Raf. J. of Comp. & Math's. , Vol. 4, No. 2, 2007

Solving System of a Linear Fractional Differential Equations by Using Laplace Transformation

Shaker M. Rashid **Yaseen T. Mustafa**

College of Education

University of Dohuk

Received on: 01/03/2007 **Accepted on: 28/06/2007**

ABSTRACT

In this paper, we provide a solution to the system of non-integer differential equation of order $0 < q < 1$, by the technique of Laplace transformation and with interest to property of Mittag-Leffler function, with the help of the programming technique of Maple.

Keywords: Mittag-Leffler function, fractional differential equations, Laplace transform, Maple package.

حل منظومة المعادلات التفاضلية الكسرية الخطية باستعمال تحويل لابلاس

شاكر رشيد **ياسين مصطفى**

كلية التربية، جامعة دهوك

تاريخ الاستلام: 2007/03/01 **تاريخ القبول: 2007/06/28**

الملخص

يتناول هذا البحث حلا لمنظومة من المعادلات التفاضلية ذات الرتب غير الصحيحة (الكسرية) من الرتبة $0 < q < 1$، وذلك بإستعمال تحويلات لابلاس و بالأستفادة من خواص دالة ميتاك – ليفلير، وبالأستعانة باللغة البرمجية مابل.

الكلمات المفتاحية: دالة ميتاك – ليفلير، المعادلات التفاضلية ذات الرتب الكسرية، تحويلات لابلاس، حزمة مابل.

1- Introduction:

Fractional order systems, or systems containing fractional derivatives and integrals, have been studied by many in the engineering area and also by mathematicians. It should be noted that there is a growing number of physical systems whose behavior can be compactly described using fractional system theory. Of specific interest to electrical engineers are long lines (Heaviside, 1922), electrochemical processes (Ichise, Nagayanagi, and Kojima, 1971; Sun, Onaral, and Tsao, 1984), and chaos (Hartley, Lorenzo, and Qammar, 1995) for more details see **[2]**. System of fractional differential equation can be written as follows:

$$\begin{aligned} D_t^q x(t) &= d_t^q x(t) = ay(t) + F(t) \\ D_t^q y(t) &= d_t^q y(t) = bx(t) + G(t) \end{aligned} \qquad (1)$$

In **[2]**, the author used Laplace transform to find the solution for the Fundamental linear fractional order differential equation, where the notation has been defined in Lorenzo and Hartely (1998) **[3]**, it will be assumed for clarity that the problem starts at *t=0*, it is also assumed that all initial conditions are zeros. In this paper, we used this assumption to solve system (1), so we suppose that all initial conditions are set to zero, and by recoursing to properties of Mittage-Leffler function we formulate solutions for the system (1) (as a special solution). Using the Maple language **[4]** for calculating the value of solution for arbitrary q.

We set forth some definitions and theorems; see **[5]** and **[7].**

Definition 1.1: The function $F(t)$ is said to be sectionally continuous over the interval a $\leq t \leq$ b if that interval can be divided into a finite number of subintervals c $\leq t \leq$ d such that in each subinterval:
(***i***) $F(t)$ is continuous in the open interval c $< t <$ d
(***ii***) $F(t)$ approaches a limit as t approaches end point from within the interval; that is, $\lim_{t \to c^+} F(t)$ and , $\lim_{t \to d^-} F(t)$ exist .

Definition 1.2: The function $F(t)$ is said to be of exponential order as $t \to \infty$ if constants M and b and affixed t-value to exist such that $|F(t)| < M \exp(bt)$ for $t \geq t_0$.

Definition 1.3: Function of class A is any function that is:
(***i***) Sectionally continuous over every finite interval in the range $t \geq 0$.
(***ii***) Of exponential order as $t \to \infty$.

Theorem 1.4: If $F(t)$ is a function of class A, $L\{F(t)\}$ exists.

Note 1.5: It is important to realize that this theorem states only that for $L\{F(t)\}$ to exist, it is sufficient that $F(t)$ be of class A. The condition is not necessary, Classic example showing that function other than those of class A do have Laplace transforms is $F(t)= t^{-0.5}$. This function is not sectionally continuous in every finite interval in the rang $t \geq 0$, because $F(t) \to \infty$ as $t \to 0$, but $t^{-0.5}$ is integrable from 0 to any positive t_0. Also $t^{-0.5} \to 0$ as $t \to \infty$ so its of exponential order hence $L\{ t^{-0.5}\}$ exists.

Theorem 1.6: if $F(t)$ a function of class A and if $L\{F(t)\} = f(s)$, then $\lim_{s \to \infty} f(s) = 0$.

Theorem 1.7: If $L^{-1}\{f(s)\}=F(t)$, c ≥ 0, and $F(t)$ be assigned values for $-c \le t \le 0$, then $L^{-1}\{\exp(-cs)f(s)\}=F(t-c)\alpha(t-c)$, where $\alpha(t-c)$ is a step function.

Remark 1.8: If $F(t), F'(t), \ldots, F^{(n-1)}(t)$ are continuous for $t \ge 0$ and of exponential order as $t \to \infty$, and if $F^{(n)}(t)$ is of class A; then

$$L\{F^{(n)}(t)\} = s^n f(s) - \sum_{k=0}^{n-1} s^{n-1-k} F^{(k)}(0)$$

$$L\{\underset{t}{\overset{q}{D}} F(t)\} = s^q f(s) - \sum_{k=0}^{n-1} s^{q-1-k} F^{(k)}(0),$$

where n is an integer and $n-1<q<n$.

The reader can observe that there is no difference between the two forms when all the initial conditions are zero; **[1], [5].**

To Compute the inverse Laplace transformation of $f(s)g(s)$ ($L^{-1}\{f(s)\, g(s)\}$) we use the concepts of Convolution Theorem.

Theorem (Convolution Theorem) 1.9: If $L^{-1}\{f(s)\} = F(t)$, if $L^{-1}\{g(s)\} = G(t)$ and if $F(t)$ and $G(t)$ are functions of class A; then

$$F(t)G(t) = L^{-1}\{f(s)g(s)\} = \int_0^t G(\beta)F(t-\beta)d\beta \qquad (2)$$

Of course, F and G are interchangeable in (2) because f and g in (2) are symmetrically.

Mittag-Leffler function concepts 1.10:

The Mittag-Leffler function $E_\lambda(z)$, named after its originator, the Swedish mathematician Gosta Mittag-Leffler (1846-1927) **[8]**, is defined by

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)} \qquad (3)$$

where z is a complex variable and $\lambda \ge 0$ (for $\lambda = 0$ the radius of convergence of equation (3) is finite, and one has by definition $E_0(z) = \frac{1}{(1-z)}$). The Mittag-Leffler function is a generalization of the exponential function, to which it reduces for $\lambda = 1$, $E_1(z) = \exp(z)$. For $0 < \lambda < 1$ it interpolates between a pure exponential and a hyperbolic function $E_0(z) = \frac{1}{(1-z)}$. The generalized Mittag-Leffler function is

$$E_{\lambda,\eta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \eta)}, \qquad (4)$$

So that $E_{\lambda,1}(z) = E_{\lambda}(z)$, in the simplest λ, η ≥ 0; **[6]**.

2-Main result:

Theorem 2.1: If $F(t)$ and $G(t)$ are functions of class A, in fractional differential system, satisfies the convolution theorem, then (1) has a solution for $0 < q < 1$.

Proof: rewrite the system (1)

$$\begin{aligned} {}_{t}D^{q} X(t) &= {}_{t}d^{q} x(t) = ay(t) + F(t) \\ {}_{t}D^{q} Y(t) &= {}_{t}d^{q} y(t) = bx(t) + G(t) \end{aligned} \qquad \text{(I)}$$

Using Laplace transform we have

$$\begin{aligned} S^q x(s) &= a\, y(s) + f(s) \\ S^q y(s) &= bx(s) + g(s) \end{aligned} \qquad \text{(II)}$$

With initial conditions all zeros. We have

$$\begin{aligned} S^q x(s) - a\, y(s) &= f(s) \\ -bx(s) + S^q y(s) &= g(s) \end{aligned} \qquad \text{(III)}$$

using crammer rule, we find that the solution of the algebraic system (III)

$$x(s) = \frac{\begin{vmatrix} f(s) & -a \\ g(s) & s^q \end{vmatrix}}{\begin{vmatrix} s^q & -a \\ -b & s^q \end{vmatrix}} = \frac{f(s)s^q + ag(s)}{s^{2q} - ab} \qquad \text{(IV)}$$

$$y(s) = \frac{\begin{vmatrix} s^q & f(s) \\ -b & g(s) \end{vmatrix}}{s^{2q} - ab} = \frac{s^q g(s) + bf(s)}{s^{2q} - ab} \qquad \text{(V)}$$

we can expand $\frac{1}{s^{2q} - ab} = \frac{1}{s^{2q}} + \frac{ab}{s^{4q}} + \frac{(ab)^2}{s^{6q}} + \frac{(ab)^3}{s^{8q}} + \dots\dots\dots. = \sum_{n=0}^{\infty} \frac{(ab)^n}{s^{q(2n+2)}} = r(s)$

so $\frac{s^q}{s^{2q} - ab} = \frac{1}{s^q} + \frac{ab}{s^{3q}} + \frac{(ab)^2}{s^{5q}} + \dots\dots\dots. = \sum_{n=0}^{\infty} \frac{(ab)^n}{s^{q(2n+1)}} = M(s)$

Moreover $\frac{1}{s^q} = L\left\{\frac{t^{q-1}}{\Gamma(q)}\right\}$, $q > 0$

Then

$$L^{-1}\left\{\frac{s^q}{s^{2q} - ab}\right\} = \frac{t^{q-1}}{\Gamma(q)} + \frac{(ab)t^{3q-1}}{\Gamma(3q)} + \frac{(ab)^2 t^{5q-1}}{\Gamma(5q)} + \dots = t^{q-1} \sum_{n=0}^{\infty} (ab)^n \frac{t^{2nq}}{\Gamma(2nq + q)}$$

and

$$L^{-1}\left\{\frac{1}{s^{2q}-ab}\right\}=\frac{t^{2q-1}}{\Gamma(2q)}+\frac{(ab)t^{4q-1}}{\Gamma(4q)}+\frac{(ab)^2t^{6q-1}}{\Gamma(6q)}+...=t^{2q-1}\sum_{n=0}^{\infty}\frac{(ab)^n t^{2nq}}{\Gamma(2nq+2q)}=N(t)$$

Return to equation (IV):

$$L^{-1}\{x(s)\}=L^{-1}\left\{\frac{f(s)s^q}{s^{2q}-ab}\right\}+aL^{-1}\left\{\frac{g(s)}{s^{2q}-ab}\right\}$$
, so by using convolution theorem we have

$$X(t)=L^{-1}\{x(s)\}=\int_0^t F(t-\beta)M(\beta)d\beta+a\int_0^t G(t-\beta)N(\beta)d\beta$$

$$=\int_0^t F(t-\beta)\left(\beta^{q-1}\sum_{n=0}^{\infty}\frac{(ab)^n\beta^{2nq}}{\Gamma(2nq+q)}\right)d\beta+a\int_0^t G(t-\beta)\left(\beta^{2q-1}\sum_{n=0}^{\infty}\frac{(ab)^n\beta^{2nq}}{\Gamma(2nq+2q)}\right)d\beta$$

$$X(t)=\sum_{n=0}^{\infty}\int_0^t\frac{F(t-\beta)\beta^{q-1}(ab)^n\beta^{2nq}}{\Gamma(2nq+q)}d\beta+a\sum_{n=0}^{\infty}\int_0^t\frac{G(t-\beta)\beta^{2nq}\beta^{2q-1}(ab)^n}{\Gamma(2nq+2q)}d\beta \qquad \text{(VI)}$$

which can be solved after integration for β.

Now, similarly we can find solution of $y(t)$ that is:

$$L^{-1}\{y(s)\}=y(t)=L^{-1}\left\{\frac{s^q g(s)}{s^{2q}-ab}\right\}+bL^{-1}\left\{\frac{f(s)}{s^{2q}-ab}\right\}$$

$$Y(t)=\int_0^t G(t-\beta)M(\beta)d\beta+b\int_0^t F(t-\beta)N(\beta)d\beta$$
; So

$$Y(t)=\sum_{n=0}^{\infty}\int_0^t\frac{G(t-\beta)\beta^{q-1}(ab)^n\beta^{2nq}}{\Gamma(2nq+q)}d\beta+b\sum_{n=0}^{\infty}\int_0^t\frac{F(t-\beta)\beta^{2nq}\beta^{2q-1}(ab)^n}{\Gamma(2nq+2q)}d\beta \qquad \text{(VII)}$$

Corollary 2.2: Locally solution of system (1) consists of implicitly Mittag-leffler function.

Proof: To prove that we set $K=2nq+2q-1$, $R=2nq+q-1$, for $n = 0, 1, 2,\ldots$, with $0 < q < 1$.

Now the solution (VI) and (VII) have the new forms:

$$X(t)=\int_0^t E_{1,1}(\beta,R)F(t-\beta)(ab)^{\frac{R-q+1}{2q}}d\beta+a\int_0^t E_{1,1}(\beta,K)G(t-\beta)(ab)^{\frac{K-2q+1}{2q}}d\beta$$

$$Y(t)=\int_0^t E_{1,1}(\beta,R)G(t-\beta)(ab)^{\frac{R-q+1}{2q}}d\beta+b\int_0^t E_{1,1}(\beta,K)F(t-\beta)(ab)^{\frac{K-2q+1}{2q}}d\beta$$

Where $E_{1,1}(\beta,K)=\sum_{k=2q-1}^{\infty}\frac{\beta^K}{\Gamma(K+1)}$ is a Mittag-leffler function for:

$$K = \begin{cases} \dfrac{(2m+2)}{L} & \text{for } L = 8 \\ n & \text{for } L = 2 \\ \dfrac{m}{L} & \text{for } L \text{ is even in otherwise} \\ \dfrac{(2(m+1)+3)}{L} & \text{for } L \text{ is odd} \end{cases}$$

And $E_{1,1}(\beta, R) = \sum_{R=q-1}^{\infty} \frac{\beta^R}{\Gamma(R+1)}$; is a Mittag-leffler function for:

$$R = \begin{cases} \dfrac{(2m+3)}{L} & \text{if } L \text{ is even} \\ \dfrac{2(m+1)}{L} & \text{if } L \text{ is odd} \end{cases}$$

Where $L = \frac{1}{q}$, and m = -4, -3, -2,…,10.

3-Example of application:

If we take $F(t) = 1$, $G(t) = t$ and for $a = b = 1$. Then system (1) become as follows:-

$$D_t^q x(t) = d_t^q x(t) = y(t) + 1$$

$$D_t^q y(t) = d_t^q y(t) = x(t) + t$$

and by our main result the solutions of the previous system depending on q as follows:-

Value of q	The solution {X(t),Y(t)}
1/2	$x(t) = 2\sqrt{t} + \ln(-1+\sqrt{t}) - \ln(\sqrt{t}+1) + t + \ln(-1+t)$ $y(t) = \frac{2}{3}t^{3/2} + 2\sqrt{t} + \ln(-1+\sqrt{t}) - \ln(\sqrt{t}+1) + \ln(-1+t)$
1/4	$x(t) = \frac{4}{3}t^{3/4} + 4t^{1/4} + 2\ln\left(-1+t^{1/4}\right) - 2\ln\left(t^{1/4}+1\right)$ $+\frac{2}{3}t^{3/2} + t + 2\sqrt{t} + 2\ln(-1+\sqrt{t})$ $y(t) = \frac{4}{7}t^{7/4} + \frac{4}{5}t^{5/4} + \frac{4}{3}t^{3/4} + 4t^{1/4} + 2\ln\left(-1+t^{1/4}\right) - 2\ln\left(t^{1/4}+1\right)$ $+2\sqrt{t} + \ln\left(-1+\sqrt{t}\right) - \ln\left(\sqrt{t}+1\right) + \ln\left(-1+t\right)$

1/6	$x(t)=\frac{6}{5}t^{5/6}+2\sqrt{t}+6t^{1/6}+3\ln\left(-1+t^{1/6}\right)-3\ln\left(t^{1/6}+1\right)+\frac{3}{5}t^{5/3}$ $+\frac{3}{4}t^{4/3}+t+\frac{3}{2}t^{2/3}+3t^{1/3}+3\ln\left(-1+t^{1/3}\right)$ $y(t)=\frac{6}{11}t^{11/6}+\frac{2}{3}t^{3/2}+\frac{6}{7}t^{7/6}-3\ln\left(t^{1/6}+1\right)+3\ln\left(-1+t^{1/6}\right)+2\sqrt{t}+\frac{6}{5}t^{5/6}$ $+6t^{1/6}+\ln\left(-1+t\right)+3t^{1/3}+2\ln\left(-1+t^{1/3}\right)-\ln\left(1+t^{1/3}+t^{2/3}\right)+\frac{3}{2}t^{2/3}$

1/8	$x(t)=\frac{8}{7}t^{7/8}+\frac{8}{5}t^{5/8}+\frac{8}{3}t^{3/8}+8t^{1/8}+4\ln\left(-1+t^{1/8}\right)-4\ln\left(8t^{1/8}+1\right)$ $+\frac{4}{7}t^{7/4}+\frac{2}{3}t^{3/2}+\frac{4}{5}t^{5/4}+t+\frac{4}{3}t^{3/4}+2\sqrt{t}+4t^{1/4}+4\ln\left(-1+t^{1/4}\right)$ $y(t)=\frac{8}{11}t^{11/8}+\frac{8}{9}t^{9/8}+\frac{8}{7}t^{7/8}-4\ln\left(t^{1/8}+1\right)+\frac{8}{15}t^{15/8}+\frac{8}{13}t^{13/8}+4\ln\left(-1+t^{1/8}\right)$ $+\frac{8}{3}t^{3/8}+\frac{8}{5}t^{5/8}+8t^{1/8}+\frac{4}{3}t^{3/4}+2\ln\left(-1+t^{1/4}\right)-2\ln\left(t^{1/4}+1\right)+2\sqrt{t}+\ln\left(-1+\sqrt{t}\right)$ $-\ln\left(\sqrt{t}+1\right)+4t^{1/4}+\ln\left(-1+t\right)$
1/10	$x(t)=\frac{10}{9}t^{9/10}+\frac{10}{7}t^{7/10}+2\sqrt{t}+\frac{10}{3}t^{3/10}+10t^{1/10}+5\ln\left(-1+t^{1/10}\right)$ $-5\ln\left(t^{1/10}+1\right)+\frac{5}{9}t^{9/5}+\frac{5}{8}t^{8/5}+\frac{5}{7}t^{7/5}+\frac{5}{6}t^{6/5}+t+\frac{5}{4}t^{4/5}+\frac{5}{3}t^{3/5}$ $+\frac{5}{2}t^{2/5}+5t^{1/5}+5\ln\left(-1+t^{1/5}\right)$ $y(t)=\frac{10}{11}t^{11/10}+\frac{10}{9}t^{9/10}+\frac{10}{7}t^{7/10}-5\ln\left(t^{1/10}+1\right)+\frac{2}{3}t^{3/2}+\frac{10}{13}t^{13/10}+\frac{10}{19}t^{19/10}$ $+5\ln\left(-1+t^{1/10}\right)+\frac{10}{17}t^{17/10}+\frac{10}{3}t^{3/10}+2\sqrt{t}+10t^{1/10}+\ln\left(-1+t\right)$ $-\ln\left(-2t^{2/5}-t^{1/5}+\sqrt{5}t^{1/5}-2\right)-\ln\left(2t^{2/5}+t^{1/5}+\sqrt{5}t^{1/5}+2\right)+\frac{5}{4}t^{4/5}+\frac{5}{3}t^{3/5}$ $+\frac{5}{2}t^{2/5}+4\ln\left(-1+t^{1/5}\right)+5t^{1/5}$

REFERENCES

[1] Adam, Loverro (2004) “Fractional Calculus: History, Definitions and Applications for the Engineer”, Notre Dame, IN 46556, U.S.A.

[2] Carl, F. Lorenzo and Tom, T. Haetley, TM- (1999) “Fractional system identifications”, An Approach using continous order-Distributions- Nasa., ohio.

[3] Carl, F. Lorenzo and Tom, T. Haetley, TP- 1998- 208693, “A solution to the Fundemental linear fractional order differential equation”, Nasa., ohio.

[4] Char, B.W., (2001), **Maple 7 learning guide** , Waterloo Maple Inc..

[5] Earl, D. Rainville; Phillip E. Bedient and Richard E. Bediend, (1997), “Elementary Differential Equations”, Macmillan publishing comany, USA.

[6] Érdelyi, A., Magnus W., Oberhettinger F. and Tricomi F.G., 1955, **Higher Transcendental Functions**, McGraw-Hill, New York, vol. III.

[7] John, L. van Lwaarden, (1985), “Ordinary D. E. with numerical Techniques”, USA.

[8] Miller, K. S., 1993, "Integral Transform”, Spec. Funct. 41.