New Initial Parameter for the Constrained Optimization Method

Abbas Y. Al-Bayati **Ban Ahmed Mitras**
profabbasalbayati@yahoo.com *dr.banah.mitras@gmail.com*
College of Computer Sciences and Mathematics
University of Mosul, Iraq

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ABSTRACT

In this paper, we have investigated a new initial parameter in the nonlinear constrained optimization method. The aim of this new method is to make a balance between interior and exterior method for constrained optimization. The new technique has been programmed to solve some of standard problems in the non-linear optimization. The results are too effective when compared with other standard optimization methods like interior and exterior methods.

Keyword: constrained optimization, penalty method, Barrier method.

معلمة ابتدائية جديدة لطريقة الأمثلية المقيدة

بان احمد متراس **عباس يونس البياتي**

كلية علوم الحاسبات والرياضيات، جامعة الموصل

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الملخص

في هذا البحث تم استحداث معلمة ابتدائية جديدة في الأمثلية غير الخطية المقيدة والغرض من الطريقة المقترحة هو العمل على أجراء توازن بين الطريقة الداخلية والخارجية في الأمثلية المقيدة. وإن التقنية الجديدة تمت برمجتها لحل بعض مسائل الأمثلية القياسية المعروفة وتوصلنا إلى أن نتائجها اكثر كفاءة من الطرائق السابقة في هذا المجال مثل طريقة النقطة الداخلية والخارجية بصيغها كافة.

الكلمات المفتاحية: الأمثلية المقيدة، طريقة الجزاء، طريقة الحاجز.

1. Introduction:

Consider the constrained optimization problem

$$\text{Minimize} \quad f(x) \tag{1}$$

Where x required satisfying the general equality constraints

$$c_i(x) = 0 \qquad 1 \le i \le m \tag{2}$$

and the inequality constraints

$$c_i(x) \ge 0 \qquad m+1 \le i \le n \tag{3}$$

where f and c_i map R^n into R. We assume that f(x) and the $c_i(x)$ are twice continuously differentiable. (Toint et. al., 1997)

In this paper we have used exterior-interior point method. The exterior method is used for equality and inequality constraints with the new objective function:

$$\phi_k(x,r) = f(x) + g(r_k)\alpha(x_k) \qquad (4)$$

where $g(r_k)$ is a function of parameter $\alpha(x_k)$ and the remainder of the second term is the penalty function.

The interior method is suitable for equality constraints with the new objective function:

$$\phi_k(x,r) = f(x) + g(r_k)B(x_k) \qquad (5)$$

where $g(r_k)$ is a function of parameter $B(x_k)$, the second term is the Barrier function. (Abdy and Dempster, 1983)

Although both exterior and interior-point methods have many points of similarity, they represent two different points of view. In an exterior-point procedure, we start from an infeasible point and gradually approach feasibility. While doing so, we move away from the unconstrained optimum of the objective function. In an interior-point procedure, we start at a feasible point and gradually improve our objective function, while maintaining feasibility. The requirement that we begin at a feasible point and remain within the interior of the feasible inequality constrained region is the chief difficulty with interior-point methods. In many problems we have no easy way to determine a feasible starting point, and a separate initial computation may be needed. Also, if equality constraints are present, we do not have a feasible inequality constrained region in which to maneuver freely. Thus interior-point methods cannot handle equalities. (Biggs, 1989)

We many readily handle equalities by using a “mixed” method in which we use interior-point penalty functions for inequality constraints only. Thus, if the first m constraints are inequalities and constraints (m+1) to n are equalities, our problem becomes:

$$\text{Minimize} \quad \phi(x,r) = f(x) + g(r_k)B(x_k) + \frac{1}{g(r_k)}\alpha(x_r)$$

The function $\phi(x,r)$ is then minimized for a sequence of monotonically decreasing r>0. (Greig, 1980)

2. A Mixed Exterior-Interior Point Method:

We can solve the constrained problem given in eq.(1) to eq.(3) by constructing a new objective function $\phi(x,r_k)$ which is defined in eq.(5).Now our aim is to minimize the function $\phi(x,r_k)$ by starting from a feasible point x_0 and with initial value $r_0=1$ and the method reducing r_k is simple iterative method such that:

$$r_{k+1} = \frac{r_k}{\mu}, \tag{6}$$

where μ is a constant equal to *10* and the search direction d_k in this case can be defined

$$d_k = -H_k g_k ,` \tag{7}$$

where H_k is a positive definite symmetric approximation matrix to the inverse Hessian matrix G^{-1} and g is the gradient vector of the function $\phi(x, r_k)$.

The next iteration is set to further point

$$x_{k+1} = x_k + \lambda_k d_k , \tag{8}$$

where λ is a scalar chosen in such that $f_{k+1} < f_k$. We thus test $c_i(x_{k+1})$ to see that it is positive for all i. We find a feasible x_{k+1} and we can then proceed with the interpolation. Then the matrix H_k is updated by a correction matrix to get

$$H_{k+1} = H_k + \phi_k \tag{9}$$

where ϕ_k is a correction matrix which satisfies quasi-Newton condition namely $(H_{k+1} y_k = \rho v_k)$ where v_k and y_k are difference vector between two successive points and gradients, respectively and ρ is any scalar.

The initial matrix H_0 chosen to be identity matrix *I*. H_k is updated through the formula of BFGS update. (Fletcher, 1970).

$$H_{k+1} = H_k^{(1)} + H_k^{(2)} \tag{10}$$

where

$$H_k^{(1)} = H_k - \frac{H_k y_k y_k^T H_k}{y_k^T H_k y_k} + ww^T \tag{11}$$

$$H_k^{(2)} = \frac{v_k v_k^T}{v_k^T y_k} \tag{12}$$

and

$$w = (y_k^T H_k y_k)^{0.5} \left(\frac{v_k}{v_k^T y_k} - \frac{H_k y_k}{y_k^T H_k y_k}\right) \tag{13}$$

and terminate of the method if

$$|x_i - x_{i-1}| < \varepsilon \tag{14}$$

where $\varepsilon = 0.000001$, and

$$r_{k+1} = \frac{r_k}{10} \tag{15}$$

(Bazarra and Shetty, 2000)

3. The Interior-Exterior Method: (Nicholas et al., 1997)

Step (1): Find an initial approximation x_0 in the interior of the feasible region for the inequality constraints i.e. $g_i(x_0)<0$.

Step (2): Set $k=1$ and $r_0=1$ is the initial value of r_k.

Step (3): Set $d_k=-H_k g_k$

Step (5): Set $x_{k+1} = x_k + \lambda_k d_k$, where λ is a scalar.

Step (6): Check for convergence i.e. if eq.(14) is satisfied then stop.

Step (7): Otherwise, set $r_{k+1} = \frac{r_k}{10}$ and take $x=x^*$ and set $k=k+1$ and go to Step 3.

4. The New Interior-Exterior Method:

The numerical value of r_k has to be chosen carefully in order to achieve a faster convergence. The exterior method $g(r_k)\alpha(x) \to 0$ as $g(r_k) \to \infty$, and interior method $g(r_k)B(x) \to 0$ as $g(r_k) \to 0$ in order to make a balance for two methods, we have to find r_k such that depend on $\alpha(x),\ B(x)$

The initial value r_0 which is derived as

$$\phi(x, r_k) = f(x) + \sinh(r_k)B(x) + \frac{1}{\sinh(r_k)}\alpha(x) \tag{16}$$

$$= f(x) + \sinh(r_k)\frac{1}{c_k} + \frac{1}{\sinh(r_k)}[h_k]^2 \tag{17}$$

$$\nabla\phi(x, r_k) = \nabla f(x) - \sinh(r_k)\frac{\nabla c_k}{[c_k]^2} + \frac{2h_k \nabla h_k}{\sinh(r_k)} \tag{18}$$

Such that $\nabla\phi(x, r_k) = 0$

We have

$$= \nabla f(x) - \sinh(r_k)\frac{\nabla c_k}{[c_k]^2} + \frac{2h_k \nabla h_k}{\sinh(r_k)} = 0 \tag{19}$$

If we take $\sinh(r_k)>0$, then we have

$$\sinh(r_k)\nabla f(x) - \sinh^2(r_k)\frac{\nabla c_k}{[c_k]^2} + 2h_k \nabla h_k = 0 \tag{20}$$

This implies

$$\sinh^2(r_k)\frac{\nabla c_k}{[c_k]^2} - \sinh(r_k)\nabla f(x) - 2h_k \nabla h_k = 0 \qquad (21)$$

$$\sinh(r_k) = \frac{\nabla f(x) \mp \sqrt{[\nabla f(x)]^2 + 8h_k \nabla h_k \frac{\nabla c_k}{[c_k]^2}}}{2\frac{\nabla c_k}{c_k^2}} \qquad (22)$$

If three points can be selected to find the minimum value of r_k, it is usually sufficient to approximate the function with the quadratic eq.(22), the optimum value of r is then given by one of the following roots to eq.(22).

$$r_{\min} = \sinh^{-1}\left[\frac{\nabla f(x) \mp \sqrt{[\nabla f(x)]^2 + 8h_k \nabla h_k \frac{\nabla c_k}{[c_k]^2}}}{2\frac{\nabla c_k}{c_k^2}}\right] \qquad (23)$$

Let $$A = \frac{\nabla f(x) \mp \sqrt{[\nabla f(x)]^2 + 8h_k \nabla h_k \frac{\nabla c_k}{[c_k]^2}}}{2\frac{\nabla c_k}{c_k^2}} \qquad (24)$$

Then

$$r_{\min} = \sinh^{-1}(A) = \ln(A + \sqrt{A^2 + 1}) \qquad (25)$$

In the above suggestion corresponding to the assumption for deriving a new parameter to make a balance between the previous method, we have suggested the following new method.

5. The New Proposed Method:

Step (1): Find an initial approximation x_0 in the interior of the feasible region for the inequality constraints i.e. $g_i(x_0)<0$.

Step (2): Set $k=1$.

Step (3): Find the initial value of r_o by using eq.(25).

Step (4): Set $d_k=-H_k g_k$

Step (5): Set $x_{k+1} = x_k + \lambda_k d_k$, where λ is a scalar.

Step (6): Check for convergence i.e. if eq.(14) is satisfied then stop. Otherwise go to step 7.

Step (7): Set $r_{k+1} = \dfrac{r_k}{10}$ and take *x=x** and set *k=k+1* and go to step 4.

6. Results and Calculations:

In order to assess the performance of the new method is tested over (6) non-linear test functions with $1 \le n \le 3$ and $1 \le c_i(x) \le 7$.

All the results are obtained using pentium 3. All programs are written in FORTRAN language and for all cases the stopping criterion taken to be

$$|x_i - x_{i-1}| < \delta , \qquad \delta = 10^{-5}$$

In this paper, the two methods used the same exact line search strategy which is the quadratic interpolation technique directly adapted from (Bunday, 1984).

The comparative performance for the two methods is evaluated by considering NOF, NOI, and NOG, where NOF is the number of function evaluations, NOI is the number of iterations and NOG is the number of gradient evaluations.

Discussion:

In table (1) , we have compared our new method with exterior-interior point method.

From table (2), it is clear that taking exterior-interior point algorithm, as the standard (100%). The new method has an improvement on the standard exterior-interior point method in about (18%) NOF, (20%) NOI and (17%) NOG.

Table (1)

Comparative performance of the two algorithms

Test function	Exterior-interior algorithm			New algorithm		
	NOI	(NOF)	NOG	NOI	(NOF)	NOG
1.	2	(164)	56	2	(50)	30
2.	10	(795)	127	8	(715)	105
3.	10	(807)	131	8	(798)	114
4.	5	(2726)	304	3	(1929)	203
5.	7	(99)	34	4	(96)	34
6.	10	(803)	138	10	(901)	146
Total	44	(5394)	790	35	(4489)	632

Table (2)

Improvement Ratio New

	Exterior-interior algorithm	New algorithm
NOF	100%	83.3
NOI	100%	79.5

NOG	**100%**	**83.2**

7. Appendix:

Test functions:

1. $\min f(x) = x_1^2 x_2$ **(4,4)**

s.t.

$$x_1 x_2 - (\frac{x_1^2}{2}) = 6$$

$$x_1 + x_2 \geq 0$$

Bazaraa, (2000)

2. $\min f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$ **s.p(7,9)**

s.t.

$$x_1 - 2x_2 = -1$$

$$\frac{-x_1^2}{4} + x_2^2 + 1 \geq 0$$

3. $\min f(x) = x_1 x_2$ **(1,9)**

s.t.

$$25 - x_1^2 - x_2^2 = 0$$

$$x_1 + x_2 \geq 0$$

4. $\min f(x) = x_1 x_4 (x_1 + x_2 + x_3) + x_3$ **s.p(4,3,3,3)**

s.t.

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40$$

$$x_1 x_2 x_3 \geq 25$$

$$5 \geq x_i \geq 1$$

5. $\min f(x) = x_1^2 + x_2^2$ **s.p.(0.9,2)**

s.t.

$$x_1 + 2x_2 = 4$$

$$x_1^2 + x_2^2 \leq 5$$

$$x_i \geq 0$$

6. $\min f(x) = (x_1 - 3)^2 + (x_2 - 2)^2$ **s.p(0.1,2)**

s.t.

$$x_1 + 2x_2 = 4$$

$$x_1^2 + x_2^2 \leq 5$$

$$x_i \geq 0$$

(see Gottefered, 1973)

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