# On Regularity and Flatness

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**ABSTRACT**

A ring R is called a right SF-ring if all its simple right R-modules are flat. It is well known that Von Neumann regular rings are right and left SF-rings. In this paper we study conditions under which SF-rings are strongly regular. Finally, some new characterstic properties of right SF-rings are given.

**Keywords:** modules, flat, Von Neumann regular rings.

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**الملخص**

يقال للحلقة R بأنها من النمط –SF اليمنى ، إذا كان كل مقاس بسيط ايمن فيها مسطحاً. من المعروف أن كل حلقة منتظمة بمفهوم فون نيومان تكون حلقة من النمط –SF اليمنى واليسرى. في هذا البحث أعطينا شروطاً أخرى لكي تكون كل حلقة من النمط –SF اليمنى حلقة منتظمة بقوة . ومن النتائج الأخرى التي حصلنا عليها هي خواص أخرى جديدة للحلقات من النمط –SF اليمنى.

**الكلمات المفتاحية:** المقاسات، المسطحة، حلقة منتظمة بمفهوم فون نيومان.

## 1. INTRODUCTION

In this paper all rings are assumed to be associative with identity, and all modules are unital right R-modules.

Following [2], a ring R is called a right (left) SF-ring if all of its simple right (left) R-modules are flat . It is well known that a ring R is Von Neumann regular if and only if every right (left) R-module is flat [3]. Ramamurthi in [8] asked whether left and right SF-ring is Von Neumann regular. Many authors have given various conditions for SF-rings to be regular (see, e.g. Chen [1], Ming [4], Rege [9] and Xu-[10] ). In this paper, to the list of equivalent conditions, we shall add several news. We recall that:

**1-** A ring R is called reduced if R contains no non-zero nilpotent elements.

**2**-R is said to be Von Neumann regular (or just regular) if $a \in aRa$ for every $a \in R$, and R is called strongly regular if $a \in a^2R$ . Clearly, every strongly regular ring is a regular reduced ring.

**3**-R is said to be right duo-ring if every right ideal is a two-sided ideal.

**4**-r*(a)* and L*(a)* will denote right and left annihilator of *a* respectively.

**5**-Following [9], for any ideal I of R, R/I is flat if and only if for each $a \in I$ , there exists $b \in I$ such that $a = ba$.

**6**-Y and J will stand for the right singular ideal and Jacobson radical of R.

## 2. RINGS WHOSE SIMPLE MODULES ARE FLAT

Following [7], a ring R is called ERT-ring if every essential right ideal of R is a two-sided ideal.

Ming [6] proved the following:

**Proposition 2.1.** If R is a right duo-ring then R/Y is a reduced ring.

We use a similar method of proof in Prop.2.1 to establish the following lemma.

**Lemma 2.2:** If R is an ERT-ring , then R/Y is a reduced ring.

**Proof.** Suppose that R/Y is not reduced, then there exists an element
$Y \neq a + Y \in R/Y$ , $a \in R$ , such that $(a+Y)^2 = Y$. This implies that $a \notin Y$ and $a^2 \in Y$ . So $r(a^2)$ is essential right ideal of R. Since R is ERT, then $r(a^2)$ is a two-sided ideal . Let I be any subideal of $r(a^2)$
Such that
I is̸ essential in $(a)$I , this means that $r \cap (a)r \subseteq Ia$, then $(a)r \subseteq r(a^2)$ and hence in R , this contradicts $a \notin Y$.

The following theorem gives the condition of being right SF-rings are strongly regular.

**Theorem 2.3:** Let R be a ring. Then the following are equivalent.
(1) R is strongly regular.
(2) R is a right SF- and ERT ring.

**Proof**. (1)$\Longrightarrow$(2) is obvious.
(2) $\Longrightarrow$(1) By Lemma 2.2, R/Y is a reduced ring. We claim that Y=0. Suppose that $Y \neq 0$ then by [5], there exists $0 \neq y \in Y$ such that $y^2 = 0$.

Let M be a maximal right ideal containing r (y). Since r (y) is an essential two-sided ideal of R , then M must be an essential two-sided ideal of R. On the other hand, since R/M is flat module , and since $y \in M$ , there exists $c \in M$ such that y=yc, whence $1\text{-}c \in r(y) \subseteq M$, yielding $1 \in M$ which contradicts $M \neq R$. This proves that R is a reduced ring. In order to show that R is regular we need to prove that $aR+r(a) = R$ for any $a \in R$. Suppose that $aR + r(a) \neq R$ , then there exists a maximal right ideal L containing $aR + r(a)$. But $a \in L$ and R/M is flat , there exists $b \in L$ such that $a=ba$, whence $1\text{-}b \in L(a) = r(a) \subseteq M$. Yielding $1 \in M$ which contradicts $L \neq R$. In particular $ar+d = 1$, for some $r \in R$ and $d \in r(a)$, whence $a^2r=a$ . This proves that R is a strongly regular ring.

We now consider an other condition for right SF-ring to be strongly regular.

**Theorem 2.4:** Let R be a right SF-ring with every nilpotent element of R is central. Then R is strongly regular.

**Proof.** Let $a$ be a non-zero element in R with $a^2=0$, and let M be a maximal right ideal containing $r(a)$ . Since $a \in r(a) \subseteq M$ , and since R/M is flat , there exists $b \in M$ such that $a = ba$. This implies that $1-b \in L(a)$. But every nilpotent is central gives $r(a)=L(a)$. Whence $1-b \in r(a) \subseteq M$, yielding $1 \in M$ , and this contradicts $M \neq R$. Therefore, R is a reduced ring. By a similar method of proof used in Theorem 2.3,R is strongly regular.

## 3. BASIC PROPERTIES

Recall that a ring R is a right uniform if every right ideal of R is essential .

We are now in a position to give new characteristic properties of a right SF-ring.

**Theorem 3.1:** If R is a right SF- ring , then

1- If $L(a) = 0$ , then $a$ is a right invertable .

2- Every reduced ideal of R is strongly regular.

3- If J is reduced , then J = 0 .

4- If R is a right uniform ring, then R is a division ring.

**Proof.**

(1) Let $a \in R$ with $L(a)=0$. If $aR \neq R$ , there exists a maximal right ideal M containing $a$R. Since $a \in M$ and R/M is flat , there exists $b \in M$, such that $a=ba$. Whence $1-b \in L(a) = 0$, yielding $L \in M$, which contradicts $M \neq R$. Therefore $aR=R$.

(2) Follows from Theorem 2.3.

(3) Let $a \in J$, then by (2) J is strongly regular, and hence there exists $b \in J$ such that $a=a^2b$ . But $a \in J$ gives $(1-ab)\, u = 1$ for some $u \in R$, this implies that $(a-a^2b)u=a$. Thus $a$=0, consequently, J=0.
(4) Suppose that $Y \neq 0$, then there exists a maximal right ideal M containing Y. For any $0 \neq y \in Y$, gives $y \in M$, but R/M is flat, then there exists $x \in M$ such that y=xy, whence $y \in r(1-x)$. On the other hand, since R is a right uniform , then r(1-x) is an essential right ideal of R. Thus $1-x \in Y \subseteq M$, this implies that $1 \in M$, contradicting $M \neq R$. Therefore, Y=0. On the other hand, since R is uniform, then for every $a \in R$, $r(a) = 0$, then by (1) , R is a division ring.

Before closing this section, we present the following result.

**Proposition 3.2:** Let R be a reduced right SF- ring, for any $a$, $b \in R$ with $a$.b=0 , then $r(a) + r(b) = R$.

**Proof.** Suppose that $a$.b=0 and $r(a) + r(b) \neq R$. Then there exists a maximal right ideal M containing $r(a) + r(b)$. Since $a \in r(b) \subseteq M$, and since R/M is flat, there exists $c \in M$ such that $a = ca$, whence $1-c \in L(a) = r(a) \subseteq M$, yielding $L \in M$, which contradicts $M \neq R$.
Therefore $r(a) + r(b) = R$.

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