Hosoya Polynomial, Wiener Index, Coloring and Planar of Annihilator Graph of Z_n

Mohammed S. Ahmed

mohammed.shaker.msa@gmail.com

*Department of Mathematics,
College of Computer Science and
Mathematics,
University of Mosul, Mosul, Iraq*

Akram S. Mohammed

*Department of Mathematics,
College of Computer Science and
Mathematics,
University of Tikrit, Tikrit, Iraq*

Nabeel E. Arif

Received on: 13/04/2020

Accepted on: 20/05/2020

ABSTRACT

Let R be a commutative ring with identity. We consider $\Gamma_B(R)$ an annihilator graph of the commutative ring R . In this paper, we find Hosoya polynomial, Wiener index, Coloring, and Planar annihilator graph of Z_n denote $\Gamma_B(Z_n)$, with $n = p^m$ or $n = p^m q$, where p, q are distinct prime numbers and m is an integer with $m \geq 1$.

Keywords: Annihilator graph of ring, Zero-divisor graph, Hosoya polynomial, Wiener index, coloring graph and planar graph.

متعددة حدود هوسويا، دليل وينر، التلوين والاستواء لبيان تالف الحلقة الإبدالية Z_n

نبيل عز الدين عارف

أكرم سالم محمد

محمد شاكر أحمد

*قسم الرياضيات، كلية علوم الحاسوب والرياضيات،
جامعة تكريت، تكريت، العراق*

*قسم الرياضيات، كلية علوم الحاسوب والرياضيات،
جامعة الموصل، الموصل، العراق*

تاريخ قبول البحث: ٢٠٢٠/٠٥/٢٠

تاريخ استلام البحث: ٢٠٢٠/٠٤/١٣

الملخص

لتكن R حلقة إبدالية بعنصر محايد. نعتبر $\Gamma_B(R)$ بيان لتالف R . في هذا البحث، وجدنا متعددة حدود هوسويا، دليل وينر، التلوين والاستواء لبيان تالف Z_n نرمز له $\Gamma_B(Z_n)$ ، حيث $n = p^m$ أو $n = p^m q$ بحيث p, q عدداً أوليان مختلفان و m عدد صحيح أكبر أو يساوي واحد.

الكلمات المفتاحية: بيان تالف الحلقة، بيان القواسم الصفرية، متعددة حدود هوسويا، دليل وينر، تلوين البيان واستواء البيان.

1. Introduction

Let R be a commutative ring with identity the annihilator of R is the set of all element $x \in R$ satisfy $ann(R) = \{x \in R: x.y = 0, \forall y \in R\}$ [6], and let $ann(R)$ be the set of all annihilator in R . We consider a simple graph $\Gamma_B(R)$ to R with vertices $ann(R)$, for every two distinct vertices x, y are adjacent if and only if $\{x.y = 0: x, y \in ann(R)\}$, and let $Z(R)$ be the set of all zero-divisors in R , and $Z(R)^*$ is the set of all non-zero zero-divisors in it. A simple graph $\Gamma(R)$ with vertices $Z(R)^*$, for every two distinct vertices x, y are adjacent if and only if $\{x.y = 0: x, y \in Z(R)^*\}$.

The notion of an annihilator graph of a commutative ring was first introduced in 1988 by Beck [5], where he was interested in colorings, this investigation was then continued by Anderson and Nasser [3] zero-divisor graph of a commutative ring, further that Anderson and Livingston [2]. They denoted that by $\Gamma(R)$. It is clear that from Beck's definition of annihilator graph of a commutative ring and Anderson's definition of a zero-divisor graph of a commutative ring can be defined Annihilator graph of a commutative ring can be defined $\Gamma_B(R) = ((\Gamma(R) \cup \{ann(R^*) - Z(R)^*\}) + k_1)$. Such that: $\Gamma(R)$ zero-divisor graph of the ring, $ann(R^*)$ set of all vertices in R non-zero, $Z(R)^*$ set of all non-zero zero-divisors in R and $k_1 = 0$.

A graph G is called a connected graph if there is at least one path between any pair of vertices in G , otherwise it is called disconnected [7]. For vertices x, y of G , let $d(x, y)$ be the length of the shortest path from x to y (and it is called distance between two vertices x, y in G). The maximum distance between any two vertices x, y in G is called the diameter graph G [7], that is $diam(G) = \max_{x, y \in V(G)} \{d(x, y)\}$, where $V(G)$ is the set of all vertices of G . A graph G is complete if every two of its vertices are adjacent, so the complete graph of order n is denoted by K_n . If the vertex set of a graph G can be split into two disjoint sets A and B (such that the induced subgraph that generated by either A or B is null graph), then we said G is a bipartite graph. This graph is also said to be a complete bipartite graph is a bipartite graph in graph if each vertex in the set A has joined to every vertex in the set B with just one edge.

Hosoya polynomial of the graph G is defined by $H(G; x) = \sum_{k=0}^{diam(G)} d(G, k)x^k$, where $d(G, k)$ is the number of pairs of vertices of a graph G , that are at distance k apart, for $k = 0, 1, 2, \dots, diam(G)$. The Wiener index of G is defined as the sum of all distances between vertices of the graph G , and denoted by $W(G)$, we can also find this index by differentiating Hosoya polynomial with respect to x at $x=1$, by symbols we can write: $W(G) = \left. \frac{d}{dx} H(G; x) \right|_{x=1}$, See [8,12].

Let $\chi(G)$ denote the chromatic number of vertices, i.e., the minimal number of colors, which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors [7]. We let $\tilde{\chi}(G)$ denote the chromatic number of edges, i.e., the minimal number of colors, which can be assigned to the edges of G in such a way that every two adjacent edges have different colors [7]. And last we assumed $f(G)$ denote the chromatic number of faces, i.e., the minimal number of colors, that can be assigned to the faces of planar graph G in such a way that every two adjacent faces have different colors [7]. A planar graph G is a graph that can be drawn in the plane without crossings for any two edges in G [7]. There are many studies in the graph properties and commutative ring. See [1],[4],[10]&[11].

2. Some Properties of Graph $\Gamma_B(Z_{p^m})$

We will start this item by a lemma.

Lemma 2.1: The vertex (0) connect with every vertex of the graph $\Gamma_B(Z_n)$.

Proof: Since $0 \cdot a = 0, \forall a \in Z_n$, so it is the vertex (0) connect with every vertex of the graph $\Gamma_B(Z_n)$.

Lemma 2.2 [7]: Let G be a connected graph of order p , then:

$$\sum_{k=0}^{diam(G)} d(G, k) = \binom{p+1}{2} = \frac{1}{2}p(p+1).$$

Theorem 2.3: The Hosoya polynomial of graph $\Gamma_B(Z_{p^m})$ where p is a prime number and m is an integer with $m \geq 1$.

$$H(\Gamma_B(Z_{p^m}); x) = p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right] x + \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right] x^2.$$

Proof: From the definition of the graph $\Gamma_B(R)$, since the vertex (0) connect with every vertex of the graph $\Gamma_B(Z_n)$, so the order of the graph $\Gamma_B(Z_n)$ which represents absolute term of Hosoya polynomial of the graph $\Gamma_B(Z_{p^m})$.

Now, we find the coefficient of x that represent size of the graph $\Gamma_B(Z_{p^m})$ using the definition of the graph $\Gamma_B(R)$ is the sum of $(p^m - 1)$ of the edges (since the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m})$ from the Lemma (2.1), with a_1 of the graph $\Gamma(Z_{p^m})$ [9] where as $\left(a_1 = \frac{1}{2} \left[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2 \right] \right)$ so we get.

$$\begin{aligned} a_1 + (p^m - 1) &= \frac{1}{2} \left[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2 \right] + (p^m - 1) \\ &= \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right]. \end{aligned}$$

Now, we find the coefficient of x^2 as the diameter of the graph $\Gamma_B(Z_{p^m})$ is two from the Lemma (2.1) and using Lemma (2.2) so we get:

$$\begin{aligned} \sum_{k=0}^{\text{diam}(\Gamma_B(Z_{p^m}))} d(\Gamma_B(Z_{p^m}), k) &= \binom{p^m + 1}{2} \\ \Rightarrow \frac{p^m(p^m+1)}{2} &= d(\Gamma_B(Z_{p^m}), 0) + d(\Gamma_B(Z_{p^m}), 1) + d(\Gamma_B(Z_{p^m}), 2) \\ d(\Gamma_B(Z_{p^m}), 2) &= \frac{p^m(p^m+1)}{2} - d(\Gamma_B(Z_{p^m}), 0) - d(\Gamma_B(Z_{p^m}), 1) \\ &= \frac{p^m(p^m+1)}{2} - p^m - \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right] \\ &= \frac{1}{2} \left[p^{2m} + p^m - 2p^m - mp^m - p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right] \\ &= \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right]. \blacksquare \\ \therefore H(\Gamma_B(Z_{p^m}); x) &= p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right] x \\ &\quad + \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right] x^2. \end{aligned}$$

Corollary 2.4: The Wiener index of graph $\Gamma_B(Z_{p^m})$ where p is prime number and m is an integer with $m \geq 1$.

$$W(\Gamma_B(Z_{p^m})) = \frac{1}{2} \left[2p^{2m} - (m+3)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right].$$

Proof: Since wiener index is the first derivative polynomial of Hosoya after compensation for a value $x = 1$ so we get:

$$\begin{aligned} \therefore W(\Gamma_B(Z_{p^m})) &= \left. \frac{d}{dx} H(\Gamma_B(Z_{p^m}); x) \right|_{x=1} \\ \therefore W(\Gamma_B(Z_{p^m})) &= \left. \frac{d}{dx} \left(p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right] x + \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right] x^2 \right) \right|_{x=1} \\ &= 0 + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right] \end{aligned}$$

$$\begin{aligned}
 & + \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right] x \Big|_{x=1} \\
 & = \frac{1}{2} \left[mp^m + p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2p^{2m} - 2mp^m - 4p^m \right. \\
 & \quad \left. + 2mp^{m-1} + 2p^{\lfloor \frac{m}{2} \rfloor} \right] \\
 & = \frac{1}{2} \left[2p^{2m} - (m+3)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right]. \blacksquare
 \end{aligned}$$

Example 1: The Hosoya polynomial and wiener index of graph $\Gamma_B(Z_{16})$.

The graph is clear $\Gamma_B(Z_{16})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 2$ and $m = 4$.

$$\begin{aligned}
 \therefore H(\Gamma_B(Z_{p^m}); x) &= p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} \right] x \\
 & \quad + \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right] x^2.
 \end{aligned}$$

$$\therefore H(\Gamma_B(Z_{16}); x) = 16 + 22x + 98x^2.$$

$$\therefore W(\Gamma_B(Z_{p^m})) = \frac{1}{2} \left[2p^{2m} - (m+3)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} \right].$$

$$\therefore W(\Gamma_B(Z_{16})) = 218.$$

Theorem 2.5: (Coloring of graph $\Gamma_B(Z_{p^m})$).

$$\text{A- Chromatic number of vertices of the graph } \Gamma_B(Z_{p^m}) = \begin{cases} p^{\frac{m-1}{2}} + 1, & m \text{ is an odd.} \\ p^{\frac{m}{2}}, & m \text{ is an even.} \end{cases}$$

B- Chromatic number of edges of the graph $\Gamma_B(Z_{p^m})$ is $p^m - 1$.

Proof: A- Case 1: if m is an odd.

Since the multiplication of the number $p^{\frac{m+1}{2}}$ by one of its complication $(2p^{\frac{m+1}{2}}, 3p^{\frac{m+1}{2}}, \dots, p^{\frac{m-1}{2}} \cdot p^{\frac{m+1}{2}} = 0)$, that the product is one of its complications of the number p^m which is equal to (0) in the ring Z_{p^m} . Or multiplication one complication of the number $p^{\frac{m+1}{2}}$ in another complication of the number $p^{\frac{m+1}{2}}$ that the product is one of the complications of the number p^m which is equal to (0) in the ring Z_{p^m} as in the Figure (2.1).

Clearly the complete graph $k_{\frac{m-1}{p^{\frac{m-1}{2}}}}$ be subgraph from the graph $\Gamma_B(Z_{p^m})$ (when m is an odd), and also the multiplication of the number $p^{\frac{m-1}{2}}$ or one of its complications $(2p^{\frac{m-1}{2}}, 3p^{\frac{m-1}{2}}, \dots, p^{\frac{m+1}{2}} \cdot p^{\frac{m-1}{2}} = 0)$ by the number $p^{\frac{m+1}{2}}$ or one of its complications is the product p^m or one of its complications which is equal to (0), in the ring Z_{p^m} . And thus the complete graph $k_{\frac{m-1}{p^{\frac{m-1}{2}}+1}}$ is the largest complete subgraph that exist in the graph $\Gamma_B(Z_{p^m})$. And since the chromatic number of vertices of a complete graph $k_{\frac{m-1}{p^{\frac{m-1}{2}}+1}}$ is $(p^{\frac{m-1}{2}} + 1)$ [7], so it is the chromatic number of the graph $\Gamma_B(Z_{p^m})$ is $(p^{\frac{m-1}{2}} + 1)$ and also of vertices (when m is an odd).

A- Case 2: if m is an even:

Since the multiplication of the number $p^{\frac{m}{2}}$ by one of its complications $(2p^{\frac{m}{2}}, 3p^{\frac{m}{2}}, \dots, p^{\frac{m}{2}} \cdot p^{\frac{m}{2}} = 0)$, that the product is one of its complications of the number p^m which is equal to (0) in the ring Z_{p^m} . Or multiplying one complication of the number $p^{\frac{m}{2}}$ in another complication of the number $p^{\frac{m}{2}}$, that the product is one of its complications of the number p^m , which is equal to (0) in the ring Z_{p^m} as in the Figure (2.2).

Fig (2.2)

Clearly the complete graph $k_{\frac{m}{p^{\frac{m}{2}}}}$ is the largest complete subgraph that exist in the graph $\Gamma_B(Z_{p^m})$ (when m is an even). And since the chromatic number of vertices of a complete graph $k_{\frac{m}{p^{\frac{m}{2}}}}$ is $p^{\frac{m}{2}}$ [7], so it is the chromatic number of the graph $\Gamma_B(Z_{p^m})$ is $p^{\frac{m}{2}}$ and also of vertices (when m is an even).

B- From the Lemma (2.1) the vertex (0) connect with every vertex in the graph $\Gamma_B(Z_{p^m})$ then the degree of the vertex (0) is $(p^m - 1)$ so the chromatic number of the edges is $(p^m - 1)$.

Theorem 2.6 [7]: (kuratowski's Theorem), The graph G is planar if and only if it does not contain G on subgraph that is homeomorphic to k_5 or $k_{3,3}$.

Theorem 2.7:

- a- The graph $\Gamma_B(Z_{p^m})$ contains a subgraph that is homeomorphic to $k_{\frac{m-1}{p^{\frac{m-1}{2}}+1}}$ and $k_{\left(\frac{m+1}{p^{\frac{m-1}{2}}}-p^{\frac{m-1}{2}}\right),p^{\frac{m-1}{2}}}$ (when m is an odd).
- b- The graph $\Gamma_B(Z_{p^m})$ contains a subgraph that is homeomorphic to $k_{\frac{m}{p^{\frac{m}{2}}}}$ and $k_{\left(\frac{m+2}{p^{\frac{m}{2}}}-p^{\frac{m-2}{2}}\right),p^{\frac{m-2}{2}}}$ (when m is an even).

Proof:

- a- From the Theorem (2.5-A-1) we get the first part of the Theorem directly.

Since the multiplication of the number $p^{\frac{m+1}{2}}$ or one of its complications $\left(2p^{\frac{m+1}{2}}, 3p^{\frac{m+1}{2}}, \dots, p^{\frac{m-1}{2}} \cdot p^{\frac{m+1}{2}} = 0\right)$ by number $p^{\frac{m-1}{2}}$ or one of its complications $\left(2p^{\frac{m-1}{2}}, 3p^{\frac{m-1}{2}}, \dots, p^{\frac{m+1}{2}} \cdot p^{\frac{m-1}{2}} = 0\right)$ be the product p^m or one of its complications of the number p^m which is equal to (0) in the ring Z_{p^m} . Thus, the graph $\Gamma_B(Z_{p^m})$ contains a subgraph homeomorphic with complete bipartite graph $k_{\left(\frac{m+1}{p^{\frac{m+1}{2}}}-p^{\frac{m-1}{2}}\right),p^{\frac{m-1}{2}}}$ it is the largest complete bipartite graph there is in the graph $\Gamma_B(Z_{p^m})$ as in the Figure (2.3).

The graph $\Gamma_B(Z_{p^m})$

Fig (2.3)

- b- From the Theorem (2.5-A-2) we get the first part of the Theorem directly.

Since the multiplication of the number $p^{\frac{m+2}{2}}$ or one of its complications $\left(2p^{\frac{m+2}{2}}, 3p^{\frac{m+2}{2}}, \dots, p^{\frac{m-2}{2}} \cdot p^{\frac{m+2}{2}} = 0\right)$ by number $p^{\frac{m-2}{2}}$ or one of its complications $\left(2p^{\frac{m-2}{2}}, 3p^{\frac{m-2}{2}}, \dots, p^{\frac{m+2}{2}} \cdot p^{\frac{m-2}{2}} = 0\right)$ be the product p^m or one of its complications of the number p^m which is equal to (0) in the ring Z_{p^m} . Thus, the graph $\Gamma_B(Z_{p^m})$ contains a subgraph homeomorphic complete bipartite graph $k_{\left(\frac{m+2}{p^{\frac{m+2}{2}}}-p^{\frac{m-2}{2}}\right),p^{\frac{m-2}{2}}}$ it is the largest complete bipartite graph there is in the graph $\Gamma_B(Z_{p^m})$ as in the Figure (2.4).

 The graph $\Gamma_B(Z_{p^m})$

Fig (2.4)

Remarks:

1. From the Theorem (2.7-a), the only graphs $\Gamma_B(Z_p)$ and $\Gamma_B(Z_8)$ from the formula $\Gamma_B(Z_{p^m})$ when m is an odd it does not contain subgraph homeomorphic k_5 or $k_{3,3}$ therefore it is planar graphs by kuratowski's Theorem.
2. From the Theorem (2.7-b), the only graphs $\Gamma_B(Z_4)$, $\Gamma_B(Z_9)$ and $\Gamma_B(Z_{16})$ from the formula $\Gamma_B(Z_{p^m})$ when m is an even it does not contain subgraph homeomorphic k_5 or $k_{3,3}$ therefore it is planar graphs by kuratowski's Theorem.
3. The only graphs $\Gamma_B(Z_4)$, $\Gamma_B(Z_8)$, $\Gamma_B(Z_9)$, $\Gamma_B(Z_{16})$ and $\Gamma_B(Z_p)$ they are colorable for faces.

Example 2: The chromatic number of the graphs $\Gamma_B(Z_{16})$ and $\Gamma_B(Z_{27})$.

The graph is clear $\Gamma_B(Z_{16})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 2$ and $m = 4$ and the graph is clear $\Gamma_B(Z_{27})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 3$ and $m = 3$.

The chromatic number of vertices the graph $\Gamma_B(Z_{p^m})$ is $p^{\frac{m}{2}}$ (when m is an even).

$$\therefore \chi(\Gamma_B(Z_{16})) = 4.$$

The chromatic number of edges the graph $\Gamma_B(Z_{p^m})$ is $p^m - 1$.

$$\therefore \tilde{\chi}(\Gamma_B(Z_{16})) = 15.$$

From Theorem (2.7-b) we get the graph $\Gamma_B(Z_{16})$ contains a subgraph that is homeomorphic to k_4 and $k_{6,2}$ then the graph $\Gamma_B(Z_{16})$ it is planar by kuratowski's Theorem.

$$\therefore f(\Gamma_B(Z_{16})) = 3.$$

The chromatic number of vertices the graph $\Gamma_B(Z_{p^m})$ is $p^{\frac{m-1}{2}} + 1$ (when m is an odd).

$$\therefore \chi(\Gamma_B(Z_{27})) = 4.$$

The chromatic number of edges the graph $\Gamma_B(Z_{p^m})$ is $p^m - 1$.

$$\therefore \tilde{\chi}(\Gamma_B(Z_{27})) = 26.$$

From Theorem (2.7-a) we get the graph $\Gamma_B(Z_{27})$ contains a subgraph that is homeomorphic to $k_{6,3}$ then the graph $\Gamma_B(Z_{27})$ it is not planar by kuratowski's Theorem.

3. Some Properties of graph $\Gamma_B(Z_{p^m q})$.

Theorem 3.1: The Hosoya polynomial of graph $\Gamma_B(Z_{p^m q})$ where p, q are distinct prime numbers and m is an integer with $m \geq 1$.

$$H(\Gamma_B(Z_{p^m q}); x) = p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x \\ + \left(\frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x^2.$$

Proof: From the definition of the graph $\Gamma_B(R)$ since the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m q})$ so the order of the graph $\Gamma_B(Z_{p^m q})$ which represents absolute term Hosoya polynomial of graph $\Gamma_B(Z_{p^m q})$.

Now, we find the coefficient of x that represent size of the graph $\Gamma_B(Z_{p^m q})$ using the definition of the graph $\Gamma_B(R)$ is the sum of $(Z_{p^m q} - 1)$ of the edges (since the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m q})$ from the Lemma (2.1), with a_1 of the graph $\Gamma(Z_{p^m q})$ [9] where as $\left(a_1 = \frac{1}{2} [2mq(p - 1) - (m + 1)p + m] p^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 1 \right)$ so we get.

$$a_1 + (p^m q - 1) = \frac{1}{2} [2mq(p - 1) - (m + 1)p + m] p^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 1 + p^m q - 1 \\ = \frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor}.$$

Now, we find the coefficient of x^2 as the diameter of the graph $\Gamma_B(Z_{p^m q})$ is two from the Lemma (2.1) and using Lemma (2.2) so we get.

$$\sum_{k=0}^{\text{diam}(\Gamma_B(Z_{p^m q}))} d(\Gamma_B(Z_{p^m q}), k) = \binom{p^m q + 1}{2} \\ \Rightarrow \frac{p^m q(p^m q + 1)}{2} = d(\Gamma_B(Z_{p^m q}), 0) + d(\Gamma_B(Z_{p^m q}), 1) + d(\Gamma_B(Z_{p^m q}), 2) \\ d(\Gamma_B(Z_{p^m q}), 2) = \frac{p^m q(p^m q + 1)}{2} - d(\Gamma_B(Z_{p^m q}), 0) - d(\Gamma_B(Z_{p^m q}), 1) \\ = \frac{p^m q(p^m q + 1)}{2} - p^m q - \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right). \\ = \frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor}. \blacksquare \\ \therefore H(\Gamma_B(Z_{p^m q}); x) = p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x \\ + \left(\frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x^2.$$

Corollary 3.2: The Wiener index of $\Gamma_B(Z_{p^m q})$ where p, q are distinct prime numbers and m is an integer with $m \geq 1$.

$$W(\Gamma_B(Z_{p^m q})) = \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor}$$

Proof: Since wiener index is the first derivative polynomial of Hosoya after compensation for a value $x = 1$ so we get:

$$\therefore W(\Gamma_B(Z_{p^m q})) = \frac{d}{dx} H(\Gamma_B(Z_{p^m q}); x) \Big|_{x=1}$$

$$\therefore W(\Gamma_B(Z_{p^m q})) = \frac{d}{dx} \left(p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x \right. \\ \left. + \left(\frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x^2 \right) \Big|_{x=1}$$

$$\begin{aligned}
 &= \left(0 + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) \right. \\
 &+ \left. \left([q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x \right|_{x=1} \\
 &= \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor}. \blacksquare
 \end{aligned}$$

Example 3: The Hosoya polynomial and wiener index of graph $\Gamma_B(Z_{18})$.

The graph is clear $\Gamma_B(Z_{18})$ of formula $\Gamma_B(Z_{p^m q})$, where $p = 3, q = 2$ and $m = 2$.

$$\begin{aligned}
 \therefore H(\Gamma_B(Z_{p^m q}); x) &= p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x \\
 &+ \left(\frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor} \right) x^2 \\
 \therefore H(\Gamma_B(Z_{18}); x) &= 18 + 30x + 123x^2. \\
 \therefore W(\Gamma_B(Z_{p^m q})) &= \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{\lfloor \frac{m}{2} \rfloor}. \\
 \therefore W(\Gamma_B(Z_{18})) &= 276.
 \end{aligned}$$

Theorem 3.3: (Coloring of graph $\Gamma_B(Z_{p^m q})$).

A- Chromatic number of vertices of the graph

$$\Gamma_B(Z_{p^m q}) = \begin{cases} p^{\frac{m-1}{2}} + 2, & m \text{ is an odd.} \\ p^{\frac{m}{2}} + 1, & m \text{ is an even.} \end{cases}$$

B- Chromatic number of edges of the graph $\Gamma_B(Z_{p^m q})$ is $p^m q - 1$.

Proof: A- Case 1: if m is an even:

From the Theorem (2.5-A-1). Since the subgraph $k_{\frac{p^{\frac{m-1}{2}} + 1}}^{\frac{m-1}{2}}$ is the largest complete subgraph exist in the graph $\Gamma_B(Z_{p^m})$ (when m is an odd). It is also clear that the number $p^m q$ product of multiplication the number p^m or one of its complications in the number q or one of its complications thus a new vertex will be added to the complete graph $k_{\frac{p^{\frac{m-1}{2}} + 1}}^{\frac{m-1}{2}}$ so we have the complete graph $k_{\frac{p^{\frac{m-1}{2}} + 2}}^{\frac{m-1}{2}}$ is the largest complete subgraph exist in the graph $\Gamma_B(Z_{p^m q})$ hence the chromatic number of the graph $\Gamma_B(Z_{p^m q})$ is $\left(p^{\frac{m-1}{2}} + 2 \right)$ [7].

A- Case 2: if m is an even:

From the Theorem (2.5-A-2). Since the subgraph $k_{\frac{p^{\frac{m}{2}}}}^{\frac{m}{2}}$ is the largest complete subgraph, exist in the graph $\Gamma_B(Z_{p^m})$ (when m is an even). It is also clear that the number $p^m q$ product of multiplication the number p^m or one of its complications in the number q or one of its complications thus a new vertex will be added to the complete graph $k_{\frac{p^{\frac{m}{2}}}}^{\frac{m}{2}}$ so we have the complete graph $k_{\frac{p^{\frac{m}{2}} + 1}}^{\frac{m}{2}}$ is the largest complete subgraph exist in the graph $\Gamma_B(Z_{p^m q})$ hence the chromatic number of the graph $\Gamma_B(Z_{p^m q})$ is $\left(p^{\frac{m}{2}} + 1 \right)$ [7].

B- From the Lemma (2.1) so it is the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m q})$ then the degree of the vertex (0) is $(p^m q - 1)$ so it is the chromatic number of the edges is $(p^m q - 1)$.

Theorem 3.4: The graph $\Gamma_B(Z_{p^m q})$ contains a subgraph that is homeomorphic to $k_{(p^m - 1), q}$ and $k_{(p q - p^{m-1}), p^{m-1}}$.

Proof: The first part, since the multiplying the number p^m or one of its complications $(2p^m, 3p^m, \dots, (q - 1)p^m, p^m q = 0)$ by number q or one of its complications $(2q, 3q, \dots, (p^m - 1)q, p^m q = 0)$ be the product $p^m q$ or one a complications of the number $p^m q$ which is equal to (0) in the ring $Z_{p^m q}$. Thus the graph $\Gamma_B(Z_{p^m q})$ contains a subgraph homeomorphic complete bipartite graph $k_{(p^m - 1), q}$, as in the Figure (3.1).

Fig (3. 1)

The second part, since the multiplying the number $p q$ or one of its complications $(2pq, 3pq, \dots, (p^{m-1} - 1)pq, p^m q = 0)$ by number p^{m-1} or one of its complications $(2p^{m-1}, 3p^{m-1}, \dots, (pq - 1)p^{m-1}, p^m q = 0)$ be the product $p^m q$ or one a complications of the number $p^m q$ which is equal to (0) in the ring $Z_{p^m q}$. Thus the graph $\Gamma_B(Z_{p^m q})$ contains a subgraph homeomorphic complete bipartite graph $k_{(pq - p^{m-1}), p^{m-1}}$, as in the Figure (3.2).

Fig (3. 2)

Remark:

From the Theorem (3.4), the only graphs of the formula $\Gamma_B(Z_{p^m q})$ when $q = 2$ and $m = 1$ does not contain a subgraph homeomorphic $k_{3,3}$ or k_5 therefore it is planar and colorable for faces. Otherwise, the graphs of the formula $\Gamma_B(Z_{p^m q})$ contain a subgraph homeomorphic $k_{3,3}$ or k_5 therefore it is not planar graphs by kuratowski's Theorem.

Example 4: The chromatic number of the graphs $\Gamma_B(Z_{18})$ and $\Gamma_B(Z_{22})$

The graph is clear $\Gamma_B(Z_{18})$ of formula $\Gamma_B(Z_{p^m q})$, where $p = 3, q = 2$ and $m = 2$ and the graph is clear $\Gamma_B(Z_{22})$ of formula $\Gamma_B(Z_{p^m q})$, where $p = 11, q = 2$ and $m = 1$.

The chromatic number of vertices of the graph $\Gamma_B(Z_{p^m q})$ is $p^{\frac{m}{2}} + 1$ (when m is an even).

$$\therefore \chi(\Gamma_B(Z_{18})) = 4.$$

The chromatic number of edges the graph $\Gamma_B(Z_{p^m q})$ is $p^m q - 1$.

$$\therefore \tilde{\chi}(\Gamma_B(Z_{18})) = 17.$$

From Theorem (3.4) we get the graph $\Gamma_B(Z_{18})$ contains a subgraph that is homeomorphic to $K_{3,3}$ then the graph $\Gamma_B(Z_{18})$ it is not planar by kuratowski's Theorem.

The chromatic number of vertices the graph $\Gamma_B(Z_{p^m q})$ is $p^{\frac{m-1}{2}} + 2$ (when m is an odd).

$$\therefore \chi(\Gamma_B(Z_{22})) = 3.$$

The chromatic number of edges the graph $\Gamma_B(Z_{p^m q})$ is $p^m q - 1$.

$$\therefore \tilde{\chi}(\Gamma_B(Z_{22})) = 21.$$

From Theorem (3.4) we get the graph $\Gamma_B(Z_{22})$ contains a subgraph that is homeomorphic to $K_{10,2}$ it is the largest complete bipartite graph there is in the graph $\Gamma_B(Z_{22})$ then the graph $\Gamma_B(Z_{22})$ it is planar by kuratowski's Theorem.

$$\therefore f(\Gamma_B(Z_{22})) = 3.$$

REFERENCES

- [1] Ahmdi M. R. and Nezhad R. J., (2011), "Energy and Wiener Index of Zero-Divisor Graphs", Iranian Journal of Mathematical Chemistry, Vol. 2, No. 1, 45-51.
- [2] Anderson D. and Livingston P., (1999), "The Zero-Divisor Graph of a Commutative Rings", J. Algebra, 217, 434-447.
- [3] Anderson D. and Naseer M., (1993), "Beck's Coloring of a Commutative Rings", J. Algebra 159-500-514.
- [4] Anna D., (2006), "Proper Colorings and p-Partite Structures of the Zero-Divisor Graph".
- [5] Beck I., (1988), "Coloring of Commutative Rings", J. Algebra 116, 208-226.
- [6] Burton D. M., (1967), "Introduction to Modern Abstract Algebra", University of New Hampshire.
- [7] Chartrand, G. and Lesniak, L. "Graphs and Digraphs", 6th ed., Wadsworth and Brooks/ Cole. California. 2016.
- [8] Hosoya H., (1985), "On Some Counting Polynomials in Chemistry", Discrete Applied Math. Vol. 19, 239-257.
- [9] Husam. Q. M and Mohammad. N. A, (2018), "Hosoya polynomial and Wiener index of zero-divisor graph of Z_n ", Al-Rafidain Journal of Computer Sciences and Mathematics.
- [10] Luma. A. K, (2019), "The Maximal Degree of Zero- Divisor-Graph of $Z_{p^m q^2}$ and $Z_{p^m q^r}$ ", Ph.D. Thesis, Mosul University, Mosul.
- [11] Narmin. J. K, (2019), "Pseudo-Von Neumann Regular Graph of Commutative Rings with some Properties of it's Graphs", M.Sc. Thesis, Tikrit University, Tikrit.
- [12] Wiener H., (1947), "Structural Determination of the Paraffin Boiling Points", J. Am. Chem. Soc. 6917-20.