The Modified Integral Transform Method to Solve Heat Equation in a Cylindrical Coordinate

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ABSTRACT

This paper investigated a modified integral transform method used to solve heat equation in cylindrical coordinate, this modification method has been obtained based on $L_2$ integral transform (x-coordinate), we expand $L_2$ integral transform (x-coordinate) to $L_{24}$ integral transform (x,y,z,t-coordinates) and convert it to cylindrical coordinate denoted by $L_{24c}$ integral transform (r,θ,z,t-coordinates). Finally we used $L_{24c}$ integral transform to solve heat equation in cylindrical coordinate.

Key Words: Heat Equation, Integral Transform Method, Cylindrical Coordinate.

1. Introduction:

Heat transfer topic have great importance in several problems of industrial and environmental. In advance, in energy production and transformation applications, In this field there is no single application that does not include the effects of heat transfer In a way or another. It have a wide participation of power from conventional fossil fuels, sources of nuclear or the use of geothermal energy sources [9]. Many science fields and engineering confrontation linear or non-linear partial differential equations describing the physical phenomena. Many of methods (for instance, exact and approximate methods) can be used to solve differential equations. Often they of difficulty to solve these equations analytically. Such equations can be solved by integral transforms like Fourier and Laplace transforms and the importance of Fourier and Laplace transforms lies in their ability to get algebraic equations from differential equations [3]. In general several researchers used integral transform method to solve heat equation such as:

Kamel Al-Khaled solve heat equation by using finite Fourier Transform [4]. Xiao-Jun Yang found the solution of one dimensional heat-diffusion equation in cartesian coordinate by depending on a new integral transform operator [11]. Also Xiao-Jun Yang used new integral transform for solve a steady heat transfer problem [12]. Ranjit R.Dhunde and G.L.Waghmare used double Laplace Transform to solve one dimensional heat equation in cartesian coordinate [7]. Hamood Ur Rehman, Muzammal Iftikhar, Shoaib Saleem, Muhammad Younis and Abdul Mueed used Quadruple Laplace Transform to solve heat equation in cartesian coordinate [3]. V. S. Kulkarni, K. C. Deshmukh and P. H.Munjankar used finite Hankel transform to solve steady state temperature of the cylinder satisfies the heat conduction equation(r,z,t coordinate) [10]. Also the solution of steady state heat equation(r,θ,t coordinate) was found by using melin transform [1]. Also the solution of one dimension heat equation in cylindrical coordinate got by Laplace Transform [6].
2. Modified Integral Transform Method

L₂ integral transform method given by [5]:

\[ F(p) = L₂(f(x)) = \int_{0}^{\infty} xe^{-p^2x^2} f(x) dx \text{, } p \text{ is parameter} \]

the above transform extended to L₂₂ transform:

\[ F(p, q) = L₂₂(f(x, y)) = \int_{0}^{\infty} \int_{0}^{\infty} xy e^{-p^2x^2-q^2y^2} f(x, y) dx dy \text{, } p, q \text{ are parameters} \]

put L₂₂ transform in the fourth dimension:

\[ F(p, q, v, s) = L₂₄(f(x, y, z, t)) \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} xyt e^{-p^2x^2-q^2y^2-v^2z^2-s^2t^2} f(x, y, z, t) dx dy dz dt \text{, } p, q, v, s \text{ are parameters} \]

by use \( x = r\cos\theta \), \( y = r\sin\theta \), \( z = z \) convert the above integral transform to cylindrical coordinate [2]:

\[ F(p, q, v, s) = L₂₄ₖ(f(r, \theta, z, t)) \]

\[ = \int_{0}^{\infty} \int_{\pi/2}^{\infty} \int_{\pi/2}^{\infty} \int_{\pi/2}^{\infty} (r^2\sin\theta \cos\theta z t) e^{-r^2(p^2\cos^2\theta+q^2\sin^2\theta)} e^{-v^2z^2-s^2t^2} f(r, \theta, z, t) r dr d\theta dz dt \]

\( , p, q, v, s \text{ are parameters} \)

whereas \( L₂₄ₖ \) denoted to cylindrical coordinate of \( L₂₂ \) transform . In the next step, we use \( L₂₄ₖ \) which is modified integral transform method to solve heat equation in cylindrical coordinate. Also \( L₂₄ₖ \) inverse is denoted by \( L⁻¹₂₄ₖ \) and defined by:

\[ L⁻¹₂₄ₖ F(p, q, v, s) = f(r, \theta, z, t) \]

\[ = \frac{1}{2\pi} \int_{a}^{\pi/2} \int_{\beta}^{\infty} \int_{\gamma}^{\infty} \int_{\delta}^{\infty} (r^3\sin\theta \cos\theta z t) e^{r^2(p^2\cos^2\theta+q^2\sin^2\theta)} e^{v^2z^2+s^2t^2} F(p, q, v, s) ds dv dq dp \]

3. Apply \( L₂₄ₖ \) integral transform to solve heat equation

The expression of heat equation in cylindrical coordinate is [8]:

\[ \frac{\partial^2 T}{\partial r^2} + \left( \frac{1}{r} \right) \frac{\partial T}{\partial r} + \left( \frac{1}{r^2} \right) \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = \left( \frac{\rho c_p}{k} \right) \frac{\partial T}{\partial t} \]  \( (1) \)

\( \rho \) : fluid density
\( c_p \) : fluid specific heat
\( k \) : fluid thermal conductivity

with boundary and initial conditions:

\[ T(a, \theta, z, t) = 0 \text{, } T(r, 0, z, t) = 0 \text{, } T(r, \frac{\pi}{2}, z, t) = T_1 \text{, } T(r, \theta, 0, t) = 0 \text{, } T(r, \theta, b, t) = T_a \text{, } 0 \leq r \leq a \text{, } 0 \leq \theta \leq \frac{\pi}{2} \text{, } 0 \leq z \leq b \text{, } t \geq 0 \]

We take \( L₂₄ₖ \) to equation (1):

\[ L₂₄ₖ \left[ \frac{\partial^2 T}{\partial r^2} \right] + L₂₄ₖ \left[ \left( \frac{1}{r} \right) \frac{\partial T}{\partial r} \right] + L₂₄ₖ \left[ \left( \frac{1}{r^2} \right) \frac{\partial^2 T}{\partial \theta^2} \right] + L₂₄ₖ \left[ \frac{\partial^2 T}{\partial z^2} \right] = L₂₄ₖ \left[ \left( \frac{\rho c_p}{k} \right) \frac{\partial T}{\partial t} \right] \]  \( (2) \)

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Now we must find:

\[
L_{24c} \left( \frac{\partial^2 T}{\partial r^2} \right) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left( r^2 \sin\theta \cos\theta \ z \ t \right) e^{-r^2 (p^2 \cos^2\theta + q^2 \sin^2\theta)} e^{-v^2 z^2 - s^2 t^2} \left( \frac{\partial^2 T}{\partial r^2} \right) r \ dr \ d\theta \ dz \ dt
\]

by integrate twice with respect to \( r \) we get:

\[
L_{24c} \left[ \frac{\partial^2 T}{\partial r^2} \right] = L_{24c} [4r^2 ((p^2 \cos^2\theta + q^2 \sin^2\theta) T)] - L_{24c} [8(p^2 \cos^2\theta + q^2 \sin^2\theta) T] \\
- L_{24c} [6(p^2 \cos^2\theta + q^2 \sin^2\theta) T] + L_{24c} [6 \left( \frac{1}{r^2} \right) T] ... (3)
\]

similarly:

\[
L_{24c} \left[ \frac{1}{r} \frac{\partial T}{\partial r} \right] = L_{24c} [2(p^2 \cos^2\theta + q^2 \sin^2\theta) T] - L_{24c} [\left( \frac{2}{r^2} \right) T] ... (4)
\]

\[
L_{24c} \left[ \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right] = L_{24c} [(4 \sin^2\theta \cos^2\theta (p^2 - q^2)^2) T] - L_{24c} [6(p^2 - q^2) \sin^2\theta T] \\
+ L_{24c} [6(p^2 - q^2) \cos^2\theta T] + \left( \frac{1}{2q^2} \right) \left( \frac{1}{2s^2} \right) T \left( \frac{\pi}{2} \right) - L_{24c} [4 \left( \frac{1}{r^2} \right) T] ... (5)
\]

\[
L_{24c} \left[ \frac{\partial^2 T}{\partial z^2} \right] = L_{24c} [4z^2 v^4 T] - L_{24c} [4v^2 T] - L_{24c} [2v^2 T] ... (6)
\]

\[
L_{24c} \left[ \frac{\rho C_p}{k} \frac{\partial T}{\partial t} \right] = L_{24c} [2s^2 \left( \frac{\rho C_p}{k} \right) t T] - L_{24c} \left[ \frac{1}{t} \left( \frac{\rho C_p}{k} \right) T \right] ... (7)
\]

substituting (3), (4), (5), (6) and (7) in (2) we get:

\[
L_{24c} [4r^2 ((p^2 \cos^2\theta + q^2 \sin^2\theta) T)] - L_{24c} [8(p^2 \cos^2\theta + q^2 \sin^2\theta) T] \\
- L_{24c} [6(p^2 \cos^2\theta + q^2 \sin^2\theta) T] + L_{24c} \left[ 6 \left( \frac{1}{t^2} \right) T \right] + L_{24c} [2(p^2 \cos^2\theta + q^2 \sin^2\theta) T] \\
- L_{24c} \left[ \frac{2}{r^2} \right] T] + L_{24c} [(4 \sin^2\theta \cos^2\theta (p^2 - q^2)^2) T] - L_{24c} [6(p^2 - q^2) \sin^2\theta T] \\
+ L_{24c} [6(p^2 - q^2) \cos^2\theta T] + \left( \frac{1}{2q^2} \right) \left( \frac{1}{2s^2} \right) T \left( \frac{\pi}{2} \right) - L_{24c} [4 \left( \frac{1}{r^2} \right) T] \\
+ L_{24c} [4z^2 v^4 T] - L_{24c} [4v^2 T] - L_{24c} [2v^2 T] = L_{24c} [2s^2 \left( \frac{\rho C_p}{k} \right) t T] \\
- L_{24c} \left[ \frac{1}{t} \left( \frac{\rho C_p}{k} \right) T \right]
\]

arrange the above equation:

\[
L_{24c} [4r^2 ((p^2 \cos^2\theta + q^2 \sin^2\theta) T)] - L_{24c} [8(p^2 \cos^2\theta + q^2 \sin^2\theta) T] \\
- L_{24c} [6(p^2 \cos^2\theta + q^2 \sin^2\theta) T] + L_{24c} \left[ 6 \left( \frac{1}{t^2} \right) T \right] + L_{24c} [2(p^2 \cos^2\theta + q^2 \sin^2\theta) T] \\
- L_{24c} \left[ \frac{2}{r^2} \right] T] + L_{24c} [(4 \sin^2\theta \cos^2\theta (p^2 - q^2)^2) T] - L_{24c} [6(p^2 - q^2) \sin^2\theta T] \\
+ L_{24c} [6(p^2 - q^2) \cos^2\theta T] - L_{24c} \left[ 4 \left( \frac{1}{r^2} \right) T \right] + L_{24c} [4z^2 v^4 T] - L_{24c} [4v^2 T]
\]
\[-L_{24c}[2v^2 T] - L_{24c} \left[ 2s^2 \left( \frac{pc_p}{k} \right) T \right] + L_{24c} \left[ \left( \frac{1}{t} \right) \left( \frac{pc_p}{k} \right) T \right] = - \left( \frac{T \left( \frac{p}{T} \right)}{(2s^2)(2v^2)(2q^2)} \right)\]

multiply both sides of the above equation by \(\frac{1}{2p^2}\):
\[
L_{24c} \left[ 2r^2 \left( \frac{1}{p^2} \right) ((p^2 \cos^2 \theta + q^2 \sin^2 \theta)^2 T) \right] - L_{24c} \left[ 4 \left( \frac{1}{p^2} \right) (p^2 \cos^2 \theta + q^2 \sin^2 \theta) T \right]
- L_{24c} \left[ 3 \left( \frac{1}{p^2} \right) (p^2 \cos^2 \theta + q^2 \sin^2 \theta) T \right] + L_{24c} \left[ 3 \left( \frac{1}{p^2} \right) \left( \frac{1}{r^2} \right) T \right]
+ L_{24c} \left[ \left( \frac{1}{p^2} \right) (p^2 \cos^2 \theta + q^2 \sin^2 \theta) T \right] - L_{24c} \left[ \left( \frac{1}{p^2} \right) \left( \frac{1}{r^2} \right) T \right]
+ L_{24c} \left[ \left( \frac{1}{p^2} \right) (2 \sin^2 \theta \cos^2 \theta (p^2 - q^2)^2 T) \right] - L_{24c} \left[ 3 \left( \frac{1}{p^2} \right) (p^2 - q^2) \sin^2 \theta T \right]
+ L_{24c} \left[ 3 \left( \frac{1}{p^2} \right) (p^2 - q^2) \cos^2 \theta T \right] - L_{24c} \left[ 2 \left( \frac{1}{p^2} \right) \left( \frac{1}{r^2} \right) T \right] + L_{24c} \left[ 2 \left( \frac{1}{p^2} \right) z^2 v^4 T \right]
- L_{24c} \left[ 2 \left( \frac{1}{p^2} \right) v^2 T \right] - L_{24c} \left[ \left( \frac{1}{p^2} \right) v^2 T \right] - L_{24c} \left[ s^2 \left( \frac{pc_p}{k} \right) \left( \frac{1}{p^2} \right) T \right]
+ L_{24c} \left[ \left( \frac{1}{t} \right) \left( \frac{1}{2p^2} \right) \left( \frac{pc_p}{k} \right) T \right] = \left( \frac{T \left( \frac{p}{T} \right)}{(2s^2)(2v^2)(2q^2)(2p^2)} \right)\]

take \(L_{24c}^{-1}\) to the above equation:
\[
\left[ 2r^2 \left( \frac{1}{p^2} \right) ((p^2 \cos^2 \theta + q^2 \sin^2 \theta)^2 T) \right] - \left[ 4 \left( \frac{1}{p^2} \right) (p^2 \cos^2 \theta + q^2 \sin^2 \theta) T \right]
- \left[ 3 \left( \frac{1}{p^2} \right) (p^2 \cos^2 \theta + q^2 \sin^2 \theta) T \right] + \left[ 3 \left( \frac{1}{p^2} \right) \left( \frac{1}{r^2} \right) T \right]
+ \left[ \left( \frac{1}{p^2} \right) (p^2 \cos^2 \theta + q^2 \sin^2 \theta) T \right] - \left[ \left( \frac{1}{p^2} \right) \left( \frac{1}{r^2} \right) T \right]
+ \left[ \left( \frac{1}{p^2} \right) (2 \sin^2 \theta \cos^2 \theta (p^2 - q^2)^2 T) \right] - \left[ 3 \left( \frac{1}{p^2} \right) (p^2 - q^2) \sin^2 \theta T \right]
+ \left[ 3 \left( \frac{1}{p^2} \right) (p^2 - q^2) \cos^2 \theta T \right] - \left[ 2 \left( \frac{1}{p^2} \right) \left( \frac{1}{r^2} \right) T \right] + \left[ 2 \left( \frac{1}{p^2} \right) z^2 v^4 T \right]
- \left[ 2 \left( \frac{1}{p^2} \right) v^2 T \right] - \left[ \left( \frac{1}{p^2} \right) v^2 T \right] - \left[ s^2 \left( \frac{pc_p}{k} \right) \left( \frac{1}{p^2} \right) T \right]
+ \left[ \left( \frac{1}{t} \right) \left( \frac{1}{2p^2} \right) \left( \frac{pc_p}{k} \right) T \right] = \left( \frac{T \left( \frac{p}{T} \right)}{(2s^2)(2v^2)(2q^2)(2p^2)} \right)\]

the above equation reduce to:
\[
T \left( \frac{2r^2}{p^2} (p^2 \cos^2 \theta + q^2 \sin^2 \theta) - \frac{6}{p^2} (p^2 \cos^2 \theta + q^2 \sin^2 \theta) + \frac{2}{p^2} \sin^2 \theta \cos^2 \theta (p^2 - q^2)^2 + \frac{3}{p^2} (p^2 - q^2) \cos \theta + \frac{1}{p^2} z^2 v^4 - 3 \left( \frac{1}{p^2} \right) v^2 - \frac{s^2}{p^2} \left( \frac{pc_p}{k} \right) T + \left( \frac{pc_p}{k} \right) \left( \frac{1}{2p^2} \right) \right) =

\left( \frac{T \left( \frac{p}{T} \right)}{(2s^2)(2v^2)(2q^2)(2p^2)} \right) \ldots (8)\]

it is clear that:
\[
L_{24c}(k) = \frac{k}{(2s^2)(2v^2)(2q^2)(2p^2)} , \text{ so } k = L_{24c}^{-1}(\frac{k}{(2s^2)(2v^2)(2q^2)(2p^2)}) \text{ where } k \text{ is any constant.}
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rewrite the equation (8) gives:
\[ T\left(\frac{2r^2}{p^2}\right)(p^2 \cos^2 \theta + q^2 \sin^2 \theta) - \frac{2}{p^2} (p^2 \cos^2 \theta + q^2 \sin^2 \theta) + \frac{2}{p^2} \sin^2 \theta \cos^2 \theta \left( p^2 - q^2 \right)^2 + \frac{3}{p^2} (p^2 - q^2) \cos 2\theta + \frac{2}{p^2} \frac{z^2}{v^4} - 3\frac{1}{p^2}v^2 - \frac{x^2}{p^2}\left( \frac{pc}{k} \right) t + \left( \frac{pc}{k} \right) \frac{1}{2p^2 t} \right] = -T\left(\frac{\pi}{2}\right) \quad (9) \]
from boundary condition:
\[ T\left(\frac{\pi}{2}\right) = T\left( r, \frac{\pi}{2}, z, t \right) = T_1 \]
after substituting in (9) and simplicity we get:
\[ T = T(r, \theta, z, t) = -T_1\left[\frac{2r^2}{p^2}\right](p^2 \cos^2 \theta + q^2 \sin^2 \theta) - \frac{6}{p^2} (p^2 \cos^2 \theta + q^2 \sin^2 \theta) + \frac{2}{p^2} \sin^2 \theta \cos^2 \theta \left( p^2 - q^2 \right)^2 + \frac{3}{p^2} (p^2 - q^2) \cos 2\theta + \frac{2}{p^2} \frac{z^2}{v^4} - 3\frac{1}{p^2}v^2 - \frac{x^2}{p^2}\left( \frac{pc}{k} \right) t + \left( \frac{pc}{k} \right) \frac{1}{2p^2 t} \]
Which is the solution of the heat equation in cylindrical coordinate.

4. Conclusion

In this paper, we extend \( L_2 \) transform to \( L_{24} \) transform which is converted to the cylindrical coordinate denoted by \( L_{24c} \) transform. \( L_{24c} \) transform has an advantage compare with hankel transform, millen transform and quadruple Laplace transform because of \( L_{24c} \) transform solve heat equation in cylindrical coordinate \((r,0,z,t)\) which is partial differential equation with variable coefficients and hankel transform solve heat equation in cylindrical coordinate \((r,z,t)\) \([10]\), millen transform solve heat equation in cylindrical coordinate \((r,0,t)\) \([1]\), quadruple Laplace transform solve heat equation in cartesian coordinate \((x,y,z,t)\) \([3]\) which is partial differential equation with constant coefficients \([3]\).
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