The n-Hosoya Polynomials of the Square of a Path and of a Cycle

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ABSTRACT

The n-Hosoya polynomial of a connected graph G of order t is defined by:

\[ H_n(G; x) = \sum_{k=0}^{\delta_n(G)} C_n(G,k)x^k \]

Where, \( C_n(G,k) \) is the number of pairs (v,S), in which |

\[ |S| = n - 1, \quad 3 \leq n \leq t, \quad v \in V(G), \quad S \subseteq V(G) \]

such that \( d_n(v,S) = k \), for each \( 0 \leq k \leq \delta_n = \text{diam}_n(G) \).

In this paper, we find the n-Hosoya polynomial of the square of a path and of the square of a cycle. Also, the n-diameter and n-Wiener index of each of the two graphs are determined.

**Keyword:** n-diameter, n-Hosoya polynomial, n-Wiener index, path square and cycle square.
1. Introduction:

The \textbf{n-distance} \cite{1} in a connected graph $G = (V, E)$ of order $t$ is the minimum distance from a singleton, $v \in V$ to an $(n-1)$-subset $S$, $S \subseteq V$, $3 \leq n \leq t$, that is, $d_n(v, S) = \min \{d(v, u) : u \in S\}$, $3 \leq n \leq t$.

It is clear that $d_n(v, S) = 0$ when $v \in S$,
\[ d_n(v, S) \geq 1 \] when $v \not\in S$.

The \textbf{n-Wiener index} of a connected graph $G = (V, E)$ is the sum of the minimum distances of all pairs $(v, S)$ in the graph $G$, that is:
\[ W_n(G) = \sum_{(v, S) \in V \times S, |S| = n-1} d_n(v, S), \quad 3 \leq n \leq t. \]

The \textbf{n-diameter} of $G$ is defined by:
\[ \text{diam}_n G = \max \{d_n(v, S) : v \in V(G), |S| = n-1, S \subseteq V(G)\}. \]

Now, let $C_n(G, k)$ be the number of pairs $(v, S)$, $|S| = n-1, 3 \leq n \leq t, v \in V$, $S \subseteq V$, such that $d_n(v, S) = k$, for each $0 \leq k \leq \delta_n = \text{diam}_n(G)$, then the \textbf{n-Hosoya polynomial} of $G$ is defined by:
\[ H_n(G; x) = \sum_{k=0}^{\delta_n} C_n(G, k)x^k. \]

We can obtain the n-Wiener index of $G$ from the n-Hosoya polynomial of $G$ as follows:
\[ W_n(G) = \frac{d}{dx} H_n(G; x) \Big|_{x=1} = \sum_{k=1}^{\delta_n} k C_n(G, k). \]

For a vertex $v$ of a connected graph $G$, let $C_n(v, G, k)$ be the number of $(n-1)$-subsets $S$ of vertices of $G$ such that $d_n(v, S) = k$, for $n \geq 3, 0 \leq k \leq \delta_n$. The \textbf{n-Hosoya polynomial} of the vertex $v$, denoted by $H_n(v, G; x)$, is defined as:
\[ H_n(v, G; x) = \sum_{k=0}^{\infty} C_n(v, G, k)x^k. \]

It is clear that for all $k \geq 0$,
\[ \sum_{v \in V(G)} C_n(v, G, k) = C_n(G, k), \]
and
\[ \sum_{v \in V(G)} H_n(v, G; x) = H_n(G; x). \]

For more information about these concepts, see the References \cite{1, 2, 5, 6}.

The next lemma will be used in proving our results.

\textbf{Lemma 1.1:} \cite{1} Let $v$ be any vertex of a connected graph $G$. If there are $r$ vertices of distance $k \geq 1$ from $v$, and there are $s$ vertices of distance more than $k$ from $v$, then, for $n \geq 3$, ...
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\[ C_n(v, G, k) = \binom{r+s}{n-1} - \binom{s}{n-1}. \]  

...(1.1)

**Definition 1.2:** Let $G$ be a connected non-trivial graph. The square $G^2$ of the graph $G$, introduced by Harary and Ross [7], has $V(G^2) = V(G)$ with $u, v$ adjacent in $G^2$, whenever $1 \leq d_G(u,v) \leq 2$.

Notice that the square of complete graph, star graph, wheel graph, complete bipartite graph are complete graphs.

In [1,2,3,4], the $n$-Hosoya polynomials for many special graphs and many compound graphs are obtained. In this paper, we continue such works by obtaining the $n$-Hosoya polynomials of the square of paths and cycles.

2. The $n$-Hosoya Polynomial of the Square of a Path:

In this section, we obtained the $n$-Hosoya polynomial of the square $P_t^2$ of a path $P_t$ of order $t$. We shall consider two main cases of $P_t^2$ according to the parity of $t$.

**First Case:** Even $t$, $t = 2r$, $r \geq 2$.

Let $P_t: u_1, u_2, u_3, \ldots, u_t$, then $P_t^2$ is shown in Fig.2.1, and by relabeling its vertices, we have Fig. 2.2 for $P_{2r}^2$.

**Second Case:** If $t$ is odd, then there exists an integer $r$ such that $t = 2r + 1$. The graph $P_t^2$ is shown in Fig.2.3.
**Theorem 2.1:** For \( t \geq 5 \) and \( n \geq 2 \), let \( r = \left\lfloor \frac{t}{2} \right\rfloor \), then,

\[
\text{diam}_n(P^2_t) = \begin{cases} 
  r + 1 - \left\lfloor \frac{n}{2} \right\rfloor, & \text{for even } t, \\
  r + 1 - \left\lceil \frac{n}{2} \right\rceil, & \text{for odd } t.
\end{cases}
\]

**Proof:**

(1). Let \( t \) be even, then \( t = 2r \).

From Fig.2.2, we notice that \( \text{diam}(P^2_{2r}) = d(v_1, v_{r+1}) = r \), then \( \text{diam}_n(P^2_{2r}) = d_n(v_1, S) \), \( n \geq 2 \), where \( S \) consists of the first \( n-1 \) vertices from the sequence \( \{v_{r+1}, v_r, v_{r+2}, v_{r+3}, v_{r+4}, \ldots, v_2, v_1\} \).

Thus, if \( n \) is even, then

\[
S = \{v_{r+1}, v_r, v_{r+2}, v_{r+3}, v_{r+4}, \ldots, v_{r+\frac{n}{2}}, v_{r+\frac{n}{2}+1}\}, n = 4, 6, 8, \ldots, 2r.
\]

So, \( d_n(v_1, S) = r + 1 - \frac{n}{2} \).

If \( n \) is odd, then

\[
S = \{v_{r+1}, v_r, v_{r+2}, v_{r+3}, v_{r+4}, \ldots, v_{r+\frac{n+1}{2}}, v_{r+\frac{n+1}{2}+1}\}, n = 3, 5, 7, \ldots, 2r-1.
\]

So, \( d_n(v_1, S) = r + 1 - \frac{n+1}{2} \).

Therefore, \( \text{diam}_n(P^2_{2r}) = r + 1 - \left\lfloor \frac{n}{2} \right\rfloor \), for all \( n \geq 2 \).

(2). Let \( t \) be odd, then \( t = 2r+1 \).

From Fig.2.3, we notice that \( \text{diam}(P^2_{2r+1}) = d(v_1, v_{r+1}) = r \) (or \( d(v_1, v_{r+2}) \), or \( d(v_{2r+1}, v_{r+1}) \)), then \( \text{diam}_n(P^2_{2r+1}) = d_n(v_1, S) \), \( \lceil S \rceil = n-1, n \geq 2 \), where \( S \) consists of the first \( n-1 \) vertices from the sequence \( \{v_{r+1}, v_{r+2}, v_r, v_{r+3}, v_{r+4}, \ldots, v_2, v_1\} \).

Thus, if \( n \) is odd, then

\[
S = \{v_{r+1}, v_{r+2}, v_r, v_{r+3}, v_{r+4}, \ldots, v_{r+\frac{n-1}{2}}, v_{r+\frac{n+1}{2}}\}, n = 3, 5, 7, \ldots, 2r+1.
\]

So, \( d_n(v_1, S) = d(v_1, v_{r+\frac{n-1}{2}}) = r + 1 - \frac{n-1}{2} \).

If \( n \) is even, then

\[
S = \{v_{r+1}, v_{r+2}, v_r, v_{r+3}, v_{r+4}, \ldots, v_{r+\frac{n}{2}}, v_{r+\frac{n}{2}+1}\}, n = 4, 6, 8, \ldots, 2r.
\]

So, \( d_n(v_1, S) = d(v_1, v_{r+\frac{n}{2}}) = r + 1 - \frac{n}{2} \).

Therefore,

\[
\text{diam}_n(P^2_{2r+1}) = r + 1 - \left\lceil \frac{n}{2} \right\rceil, \text{ for all } n \geq 2.
\]
Remark: Throughout this work, we assume that \( \left( \frac{a}{b} \right) = 0 \), if \( a < b \).

**Theorem 2.2:** For any \( n \geq 3 \), the n-Hosoya polynomial of \( P_i^2 \), \( t \geq 6 \), is given by:

\[
H_n(P_i^2;x) = \sum_{k=0}^{\delta_n} C_n(P_i^2,k)x^k,
\]

Where, \( \delta_n = \text{diam}_n(P_i^2) \),

\[
C_n(P_i^2,0) = \binom{t-1}{n-2}, \quad \text{...}(2.2.1)
\]

\[
C_n(P_i^2,1) = \binom{t-1}{n-1} - 2\left[ \binom{t-3}{n-1} + \binom{t-4}{n-1} \right] - (t-4)\binom{t-5}{n-1}, \quad \text{...}(2.2.2)
\]

\[
C_n(P_i^2,k) = 2\left[ \binom{t-2k}{n-1} + \binom{t-2k+1}{n-1} + (t-4k+2)\binom{t-4k+3}{n-1} \right]
- 2 \sum_{i=0}^{2} \binom{t-4k+i}{n-1} - (t-4k)\binom{t-4k-1}{n-1}, \quad 2 \leq k \leq \left[ \frac{\delta_n}{2} \right]. \quad \text{...}(2.2.3)
\]

\[
C_n(P_i^2,k) = 2\left[ \binom{t-2k}{n-1} + \binom{t-2k+1}{n-1} \right], \quad \left[ \frac{\delta_n}{2} \right] + 1 \leq k \leq \delta_n. \quad \text{...}(2.2.4)
\]

**Proof:** It is clear that \( C_n(P_i^2,0) = \binom{t-1}{n-2} \).

From Fig.2.2, we notice that in \( P_i^2 \), there are two vertices of degree 2, two vertices of degree 3, and \( t - 4 \) vertices of degree 4. Thus, using formula (1.4.5) in [1], we obtain (2.2.2).

For each vertex \( w \) and given \( k \), let

\[
S_1(w,k) = \{ v \in V : d(w,v) = k \} ,
\]

\[
S_2(w,k) = \{ v \in V : d(w,v) > k \} .
\]

**First,** we shall prove (2.2.3) and (2.2.4) for **even** \( t \), assuming \( t = 2r \), \( r \geq 4 \). It is clear, from Fig. 2.2, that for \( n \geq 3 \),

\[
C_n(v_i,P_i^2,k) = C_n(v_{i+1},P_i^2,k), \quad \text{...}(2.2.5)
\]

for \( i = 1,2, \ldots, r \). Therefore, for \( 2 \leq k \leq \delta_n \),

\[
C_n(P_i^2,k) = 2 \sum_{i=1}^{r} C_n(v_i,P_i^2,k). \quad \text{...}(2.2.6)
\]

Now, let \( 2 \leq k \leq \left[ \frac{\delta_n}{2} \right] \), in which \( \delta_n \) is determined by Theorem 2.1, that is

\[
\delta_n = r + 1 - \left[ \frac{n}{2} \right].
\]

Since, \( n \geq 3 \), then \( \delta_n \leq r - 1 \), for \( r \geq 4 \).
But, in proving (2.2.3), we assume that $\delta_n \geq 4$.

According to the given value of $k$, we partition $\{v_1,v_2, \ldots, v_r\}$ into the following four cases:

(1). For $i = 1,2, \ldots, k$, we notice, from Fig. 2.2, that:

$S_t(v_i,k) = \{v_{i+k},v_{2r+2-i-k}\}$,

$S_2(v_i,k) = V(P_t^2) - \{v_1,v_2, \ldots, v_{r+k},v_{2r+2-i-k},v_{2r+3-i-k}, \ldots, v_{2r}\}$.

Thus,

$|S_t(v_i,k)| = 2$, $|S_2(v_i,k)| = t+1-2k-2i$.

So, by Lemma 1.1, we have, for $i = 1,2, \ldots, k$,

$C_n(v_i,P_t^2,k) = \binom{t+3-2k-2i}{n-1} - \binom{t+1-2k-2i}{n-1}$. \hspace{1cm} \text{(c1)}$

(2). For $i = 1,2, \ldots, k-1$, we obtain, from Fig. 2.2,

$S_t(v_{r+1-i},k) = \{v_{r-i+1},v_{r+k+1}\}$,

$S_2(v_{r+1-i},k) = V(P_t^2) - \{v_{r-k+i+1},v_{r-k+i+2}, \ldots, v_{r},v_{r+1}, \ldots, v_{r+2k}\}$.

Thus,

$|S_t(v_{r+1-i},k)| = 2$, $|S_2(v_{r+1-i},k)| = t-2k-2i$.

So, using Lemma 1.1, we obtain, for $i = 1,2, \ldots, k-1$,

$C_n(v_{r+1-i},P_t^2,k) = \binom{t+2-2k-2i}{n-1} - \binom{t-2k-2i}{n-1}$. \hspace{1cm} \text{(c2)}$

(3). For $v_{r-k+1}$, we have

$S_t(v_{r-k+1},k) = \{v_{r+1},v_{2r+2-k},v_{r+1-2k}\}$,

$S_2(v_{r-k+1},k) = V(P_t^2) - \{v_{r+2k+1},v_{r+2k+2}, \ldots, v_{r},v_{r+1}, \ldots, v_{r+2k}\}$.

Thus,

$|S_t(v_{r-k+1},k)| = 3$, $|S_2(v_{r-k+1},k)| = t-4k$.

So, using Lemma 1.1, we get,

$C_n(v_{r-k+1},P_t^2,k) = \binom{t+3-4k}{n-1} - \binom{t-4k}{n-1}$. \hspace{1cm} \text{(c3)}$

(4). For $i = k+1,k+2, \ldots, r-k$,

$S_t(v_i,k) = \{v_{i-k},v_{i+k},v_{2r+k-i+1},v_{2r-k-i+2}\}$,

$S_2(v_i,k) = V(P_t^2) - \{v_{i-k},v_{i-k+1}, \ldots, v_{i+k},v_{2r-k-i+2},v_{2r-k-i+3}, \ldots, v_{2r+k-i+1}\}$.

Thus,

$|S_t(v_i,k)| = 4$, $|S_2(v_i,k)| = t-4k-1$.

Therefore, using Lemma 1.1, we get, for $i = k+1,k+2, \ldots, r-k$,

$C_n(v_i,P_t^2,k) = \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1}$. \hspace{1cm} \text{(c4)}$

Thus, from (2.2.6) and summing up the formulas (c1)-(c4) we get for $2 \leq k \leq \left\lfloor \frac{\delta_n}{2} \right\rfloor$. 

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$$C_n(P^2, k) = 2 \left\{ \sum_{i=1}^{k} \left[ \frac{(t + 3 - 2k - 2i)}{n - 1} - \frac{(t + 1 - 2k - 2i)}{n - 1} \right] + \sum_{i=1}^{k} \left[ \frac{(t + 2 - 2k - 2i)}{n - 1} - \frac{(t - 2k - 2i)}{n - 1} \right] + \left( \frac{t - 4k + 3}{n - 1} \right) - \left( \frac{t - 4k}{n - 1} \right) + (r - 2k) \left[ \frac{(t - 4k + 3)}{n - 1} - \frac{(t - 4k - 1)}{n - 1} \right] \right\}. $$

$$= 2 \left\{ \left[ \frac{(t - 2k + 1)}{n - 1} \right] - \left[ \frac{(t - 4k + 1)}{n - 1} \right] + \left[ \frac{(t - 2k)}{n - 1} \right] - \left[ \frac{(t - 4k + 2)}{n - 1} \right] + \left( \frac{t - 4k + 3}{n - 1} \right) - \left( \frac{t - 4k}{n - 1} \right) \right\}. $$

Now, we give the proof of (2.2.4) for $\left\{ \frac{\delta_n}{2} \right\} + 1 \leq k \leq \delta_n$. Here, we have two cases:

(a). For $i = 1, 2, \ldots, r - k$,

$$S_i(v, k) = \{ v_{r + i}, v_{2r + 2 - i - k} \},$$

$$S_2(v, k) = V(P^2) - \{ v_1, v_2, \ldots, v_{r + k}, v_{2r + 2 - i - k}, v_{2r + 3 - i - k}, \ldots, v_{2r} \}. $$

Thus,

$$|S_i(v, k)| = 2, \quad |S_2(v, k)| = t + 1 - 2k - 2i. $$

So, by Lemma 1.1, we have, for $i = 1, 2, \ldots, r - k$,

$$C_n(v, P^2, k) = \left( \frac{t - 2k - 2i + 3}{n - 1} \right) - \left( \frac{t - 2k - 2i + 1}{n - 1} \right). $$

(b). For $v_{r + i - 1}, i = 1, 2, \ldots, r - k$, we have

$$S_i(v_{r + i - 1}, k) = \{ v_{r - k - i + 1}, v_{r + k + i} \},$$

$$S_2(v_{r + i - 1}, k) = V(P^2) - \{ v_{r - k - i + 1}, v_{r - k - i + 2}, \ldots, v_{r}, v_{r + 1}, \ldots, v_{r + k + i} \}. $$

Thus,

$$|S_i(v_{r + i - 1}, k)| = 2, \quad |S_2(v_{r + i - 1}, k)| = t - 2k - 2i. $$

So, by Lemma 1.1, we have, for $i = 1, 2, \ldots, r - k$,

$$C_n(v_{r + i - 1}, P^2, k) = \left( \frac{t - 2k - 2i + 2}{n - 1} \right) - \left( \frac{t - 2k - 2i}{n - 1} \right). $$

Therefore, using (2.2.6) and summing up (d1) and (d2), we get for $\left\{ \frac{\delta_n}{2} \right\} + 1 \leq k \leq \delta_n$. 

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\[ C_n(P_t^2, k) = 2 \left( \sum_{i=1}^{t-1} \left( \begin{array}{c} t-2k-2i+3 \\ n-1 \end{array} \right) - \left( \begin{array}{c} t-2k-2i+1 \\ n-1 \end{array} \right) \right) + \sum_{i=1}^{t-1} \left( \begin{array}{c} t-2k-2i+2 \\ n-1 \end{array} \right) - \left( \begin{array}{c} t-2k-2i \\ n-1 \end{array} \right) \right) \\
= 2 \left[ \left( \begin{array}{c} t-2k+1 \\ n-1 \end{array} \right) - \left( \begin{array}{c} t-2r+1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} t-2k \\ n-1 \end{array} \right) - \left( \begin{array}{c} t-2r \\ n-1 \end{array} \right) \right] \\
= 2 \left[ \left( \begin{array}{c} t-2k+1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} t-2k \\ n-1 \end{array} \right) \right], \text{ because } n \geq 3. \\
\]

**Second**, the proofs of (2.2.3) and (2.2.4) for odd \( t, \ t = 2r+1, \ r \geq 3, \) are similar to the proofs of (2.2.3) and (2.2.4) for even \( t. \)

Hence, the proof of the Theorem is completed. 

**Corollary 2.3:** The \( n \)-Wiener index of \( P_t^2 \) is given by:

\[ W_n(P_t^2) = t \left( \begin{array}{c} t-1 \\ n-1 \end{array} \right) - 2 \left( \begin{array}{c} t-3 \\ n-1 \end{array} \right) + \left( \begin{array}{c} t-4 \\ n-1 \end{array} \right) - (t-4) \left( \begin{array}{c} t-5 \\ n-1 \end{array} \right) + \sum_{k=2}^n \delta C_n(P_t^2, k), \]

in which

\[ C_n(P_t^2, k) = 2 \left[ \left( \begin{array}{c} t-2k \\ n-1 \end{array} \right) + \left( \begin{array}{c} t-2k+1 \\ n-1 \end{array} \right) \right] + (t-4m+2) \left( \begin{array}{c} t-4k+3 \\ n-1 \end{array} \right) \\
- 2 \sum_{i=0}^{n-2} \left( \begin{array}{c} t-4k+i \\ n-1 \end{array} \right) - (t-4k) \left( \begin{array}{c} t-4k-1 \\ n-1 \end{array} \right), 2 \leq k \leq \left\lfloor \frac{\delta}{2} \right\rfloor, \]

\[ C_n(P_t^2, k) = 2 \left[ \left( \begin{array}{c} t-2k \\ n-1 \end{array} \right) + \left( \begin{array}{c} t-2k+1 \\ n-1 \end{array} \right) \right] + \left\lfloor \frac{\delta}{2} \right\rfloor + 1 \leq k \leq \delta. \n\]

**3. The \( n \)-Hosoya Polynomial of the Square of a Cycle:**

There are many classes of connected graphs \( G \) in which for each \( k, \ 1 \leq k \leq \delta, \)
\( C_n(v, G, k) \) is the same for every vertex \( v \in V(G); \) such graphs are called [2] vertex-\( n \)-distance regular graphs, and for the given value of \( n, \ 2 \leq n \leq t, \)
\( H_n(G; x) = tH_n(v, G; x), \) where \( v \) is any vertex of \( G \) and \( t \) is the order of \( G. \)

The graph \( C_t^2 \) is the square of a cycle of order \( t, \) shown in Fig. 3.1. We shall find the \( n \)-diameter, \( n \)-Hosoya polynomial, and \( n \)-Wiener index of \( C_t^2. \)
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**Fig. (3.1)** The Cycle Square $C_t^2$, $t \geq 6$.

**Lemma 3.1:** $\text{diam}_n(C_t^2) = \delta_n = 1 + \left\lfloor \frac{t - n}{4} \right\rfloor$, $n \geq 2$, $t \geq 6$.

**Proof:** Let $m = \left\lfloor \frac{t}{4} \right\rfloor$, then $t = 4m + r$, $r = 0, 1, 2, 3$.

For $r = 2$, $C_t^2$ is redrawn in Fig. 3.2.

Since, $C_t^2$ is vertex n-distance regular graph, then $\text{diam}_n(C_t^2) = e_n(v_i)$.

To find the n-eccentricity of $v_i$, we partition $V(C_t^2) - \{v_i\}$ into $S_1, S_2, \ldots, S_{m+1}$, where

$S_1 = \{v_2, v_3, v_1, v_{i-1}\}$,

$S_2 = \{v_4, v_5, v_{i-2}, v_{i-3}\}$,

$S_3 = \{v_6, v_7, v_{i-4}, v_{i-5}\}$,

$\cdot$

$S_j = \{v_{2j}, v_{2j+1}, v_{i-2(j-1)}, v_{i-2j+1}\}$,

$\cdot$

$S_m = \{v_{2m}, v_{2m+1}, v_{i-2m+2}, v_{i-2m+1}\}$,

$S_{m+1} = V(C_t^2) - \left(\bigcup_{j=1}^{m} S_j \cup \{v_i\}\right)$. 


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Fig. (3.2). The Cycle Square $C_t^2$, $t = 4m + 2$, $m \geq 1$.

It is clear that each vertex of $S_j$, $1 \leq j \leq m$, is of (standard) distance $j$ from $v_1$; and each of the other vertices (if exists) of $C_t^2$ (here in Fig. 3.2, we have \{v_{2m+2}\} = S_{m+1}$) is of the distance $m+1$ from $v_1$. Notice that if $t = 4m+1$, then $S_{m+1}$ is empty, and if $t = 4m$, then $S_{m+1}$ is empty and $S_m$ consists of three elements; if $t = 4m+2$, $t = 4m+3$, then $S_{m+1}$ consists of one, respectively two, elements.

Let $k$ be the greatest positive integer such that the set $\bigcup_{i=k}^{m+1} S_i$ consists of at least $(n-1)$ vertices. Therefore, since $|S_i| \leq 4$.

$$4(k-1)+1+(n-1) \leq t,$$

$$4k \leq t-n+4,$$

$$k \leq \frac{t-n}{4}+1.$$ 

Therefore, $\text{diam}_n(C_t^2) = k = 1 + \left\lfloor \frac{t-n}{4} \right\rfloor$, (\because k is positive integer). #

**Theorem 3.2:** For any $n \geq 3$, the $n$-Hosoya polynomial of $C_t^2$, $t \geq 6$ is given by:

$$H_n(C_t^2; x) = \left( t-1 \right)_{n-2} + \sum_{k=1}^{\delta_n-1} \left( \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1} \right) x^k + C_n(C_t^2, \delta_n)x^{\delta_n},$$

Where, $C_n(C_t^2, \delta_n)$ is determined in Remark 3.3, and $\delta_n$ is determined by Lemma 3.1.

**Proof:** Let $S$ be a set of (n-1) vertices of $V(C_t^2)$ such that $v_1 \not\in S$, $v_i \in V(C_t^2)$ and $d_n(v_1, S) = k$, $2 \leq k \leq \delta_n-1$. Hence, $S$ does not contain any vertex from \{v_{t-2k+3}, \ldots, v_{t-1}, v_1, v_2, v_3, \ldots, v_{2k-1}\}, (see Fig. 3.1), but $S$ must contain, at least,
one vertex of \( \{ v_{2k}, v_{2k+1}, v_{t-2k+2}, v_{t-2k+1} \} \). Then, the number of vertices in \( C_{t}^2 \) of distance more than \( k \) from \( v_i \) is \((t-4k-1)\) and there are four vertices in \( C_{t}^2 \) of distance \( k \) from \( v_i \). Hence, by Lemma 1.1,

\[
C_n(v_i, C_{t}^2, k) = \binom{t - 4k + 3}{n - 1} - \binom{t - 4k - 1}{n - 1}, \quad \text{for } 2 \leq k \leq \delta_n - 1.
\]

Moreover, it is clear that

\[
C_n(v_i, C_{t}^2, 1) = \binom{t - 1}{n - 1} - \binom{t - 5}{n - 1}.
\]

Since \( C_n(v_i, C_{t}^2, k) = C_n(v_i, C_{t}^2, k), \ 2 \leq i \leq t, \) then

\[
C_n(C_{t}^2, k) = \begin{cases} 
\binom{t - 4k + 3}{n - 1} - \binom{t - 4k - 1}{n - 1}, & \text{for } 1 \leq k \leq \delta_n - 1.
\end{cases}
\]

**Remark 3.3:** From Fig. 3-2, we can easily obtain \( C_n(C_{t}^2, \delta_n) \), for \( n \geq 3 \).

1. If \( t = 4m + 3 \), then,

\[
C_n(C_{t}^2, \delta_n) = \begin{cases} 
t, & n = 3 \end{cases}; \quad n \geq 4.
\]

2. If \( t = 4m + 2, \ 4m + 1 \), then,

\[
C_n(C_{t}^2, \delta_n) = \begin{cases} 
\binom{t - 4\delta_n + 3}{n - 1} - \binom{t - 4\delta_n - 1}{n - 1}, & n \geq 3.
\end{cases}
\]

3. If \( t = 4m \), then,

\[
C_n(C_{t}^2, \delta_n) = \begin{cases} 
\binom{3}{n - 1}, & n = 3, 4 \end{cases}; \quad n \geq 5.
\]

**Corollary 3.4:** The \( n \)-Wiener index of \( C_{t}^2 \) is given by:

\[
W_n(C_{t}^2) = \sum_{k=1}^{\delta_n} k C_n(C_{t}^2, k), \quad \text{where } C_n(C_{t}^2, k), \ 1 \leq k \leq \delta_n \text{ is given in Theorem 3.2 and Remark 3.3}.
\]
REFERENCES


