An Extension Use of ADI Method in the Solution of Biharmonic Equation
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ABSTRACT
The Biharmonic equation is one of partial differential equations which arise from discussion of some applied sciences such as fluid dynamics. In this paper, we have adopted a numerical method to solve that equation, this method is developed basically from ADI (Alternating-Direction- Implicit) finite difference method which was used in the solution of Laplace equation.
Keywords: partial differential equation, finite difference method, Alternating- Direction- Implicit, Laplace equation, Biharmonic equation.

1-Introduction:
The ADI (Alternating- Direction-Implicit) method is one of the finite difference technique used in the solution of partial differential equations. A parabolic partial differential equation having two space coordinates, that is let \( u = (x, y, t) \) and \( v = (i, j, n) \), where \( y = j\Delta y \). A simple example arising from unsteady-state heat conduction in a flat plate is
the explicit method leads to the difference equation
\[
\frac{v_{i,j,n+1} - v_{i,j,n}}{\Delta t} = \frac{\partial^2 v_{i,j,n}}{\partial x^2} + \frac{\partial^2 v_{i,j,n}}{\partial y^2}
\]
on the other hand, the implicit method leads to the difference equation
\[
\frac{v_{i,j,n+1} - v_{i,j,n}}{\Delta t} = \frac{\partial^2 v_{i,j,n+1}}{\partial x^2} + \frac{\partial^2 v_{i,j,n+1}}{\partial y^2}
\]
which, when written out in full to the simple case of square grid with \( \Delta x = \Delta y \), has the form
\[
-\lambda v_{i-1,j,n+1} - \lambda v_{i,j-1,n+1} + (1 + 4\lambda) v_{i,j,n+1} - \lambda v_{i+1,j,n+1} - \lambda v_{i,j+1,n+1} = v_{i,j,n}
\]
where \( \lambda = \frac{\Delta t}{(\Delta x)^2} \)

Essentially, the alternating implicit direction method is to employ two difference equations are used in turn over successive time-steps each of duration \( \Delta t/2 \). The first equation is implicit only in \( x \)-direction and the second is implicit only in the \( y \)-direction. Thus, if \( v_{i,j} \) is an intermediate value of the end of the first time-step, we have
\[
\frac{v^*_{i,j} - v_{i,j,n}}{\Delta t/2} = \frac{\partial^2 v^*_{i,j}}{\partial x^2} + \frac{\partial^2 v^*_{i,j,n}}{\partial y^2}
\]
followed by
\[
\frac{v_{i,j,n+1} - v^*_{i,j}}{\Delta t/2} = \frac{\partial^2 v_{i,j,n+1}}{\partial x^2} + \frac{\partial^2 v_{i,j,n+1}}{\partial y^2}
\]
written out in full and rearranged with \( \Delta x = \Delta y \) for simplicity, these equations become
\[
\frac{1}{\lambda} v_{i-1,j,n} + 2(\frac{1}{\lambda} + 1)v^*_{i,j} - v^*_{i+1,j} = v_{i,j-1,n} + 2(\frac{1}{\lambda} - 1)v_{i,j,n} + v_{i,j+1,n}
\]
\[
- v^*_{i,j,n+1} + 2(\frac{1}{\lambda} + 1)v_{i,j,n+1} - v^*_{i,j,n+1} = v_{i-1,j,n} + 2(\frac{1}{\lambda} - 1)v^*_{i,j,n+1} + v^*_{i+1,j,n+1}
\]

2- Example 1 (solution of Laplace equation by using ADI)

Consider for simplicity the following Laplace equation \( \nabla^2 T = 0 \) under the boundary conditions given by,
\[
T = \text{const} \tan t \quad \text{along the sides} \quad x = 1 \quad \text{and} \quad y = 1
\]
An Extension Use Of…

\[ \frac{\partial T}{\partial x} = 0 \quad \text{and} \quad \frac{\partial T}{\partial y} = 0 \quad \text{along the sides} \quad x = 0 \quad \text{and} \quad y = 0 \]

Now, by using ADI we have to solve the unsteady-state case instead of Laplace equation, that means we have to solve the following equation,[1]

\[ \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \]

... (2.1)

Using the formula (1.1),(1.2), equation (2.1) can be written as two difference equations,

\[ -T_{i-1,j}^* + \left( \frac{2h^2}{\Delta \tau} + 2 \right) T_{i,j}^* - T_{i+1,j}^* = T_{i,j-1} + \left( \frac{2h^2}{\Delta \tau} - 2 \right) T_{i,j} + T_{i,j+1} \]

... (2.2)

\[ -T_{i,j-1}^{*+1} + \left( \frac{2h^2}{\Delta \tau} + 2 \right) T_{i,j}^{*+1} - T_{i,j+1}^{*+1} = T_{i-1,j}^{*+1} + \left( \frac{2h^2}{\Delta \tau} - 2 \right) T_{i,j}^{*+1} + T_{i+1,j}^{*+1} \]

... (2.3)

Where \( T \) is the absolute temperature, \( T^* \) is the intermediate temperature \( h = \Delta x = \Delta y \), space increments \( \Delta x \), time increments each of equations (2.2),(2.3) represents a tridiagonal system, which can easily be solved by Gaussian elimination method, see [7, 9] for more details.

3- Example 2 (solution of Biharmonic equation by using ADI)

Consider the following Biharmonic equation

\[ \nabla^4 \psi = 0 \]

...(3.1)

with the boundary conditions, \( \psi = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \), on the rigid boundaries.

Equation (3.1) can be written as, [5]

\[ \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \]

...(3.2)

When we implement the ADI method into the solution of equation (3.2), we have to solve the following equation,

\[ \frac{\partial \psi}{\partial t} = \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} \]

...(3.3)

Equation (3.3) under the finite-difference ADI method convert into the following form

\[ \frac{\psi_{i,j}^{n+1} - \psi_{i,j}^n}{\Delta \tau} = \frac{\partial^4 \psi_{i,j}^*}{\partial x^4} + 2r \frac{\partial^2 \psi_{i,j}^*}{\partial x^2} + r \frac{\partial^2 \psi_{i,j}^*}{\partial y^2} \]

...(4.4)
\[
\frac{\psi_{i,j}^{n+1} - \psi_{i,j}^n}{2\Delta \tau} = \delta_x^2 \psi_{i,j}^n + 2r \delta_x^2 \delta_y^2 \psi_{i,j}^n + \delta_y^2 \psi_{i,j}^{n+1}
\]  

...(3.5)

Where \( r \) is an arbitrary weight factor \( \delta_x, \delta_y \) are the difference in \( x \) and \( y \) direction, respectively. Equations (3.4),(3.5) can easily be written as a linear system \( AY = B \), where \( A \) is a five-diagonal coefficient matrix [3,8]. The finite-difference equation in each time-step can be expressed as the following linear algebraic equation of the form,

\[
2(\psi_{i,j} - \psi_{i,j}^n) = \frac{\Delta \tau}{(\Delta x)^2} [\psi_{i-1,j}^n - 4\psi_{i-1,j}^n + 6\psi_{i-1,j}^n + \psi_{i+1,j}^n + \psi_{i+2,j}^n] + \frac{2\Delta \tau r}{(\Delta y)^2} [\psi_{i,j-1}^n - 2\psi_{i,j-1}^n + \psi_{i,j+1}^n - 2\psi_{i,j+1}^n + \psi_{i,j+2}^n] + \frac{\Delta \tau r}{(\Delta y)^2} [\psi_{i,j-2}^n - 4\psi_{i,j-1}^n + 6\psi_{i,j}^n - 4\psi_{i,j+1}^n + \psi_{i,j+2}^n] 
\]

...(3.6)

\[
2(\psi_{i,j}^{n+1} - \psi_{i,j}^n) = \frac{\Delta \tau}{(\Delta x)^2} [\psi_{i-2,j}^n - 4\psi_{i-1,j}^n + 6\psi_{i-1,j}^n + \psi_{i+1,j}^n + \psi_{i+2,j}^n] + \frac{2\Delta \tau r}{(\Delta y)^2} [\psi_{i,j-1}^n - 2\psi_{i,j-1}^n + \psi_{i,j+1}^n - 2\psi_{i,j+1}^n + \psi_{i,j+2}^n] + \frac{\Delta \tau r}{(\Delta y)^2} [\psi_{i,j-2}^n - 4\psi_{i,j-1}^n + 6\psi_{i,j}^n - 4\psi_{i,j+1}^n + \psi_{i,j+2}^n] 
\]

...(3.7)

The boundary conditions related to this equation will take the following finite-difference form:

\[
\begin{align*}
\psi_{0,j} &= \psi_{N,j} = 0.0 \\
\psi_{i,j} - \psi_{i-1,j} &= 0.0, \text{ implies } \psi_{N,j} = \psi_{N-1,j} \\
\psi_{i,j} - \psi_{i,j-1} &= 0.0, \text{ implies } \psi_{i,N} = \psi_{i,N-1} 
\end{align*}
\]

...(3.8)

Simplifying the equations (3.6) and (3.7) we get:

\[
-\psi_{i-2,j}^* + 4\psi_{i-1,j}^* - (6 - \frac{2h^2}{\Delta \tau})\psi_{i,j}^* + 4\psi_{i+1,j}^* - \psi_{i+2,j}^* = 2r[\psi_{i-1,j-1}^n - \psi_{i-1,j+1}^n + \psi_{i+1,j+1}^n - \psi_{i+1,j-1}^n] + 4\psi_{i,j-1}^n - 2\psi_{i,j+1}^n - 2\psi_{i,j+2}^n + \frac{2h^2}{\Delta \tau} \psi_{i,j}^n + \frac{2h^2}{\Delta \tau} \psi_{i,j+1}^n + \frac{2h^2}{\Delta \tau} \psi_{i,j+2}^n 
\]

...(3.9)

\[
-\psi_{i,j-2}^{n+1} + 4\psi_{i,j-1}^{n+1} - (6 - \frac{2h^2}{\Delta \tau})\psi_{i,j}^{n+1} + 4\psi_{i,j+1}^{n+1} - \psi_{i,j+2}^{n+1} = \psi_{i-2,j}^* - \psi_{i-1,j}^* + 4\psi_{i+1,j}^* - \psi_{i+2,j}^* + (6 + \frac{2h^2}{\Delta \tau})\psi_{i,j}^* - 4\psi_{i,j+1}^* 
\]
\[ + \psi_{i+2,j}^* + 2r[\psi_{i-1,j-1}^n - 2\psi_{i,j-1}^n + \psi_{i+1,j-1}^n - 2\psi_{i+1,j}^n + 4\psi_{i,j}^n - 2\psi_{i+1,j+1}^n + \psi_{i+1,j+2}^n] \]
\[ \cdots \text{(3.10)} \]

Where \( h = \Delta x = \Delta y \) and \( \psi, \psi^* \) refer to stream function at the beginning and at the end of half time-step \( \Delta \tau/2 \), equation (3.9) is applied to points \( i = 1,2,\ldots, n-1 \) in the \( j^{th} \) column, with the boundary conditions (3.8), we then have the following five diagonal system for the \( j^{th} \) column,
\[
\begin{align*}
&c_0 \psi_{0,j}^n + e_0 \psi_{1,j}^* + g_0 \psi_{2,j}^n = d_0^* \\
b_0 \psi_{0,j}^* + c_0 \psi_{1,j}^n + e_0 \psi_{2,j}^* + g_0 \psi_{3,j}^* = d_1^* \\
&\vdots \\
a_i \psi_{i-2,j}^* + b_i \psi_{i-1,j}^* + c_i \psi_{i,j}^* + e_i \psi_{i+1,j}^* + g_i \psi_{i+2,j}^* = d_i^* \end{align*}
\]
\[ \cdots \text{(3.11)} \]

with
\[
d_i = 2r[\psi_{i-1,j-1}^n - 2\psi_{i,j-1}^n + \psi_{i+1,j-1}^n - 2\psi_{i+1,j}^n + 4\psi_{i,j}^n - 2\psi_{i+1,j+1}^n + \psi_{i+1,j+2}^n]
\]
\[ + \psi_{i-1,j+1}^n - 2\psi_{i,j+1}^n + \psi_{i+1,j+1}^n + 4\psi_{i+1,j+2}^n] + + r[\psi_{i,j-2} - 4\psi_{i,j-1}^n + 6\psi_{i,j}^n - 4\psi_{i,j+1}^n + \psi_{i,j+2}^n]\]

Similarly the equation (3.10) is applied to each point \( j = 1,2,\ldots, n-1 \) in the \( i^{th} \) row, with the boundary conditions (3.8) which gives the following five diagonal system for the \( i^{th} \) rows:
\[
\begin{align*}
&c_0 \psi_{0,j}^{n+1} + e_0 \psi_{1,j}^{n+1} + g_0 \psi_{2,j}^{n+1} = d_0^* \\
b_0 \psi_{0,j}^{n+1} + c_0 \psi_{1,j}^{n+1} + e_0 \psi_{2,j}^{n+1} + g_0 \psi_{3,j}^{n+1} = d_1^* \\
&\vdots \\
a_i \psi_{i-2,j}^{n+1} + b_i \psi_{i-1,j}^{n+1} + c_i \psi_{i,j}^{n+1} + e_i \psi_{i+1,j}^{n+1} + g_i \psi_{i+2,j}^{n+1} = d_i^* \end{align*}
\]
\[ \cdots \text{(3.12)} \]

with
\[
d_j^* = \psi_{i-2,j}^n - 4\psi_{i-1,j}^n + (6 + \frac{2h^2}{\Delta \tau})\psi_{i,j}^n - 4\psi_{i+1,j}^n + \psi_{i+2,j}^n + 2r[\psi_{i,j-2}^n - 2\psi_{i,j}^n + \psi_{i,j+2}^n] - \\
\]
\[ \cdots \]
The systems (3.11),(3.12) can be expressed as a linear system of the form $A^*X^* = B^*$, where $A^*$ is a five diagonal matrix of the coefficients $(a_i, b_i, c_i, e_i, g_i). X^*$ represents the vector of dependent variables, $B^*$ is the vector of constants which are known, this system can be written as follows:

$$
\begin{bmatrix}
  c_0 & e_0 & g_0 & 0 & . & . & 0 & 0
  \\
  b_1 & c_1 & e_1 & g_1 & 0 & . & . & .
  \\
  \\
  \\
  \\
  0 & . & 0 & a_{N-1} & b_{N-1} & c_{N-1} & e_{N-1} & g_{N-1}
  \\
  0 & 0 & . & 0 & a_N & b_N & c_N & 0
\end{bmatrix}
\begin{bmatrix}
  x_0^*
  \\
  x_1^*
  \\
  \vdots
  \\
  \vdots
  \\
  \vdots
  \\
  x_{N-1}^*
  \\
  x_N^*
\end{bmatrix}
= \begin{bmatrix}
  B_0^*
  \\
  B_1^*
  \\
  B_2^*
  \\
  \vdots
  \\
  B_{N-1}^*
  \\
  B_N^*
\end{bmatrix}
\tag{3.13}
$$

### 4- Five diagonal Algorithm:

The solution of the system (3.13) is exactly the same as linear algebraic equation

$$a_i v_{i-2} + b_i v_{i-1} + c_i v_i + e_i v_{i+1} + g_i v_{i+2} = d_i, \quad i = 0, 1, 2, \ldots, N
\tag{4.1}$$

which can be expressed as:

$$a_{N-1} v_{N-3} + b_{N-1} v_{N-2} + c_{N-1} v_{N-1} + e_{N-1} v_N + g_{N-1} v_{N+1} = d_{N-1}
\tag{4.2}$$

$$a_N v_{N-2} + b_N v_{N-1} + c_N v_N + e_N v_{N+1} + g_N v_{N+2} = d_N
\tag{4.3}$$

To solve the equations (4.1),(4.2) and (4.3) we consider the following difference relation,

$$v_i = \gamma_i - \beta_i v_{i-1} - \alpha_i v_{i+2}
\tag{4.5}$$

Now

$$v_{i-1} = \gamma_{i-1} - \beta_{i-1} v_i - \alpha_{i-1} v_{i+1}
$$

$$v_{i-2} = \gamma_{i-2} - \beta_{i-2} v_{i-1} - \alpha_{i-2} v_i = \gamma_{i-2} - \beta_{i-2} (\gamma_{i-1} - \beta_{i-1} v_i - \alpha_{i-1} v_{i+1}) - \alpha_{i-2} v_i
$$

$$= \gamma_{i-2} v_{i-1} + (\beta_{i-2} \beta_{i-1} - \alpha_{i-2}) v_i + \beta_{i-2} \alpha_{i-1} v_{i+1}
$$

eliminating $v_{i-1}$ and $v_{i-2}$ from (4.1) gives,
\[ a_i \left( \gamma_{i-2} - \beta_{i-2} \gamma_{i-1} + (\beta_{i-2} \beta_{i-1} - \alpha_{i-2}) v_i + \beta_{i-2} \alpha_{i-1} v_{i+1} \right) + \\
+ b_i \left( \gamma_{i-1} - \beta_{i-1} v_i - \alpha_{i-1} v_{i+1} \right) + c_i v_i + + e_i v_{i+1} + g_i v_{i+2} = d_i \]

Which gives:

\[
v_i = \frac{d_i + (a_i \beta_{i-2} - b_i) \gamma_{i-1} - a_i \gamma_{i-2}}{c_i + (a_i \beta_{i-2} - b_i) \beta_{i-1} - a_i \alpha_{i-2}} - \frac{a_i \beta_{i-2} \alpha_{i-1} - b_i \alpha_{i-1} + e_i}{c_i + (a_i \beta_{i-2} - b_i) \beta_{i-1} - a_i \alpha_{i-2}} v_{i+1} - \frac{g_i}{c_i + (a_i \beta_{i-2} - b_i) \beta_{i-1} - a_i \alpha_{i-2}} v_{i+2}
\]

...(4.6)

comparing equation (4.5) with equation (4.6), we will get:

\[
\gamma_i = \frac{d_i + (a_i \beta_{i-2} - b_i) \gamma_{i-1} - a_i \gamma_{i-2}}{c_i + (a_i \beta_{i-2} - b_i) \beta_{i-1} - a_i \alpha_{i-2}} \\
\beta_i = \frac{a_i \beta_{i-2} \alpha_{i-1} - b_i \alpha_{i-1} + e_i}{c_i + (a_i \beta_{i-2} - b_i) \beta_{i-1} - a_i \alpha_{i-2}} \\
\alpha_i = \frac{g_i}{c_i + (a_i \beta_{i-2} - b_i) \beta_{i-1} - a_i \alpha_{i-2}}
\]

5- Results and conclusions:

The results indicate that we can reach a steady-state solution by using ADI method after choosing appropriate time increment (\( \Delta \tau \)) for the both equations (2.1) and (3.3), it was found that (\( \Delta \tau = 0.00125 \)) for the equation (2.1) and (\( \Delta \tau = 0.00015 \)) for the equation (3.3), respectively. It is also clear from the figures (5.1)-(5.4) how to reach a steady-state after some iterations for different nodal points as it is seen in the figures below:
Figure (5.1)
This figure shows how to reach the steady-state for the temperature taken at the nodal point (7.4).

Figure (5.2)
This figure shows how to reach the steady-state for the temperature taken at the nodal point (8.3).

Figure (5.3)
This figure shows how to reach the steady-state for the stream function taken at the nodal point (8.2).
Figure (5.4)

This figure shows how to reach the steady-state for the stream function taken at the nodal point (3.4).
REFERENCES


