^2 + n \in N(R)$.  
Now $1 - a^2 = 1 - t^2 - n$. Then $u = 1 - t^2 - n$.  
This implies $u + n = 1 - t^2$ since $u + n$ is a unit. Then $u = 1 - t^2$ where $u = u + n$. Since $1 - t^2$ is idempotent.  
Then $1 - t^2 = 1$, we get $t^2 = 0$, hence $t = 0$. We get $a = n$. Thus $I \cap J$ is nil ideal.

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