

Some Properties of Algebra of Quotients with Bounded Evaluation of a Norm Ideal on Complex Banach Space

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ABSTRACT

Cabrera-Mohammed proved that the imbedding of a norm ideal on Hilbert space in algebra of quotients with bounded evaluation is continuous with other properties. In this paper we improve this result by using complex Banach space instated of Hilbert space.

Keyword: algebra of quotients with bounded evaluation, bounded algebra of quotients, totally prime algebra.

بعض خصائص جبر القسومات ذات القيم المقيدة لمثالية المعيار على فضاء باناخ المعقد

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الملخص

كابريرا - محمد برهنا إن الغمر لمثالية المعيار على فضاء هلبيرت في جبر القسومات ذات القيم المقيدة يكون مستمراً مع خصائص أخرى. في هذا البحث نحن نحسن هذه النتيجة باستخدام فضاء باناخ المعقد بدلاً من فضاء هلبيرت.

الكلمات المفتاحية: جبر القسومات ذات القيم المقيدة، جبر القسومات المقيدة، الجبر الأولي الكلي.

1. Introduction

Normalization of a subalgebra of Martindale of algebra of quotients of prime algebra was studied first in 1986 by Martin Mathieu in his thesis [8] by using ultraprime normed algebra which known as bounded algebra of quotients. In 1990, Cabrera-Rodriguez defined totally prime algebra and they proved every ultraprime normed algebras are totally prime algebras [5, Theorem 1, p. 5]. In 2001, Cabrera-Mohammed extended the study of normalization of a subalgebra of Martindale of algebra of quotients of ultraprime normed algebra to totally prime algebra. The new subalgebra is called algebra of quotients with bounded evaluation.

In [3], Cabrera-Mohammed proved that the imbedding of a norm ideal on Hilbert space in algebra of quotients with bounded evaluation is continuous and the algebra of quotients with bounded evaluation is a right ideal of bounded algebra of quotients.

Oure aim is to extend the work of Cabrera-Mohammed in [3, proposition 3, p. 121] by using norm ideal on complex Banach space instated of Hilbert space. All algebras which will be used in this paper are associative algebras.

2. Bounded algebra of quotients and algebra of quotients with bounded evaluation of prime normed algebra.

Recall that A is prime algebra, if $IJ = 0$ then either $I = 0$ or $J = 0$ for any I, J ideals of A . Let A be an algebra and $x, y \in A$ the linear operator $M_{x,y}$ is defined on A such that $M_{x,y}(z) = xzy$ for all $z \in A$. A normed algebra A is called ultraprime if there exists a constant $c > 0$ such that $c\|x\|\|y\| \leq \|M_{x,y}\|$ for all $x, y \in A$. [8, definition 1.2, p.13].

Let A be a prime normed algebra, the right bounded algebra of quotients of A is defined to be subalgebra of $Q^r(A)$ Martindale algebra of quotients of A . It denoted by $Q_b^r(A)$ and given by

$$Q_b^r(A) = \{q \in Q^r(A) : I \text{ ideal of } A, qI \subseteq A, L_q^I \text{ is bounded}\}$$

Let $q \in Q_b^r(A)$ the norm of $Q_b^r(A)$ is given by

$$\|q\|_r = \inf\{\|L_q^I\| : I \text{ ideal of } A, qI \subseteq A, L_q^I \text{ is bounded}\}$$

Where L_q^I is a linear operator from I into A such that $L_q^I(x) = qx$ for all $x \in I$. $Q_b^r(A)$ it well be right bounded algebra of quotients for prime algebra A with identity by [8, proposition 2.6, p. 21] with algebra seminorm $\|\cdot\|_r$ by [8, Lemma 2.12, p. 26]. When A is ultraprime normed algebra, then $Q_b^r(A)$ be ultraprime normed algebra too by [9, Theorem 4.1(b), p. 307]. The left bounded algebra of quotients has similar definition and denoted by $Q_b^l(A)$ with algebra seminorm denoted by $\|\cdot\|_l$.

The multiplication algebra $M(A)$ for algebra A is a subalgebra of $L(A)$ generated by left and right multiplication operators $\{R_a, L_a : a \in A\}$ and the identity operator Id_A . Let A be an algebra and $x, y \in A$ the bilinear operator $N_{x,y}$ from $M(A) \times M(A)$ into A defined by $N_{x,y}(F, G) = F(x)G(y)$ for all $F, G \in M(A)$. Recall that a normed algebra A is totally prime if there exists a constant $c > 0$ such that $c\|x\|\|y\| \leq \|N_{x,y}\|$ for all $x, y \in A$. The multiplication algebra $M(A)$ is a subalgebra of $BL(A)$ which is normed under operator norm see [4, p. 128].

For defining algebra of quotients with bounded evaluation for prime normed algebra we need the following definition.

Let A be a prime algebra and A^1 is the unitization for algebra A of $Q^r(A)$ when A without identity. Let I be an ideal of A , the set I^r is given by

$$I^r = \left\{ \sum_{i=1}^n M_{a_i, x_i}, n \in \mathbb{N}, a_i \in A^1, x_i \in I, 1 \leq i \leq n \right\}$$

Cabrera-Mohammed in [3, lemma 1, p. 116] proved that I^r is an ideal of $M(A)$.

Let A be prime algebra, $q \in Q^r(A)$ and I non-zero ideal of A such that $qI \subseteq A$. The linear operator $E_q^{I^r}$ from I^r into A is defined by

$$E_q^{I^r}(F) = F(q) = \sum_{i=1}^n M_{a_i, x_i}(q)$$

Let A be a prime normed algebra, the right algebra of quotients with bounded evaluation for A is defined to be subalgebra of $Q^r(A)$. It denoted by $Q_{be}^r(A)$ and given by

$$Q_{be}^r(A) = \{q \in Q^r(A) : I \text{ ideal of } A, qI \subseteq A, E_q^{I^r} \text{ is bounded}\}$$

Let $q \in Q_{be}^r(A)$ the norm of $Q_{be}^r(A)$ is given by

$$\|q\|_{re} = \inf\{\|E_q^{I^r}\| : I \text{ ideal of } A, qI \subseteq A, E_q^{I^r} \text{ is bounded}\}$$

$Q_{be}^r(A)$ it will be right algebra of quotients with bounded evaluation for prime normed algebra A with seminorm $\|\cdot\|_{re}$ by [3, theorem 1, p. 117], when A is totally prime algebra then $Q_{be}^r(A)$ it will be totally prime algebra too by [3, Theorem 5, p. 129]. The left algebra of quotients with bounded evaluation has similar definition and denoted by $Q_{be}^l(A)$ with algebra seminorm denoted by $\|\cdot\|_{le}$.

3. Properties for right algebra of quotients with bounded evaluation of a norm ideal on complex Banach space.

Let X, Y be two normed spaces on a field K , the linear operator T from X to Y is called finite rank operator if $T \in BL(X, Y)$ such that $T(X)$ is finite dimension subspace of Y . We denoted to set of all finite rank operators from X to Y by $FBL(X, Y)$ which is an subspace of $BL(X, Y)$. If $X = Y$ then $FBL(X)$ is an ideal of $BL(X)$. Let X be a non-zero normed space on field K and X' is the dual space of X . Let $y \in X$ and $f \in X'$ the operator $y \otimes f$ defined on X such that $(y \otimes f)(x) = f(x)y$ [6, 1.412, p. 73].

We say that A is norm ideal, if A is a non-zero ideal of $BL(X)$ with norm $\|\cdot\|$ satisfy the following:

- i. $\|x \otimes f\| = \|x\| \|f\|$ for all $x \in X$ and $f \in X'$.
- ii. $\|FTG\| \leq \|F\|_\infty \|T\| \|G\|_\infty$ for all $T \in A$ and $F, G \in BL(X)$

Also $\|T\|_\infty \leq \|T\|$ for all $T \in A$

[7, definition 6.1.141, p. 409]

The following theorem is improve to theorem for Cabrera-Mohammed in [3, Theorem 2, p.120] which is used complex Banach space instated of Hilbert space

Theorem 3.1

Let A be norm ideal on complex Banach space X . Then

$$(Q_b^r(A), \|\cdot\|_r) = (BL(X), \|\cdot\|_\infty)$$

For proof see [1, Theorem 2.2].

Cabrera-Mohammed in [3, proposition 3, p. 121] proved that the imbedding of norm ideal on Hilbert space in algebra of quotients with bounded evaluation is continuous with other properties. We give the same result which have given in [3, proposition 3, p. 121] but we use the complex Banach space in our proof instead of Hilbert space.

Proposition 3.2

Let $(A, \|\cdot\|)$ be norm ideal on complex Banach space X then $Q_{be}^r(A)$ is a right ideal of $BL(X)$ and $\|\cdot\|_{re}$ is norm have the following properties:

- i. $\|T\|_{re} \leq \|T\|$
- ii. $\|S\|_\infty \leq \|S\|_{re}$
- iii. $\|x \otimes y'\|_{re} = \|x\| \|y'\|$
- iv. $\|SF\| \leq \|S\|_{re} \|F\|_\infty$
- v. $\|SG\|_{re} \leq \|S\|_{re} \|G\|_\infty$

For all $T \in A, S \in Q_{be}^r(A), x \in X, y' \in X', F \in FBL(X)$ and $G \in BL(X)$.

Proof:

Proof(i)

Let $(A, \|\cdot\|)$ be norm ideal on X , so by using Theorem 3.1, we have $Q_b^r(A) = BL(X)$ therefore $Q_{be}^r(A) \subseteq BL(X)$, since $Q_{be}^r(A)$ is a subalgebra of $Q_b^r(A)$ by [3, Theorem 1, p.117]. Now, let $T \in A$ and $F \in FBL(X)$, so

$$\|T\|_{re} = \inf\{\|E_T^{I^r}\| : I \text{ ideal of } A, TI \subseteq A, E_T^{I^r} \text{ bounded}\}$$

Since $FBL(X)$ is an ideal of $BL(X)$ and every ideal of $BL(X)$ are contain $FBL(X)$ by [2, Theorem 4.3.7(v)&(vi), p. 144] we get that

$$\begin{aligned} \|T\|_{re} &= \left\| E_T^{FBL(X)^r} \right\| \\ &= \sup_{F \in FBL(X)^r} \left\{ \left\| E_T^{FBL(X)^r}(F) \right\|, \|F\| = 1 \right\} \\ &= \sup_{F \in FBL(X)^r} \{ \|F(T)\|, \|F\| = 1 \} \\ &\leq \sup_{F \in FBL(X)^r} \{ \|F\| \|T\|, \|F\| = 1 \} \\ &= \|T\| \text{ for all } T \in A. \end{aligned}$$

Proof(ii)

Let $(A, \|\cdot\|)$ be norm ideal on X . Then from [3, Theorem 1, p. 117] we get $Q_{be}^r(A) \subseteq Q_b^r(A)$ and $\|S\|_r \leq \|S\|_{re}$ for all $S \in Q_{be}^r(A)$. Form Theorem 3.1 we have $(Q_b^r(A), \|\cdot\|_r) = (BL(X), \|\cdot\|_\infty)$ so $\|G\|_\infty = \|G\|_r$ for all $G \in Q_b^r(A)$. Thus $\|S\|_\infty \leq \|S\|_{re}$ for all $S \in Q_{be}^r(A)$.

Proof(iii)

Let $x \in X$ and $y' \in X'$, then $x \otimes y'$ is finite rank operator on complex Banach space X . So

$$\begin{aligned} \|x\| \|y'\| &= \|x \otimes y'\|_\infty \\ &\leq \|x \otimes y'\|_{re} \text{ from (ii)} \\ &\leq \|x \otimes y'\| \text{ from (i)} \\ &= \|x\| \|y'\| \end{aligned}$$

From properties of norm ideal, so we get that $\|x \otimes y'\|_{re} = \|x\| \|y'\|$.

Proof (iv)

Let $S \in Q_{be}^r(A)$ and $F \in FBL(X)$

$$\begin{aligned} \|SF\| &= \|R_F^A(S)\| \\ &\leq \left\| E_S^{FBL(X)^r}(R_F^A) \right\| \\ \text{Since } F &\in FBL(X), \text{ we have } R_F^A \in FBL(X)^r. \\ &\leq \left\| E_S^{FBL(X)^r} \right\| \|R_F^A\| \text{-----(1)} \end{aligned}$$

Since $S \in Q_{be}^r(A)$, there exists an ideal I of A such that $SI \subseteq A$ and $E_S^{I^r}$ is bounded, $FBL(X)$ is an ideal of $BL(X)$ and every ideal of $BL(X)$ contain $FBL(X)$ by [2, Theorem 4.3.7(v), p. 144], so $FBL(X) \subseteq I$ and $E_S^{FBL(X)^r}$ is bounded. Now

$$\begin{aligned} \|R_F^A\| &= \sup_{W \in A} \{ \|R_F^A(W)\|, \|W\| = 1 \} \\ &= \sup_{W \in A} \{ \|WF\|, \|W\| = 1 \} \\ &= \sup_{W \in A} \{ \|Id_X WF\|, \|W\| = 1 \} \\ &\leq \sup_{W \in A} \{ \|Id_X\|_\infty \|W\| \|F\|_\infty, \|W\| = 1 \} \end{aligned}$$

Since $F, Id_X \in BL(X)$ and from properties of norm ideal.

$$= \|F\|_\infty \text{-----(2)}$$

From (1) &(2) we get

$$\|SF\| \leq \left\| E_S^{FBL(X)^r} \right\| \|F\|_\infty$$

$$\leq \|S\|_{re} \|F\|_\infty \text{ for all } S \in Q_{be}^r(A) \text{ and } F \in FBL(X).$$

Proof (v)

Let $G \in BL(X)$ and $V \in FBL(X)^r$ for proving that $\|VR_G^A\| \leq \|V\| \|G\|_\infty$, first we will prove that $FBL(X)^r R_G^A \subseteq FBL(X)^r$. Let $T \in A$. Since $V \in FBL(X)^r$, then there exists $a_i \in A$, $x_i \in FBL(X)$ and $n \in \mathbb{N}$ such that $V = \sum_{i=1}^n M_{a_i, x_i}$. Now,

$$\begin{aligned} VR_G^A(T) &= \sum_{i=1}^n M_{a_i, x_i} (R_G^A(T)) \\ &= \sum_{i=1}^n M_{a_i, x_i} (TG) \\ &= \sum_{i=1}^n a_i TG x_i \\ &= \sum_{i=1}^n a_i T c_i, \quad c_i = G x_i \\ &= \sum_{i=1}^n M_{a_i, c_i} (T) = V_1(T) \text{ where } V_1 = \sum_{i=1}^n M_{a_i, c_i} \end{aligned}$$

Since $c_i \in FBL(X)$ we have $V_1 \in FBL(X)^r$, since $G \in BL(X)$ we get $R_G^A \in M(A)$, $FBL(X)^r$ is an ideal of $M(A)$ by [3, Lemma 1, p. 116], so $VR_G^A \in FBL(X)^r$ for all $V \in FBL(X)^r$, then $FBL(X)^r R_G^A \subseteq FBL(X)^r$. Now to prove that $\|VR_G^A\| \leq \|V\| \|G\|_\infty$.

$$\|VR_G^A(T)\| = \|V(TG)\|$$

$$\leq \|V\| \|TG\|$$

$T \in A$ and $G \in BL(X)$ then $TG \in A$ and V is bounded.

$$= \|V\| \|Id_X TG\|$$

$$\leq \|V\| \|Id_X\|_\infty \|T\| \|G\|_\infty$$

From properties of norm ideal

$$\leq \|V\| \|T\| \|G\|_\infty$$

Now,

$$\|VR_G^A\| = \sup_{T \in A} \{\|VR_G^A(T)\|, \|T\| = 1\}$$

$$\leq \sup_{T \in A} \{\|V\| \|T\| \|G\|_\infty, \|T\| = 1\}$$

$$= \|V\| \|G\|_\infty \text{-----(3),}$$

for all $V \in FBL(X)^r$ and $G \in BL(X)$.

Let $S \in Q_{be}^r(A)$, to prove that $\|SG\|_{re} \leq \|S\|_{re} \|G\|_\infty$

$$\left\| E_{SG}^{FBL(X)^r} (V) \right\| = \|V(SG)\|$$

$$= \|VR_G^A(S)\|$$

$$= \left\| E_S^{FBL(X)^r} (VR_G^A) \right\|$$

From above we get $VR_G^A \in FBL(X)^r$

$$\leq \|E_S^{FBL(X)^r}\| \|VR_G^A\|$$

Since $S \in Q_{be}^r(A)$ we have $E_S^{FBL(X)^r}$ is bounded and from (3) we get

$$\leq \|E_S^{FBL(X)^r}\| \|V\| \|G\|_\infty \text{-----(4)}$$

We get that $E_S^{FBL(X)^r}$ is bounded for all $G \in BL(X)$ and $S \in Q_{be}^r(A)$, then $Q_{be}^r(A)$ is right ideal of $BL(X)$.

$$\begin{aligned} \|SG\|_{re} &= \|E_{SG}^{FBL(X)^r}\| \\ &= \sup_{V \in FBL(X)^r} \left\{ \|E_{SG}^{FBL(X)^r}(V)\|, \|V\| = 1 \right\}, \text{ from (4) we get} \\ &\leq \sup_{V \in FBL(X)^r} \left\{ \|E_S^{FBL(X)^r}\| \|V\| \|G\|_\infty, \|V\| = 1 \right\} \\ &= \|E_S^{FBL(X)^r}\| \|G\|_\infty \\ &= \|S\|_{re} \|G\|_\infty \text{ for all } S \in Q_{be}^r(A) \text{ and } G \in BL(X). \blacksquare \end{aligned}$$

With similar proving for Proposition 3.2, we have the following proposition for the left algebra of quotients with bounded evaluation of norm ideal of complex Banach space.

Proposition 3.3

Let $(A, \|\cdot\|)$ be norm ideal on complex Banach space X then $Q_{be}^l(A)$ is a left ideal of $BL(X)$ and $\|\cdot\|_{le}$ is norm have the following properties:

- i. $\|T\|_{le} \leq \|T\|$
- ii. $\|S\|_\infty \leq \|S\|_{le}$
- iii. $\|x \otimes y'\|_{le} = \|x\| \|y'\|$
- iv. $\|SF\| \leq \|S\|_{le} \|F\|_\infty$
- v. $\|GS\|_{le} \leq \|G\|_\infty \|S\|_{le}$

For all $T \in A, S \in Q_{be}^l(A), x \in X, y' \in X', F \in FBL(X)$ and $G \in BL(X)$.

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