Weighted Points and Lines in Projective Plane of order 17

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ABSTRACT

In the current research ,the points and lines of the projective plane of the seventeenth (17th)order were constructed ,this was followed by examining the arcs designated (k,n;f)- in the plane type (m,n).As a result, it is concluded that these arcs do exist having all their points of the order one or two ,but the order of their lines are m or n only .A further result was arriving at proving the theorems being concluded. The research also included a study of Monoidal arcs with some given examples.

Keywords: projective plane, Monoidal arcs.

نقاط ومستقيمات المستوي الاسقاطي ذي الرتبة 17 ذات الاوزان

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الملخص

في هذا البحث تم تركيب نقاط ومستقيمات المستوي الأسقاطي ذي الرتبة السابعة عشرة ثم حاولنا دراسة القوس -(k,n;f) في هذا المستوي ومن النوع (m,n) وتوصلنا إلى إثبات وجود تلك الأقواس التي تكون جميع نقاطها ذات اوزان واحد أو اثنين ,أما أوزان مستقيماتها فتكون m اوn فقط,مع إعطاء برهان للنظريات التي تم استنتاجها .وكذلك تناول البحث دراسة للأقواس الأحادية في هذا المستوي وأعطينا بعض الأمثلة على ذلك.

الكلمات المفتاحية: المستوي الأسقاطي، الأقواس الأحادية.

1- Introduction:

The existence of (k,n;f)- arcs is studied by D' Agostini .Also [8], [6], [7],and [4] study the existance of these arcs in the projective plane of order 3,5,7 and 9, respectively.

In this paper, we studied the existance of(k,n;f)- arcs in projective plane of order seventeen .

To explain this work we need first the following introduction about the projective plane π .

If f is a function from the set of points of the projective plane π in to the set of natural number N, the value f(p) is called the weight point p and F is a function from the set of lines of π in to N, the value F(L) is called the weight line L, i.e $F(L)=\sum_{p\in L} f(p)$.

A (k,n;f)-arc Kof π is a set of k points such that K does not contain any points of weight zero.

The line L of π is called i-secant if the total weight of L is i, X_j denotes the number of the points having weight j for j=0, 1, 2,……, v (where $v = \max_{p\in k} f(p)$) and we used W_i^j for the number of lines of weight i through a point of weight j; we also denote that the number of lines of weight i is z_i , the integers z_i are called the characters of K . If the points in the plane are only of weight zero and one, then K is a (k, n)-arc.

Let V denote the total weight of K, so by [2] we have:

$$m(q+1)\leq V\leq(n-v)(q+1)+v \text{ ------- (i).}$$

Arcs for which equality holds on the left are called minimal and arcs for which equality holds on the right are called maximal also [2] has proved to be a necessary condition for the existence of a (k,n;f)-arc K of type (m, n), 0<m<n is that

$$q \equiv 0 \mod (n-m) \text{ ------(ii)}$$

and

$$v = n-m \qquad \text{------(iii)}$$

2- <u>(k,n;f)- arc of type (n-17,n)with Li>0, i=0, 1, 2, Lj = 0, j=3,.....,17.</u>

From [1], a (k, n; f)- arc of type (m, n) requires that $q \equiv 0 \mod (n-m)$, m>0, $v\leq n-m$ and $(n-v)(q+1)\leq V \leq (n-v)(q+1)+v$.

For a (k,n;f)-arc of type (n-17,n) with n-m = 17 and for discussion of maximality and minimality of this arc, we study the following conditions:

$X_0>0, X_1>0, X_2>0, X_i = 0$,for i =3,.....,17.

For minimal V = (n-17) (q+1),from ([6],p.12) we deduce that:

$$\left.\begin{array}{ll} W_{n-17}^0 = q+1 & W_n^0 = 0 \\ W_{n-17}^1 = (16/17)\, q+1 & W_n^1 = (1/17)q \\ W_{n-17}^2 = (15/17)\, q+1 & W_n^2 = (2/17)q \end{array}\right\} \text{------- (1)}$$

<u>Lemma (2-1)</u>:

There is no point of weight zero on any n-secants of a (k,n;f)-arc.

<u>Proof:</u> Follows from equation (1),since $Wn^0=0$.#

<u>Theorem(2-2):</u> The total number of n-secants and (n-17)- secants of the (k,n;f)-arc of type (n-17,n) in PG (2, 17) is:

Z_n= (1/17)(n-17)q
and ------- (2)
Z_{n-17} = (1/17)($17q^2$+34q-nq+17)

Proof :

Let Z_{n-17} be the number of lines of weight n-17 and Z_n the number of lines of weight n ;then

$Z_{n-17}+Z_n=q^2+q+1$ --------(a1)

$(n-17)Z_{n-17}+nZn=V(q+1)=(n-17)(q+1)^2$ --------(a2)

solving equations (a1)&(a2) we get

Zn=1/17(n-17)q

$Z_{n-17}=1/17(17q^2+34q-nq+17)$.#

Now let Y be an n-secants and suppose that on Y there are **a** points of weight two and **b** points of weight one.The counting points of Y, gives the following:

a + b=q+1

and the weight of points of Y is

2a + b = n

so, we get

$$\left.\begin{aligned} b &= 2\,(q+1) - n \\ a &= n-(q+1) \end{aligned}\right\} \qquad \text{------- (3)}$$

Counting the incidences between points of weight 1and n- secant gives:

$X_1W_n^1=Z_n\,b$

by using equation (1),(2) & (3) , we have:

X_1=(n-17)(2q+2-n) ----------- (4)

Simillarly, counting incidence between points of weight 2 and the n-secants gives:

$X_2W_n^2=Z_n\,a$

Hence using (1), (2) & (3), we have:

X_2=[(n-17)(n-q-1)]/2 ----------- (5)

Since

$X_0+X_1+X_2= q^2+q+1$

then by using equations(4) & (5), we get

$2q^2+(53-3n)q+n^2-20n+53-2X_0= 0$ ------------(6)

For a solution of q in (6), we require

$(n-79)^2-(3856-16X_0)$ is a square

that is : $\delta^2=(n-79)^2-(3856-16X_0)$ -------------(7)

3-<u>Arcs in PG(2,17)</u>:

Let PG(2,17) be a projective plane of order 17.From([5]chapter-3), we attained that this plane contains 307 points, 307 lines, 18 points on every line and 18 lines through every point.

Let L_1be the line which contains the points: (1, 2, 10, 16, 87, 110, 120, 152, 176, 180, 192, 211, 233, 254, 259, 272, 279, 306).

The line L_2 can be obtain by adding one to each number of the line L_1 to get L_2 (2, 3, 11, 17, 88, 111, 121, 153, 177, 181, 193, 212, 234, 255, 260, 273, 280, 307).

Simillarly we can find all others 305 lines of this plane by adding in each step one to get the other line.

4-<u>(150,29;f)-arc of type (12,29) in PG(2,17)</u>.

To find this arc we need to find the value of X_0 which makes equation (7) be a square and then find from it n and δ i.e we try to check X_0 from 1 to 307 and after a large number of efforts we found that, when $X_0=157$, we get $\delta=34$ and n=29, and from (6) we have q=17, which means that (k, n; f)-arc of type (m, n) exist in PG (2,17).
From (4) and (5), we get
$X_1= 84$ and $X_2= 66$
Which implies that this arc is (150, 29; f)-arc of type (12, 29) and the point of weight 0 form (157, 12)-arc in PG (2, 17).

5-<u>Classification of the lines of the plane with respect to the (150,29;f)-arc of type (12,29)</u>.

Let L be 12-secant having on it **a** points of weight 2, **b** points of weight 1 and **c** points of weight zero, then
a + b + c=18 -------------------(8)
and the weights of the points on L gives
2a + b = 12 --------------------(9)

Let N be an 29-secant of a (150, 29; f)-arc, since there are no points of weight 0 on a 29-secant because $W_n^0=0$, then we can assume that on N there are **a** points of weight 2 and **b** points of weight 1, then a + b = 18 and the weight of points on N gives:
2a + b = 29
so a = 11 and b = 7
and by solving equation (8) & (9), we have the following lemma.

<u>Lemma (5-1)</u>: The lines of PG (2, 17) are partitioned in to eight classes with respect to a minimal (150, 29; f)-arc of type (12, 29) as in table-1

Table -1

Type of the lines	a	b	c
L_0	0	12	6
L_1	1	10	7
L_2	2	8	8
L_3	3	6	9
L_4	4	4	10
L_5	5	2	11
L_6	6	0	12
L_7	11	7	0

<u>Corollary(5-1-1)</u>:
There is no point of weight **1** on the 12- secant of (157,12)- arc.

<u>Corollary(5-1-2)</u>: There is no point of weight **2** on the 6- secant of (157,12)- arc.

<u>Lemma(5-2)</u>: For the existence of (150,29;f)-arc K of type (12,29) with 157 points of weight zero in PG(2,17),we have the following:
1- The number Z_{29} of 29- secants of K is 12.
2- The number Z_{12} of 12- secants of K is 295.
3- The number X_2 of points of weight 2 is 66.
4- The number X_1 of points of weight 1 is 84.
<u>Proof</u>:
Follows from equations (2), (4) & (5).#

<u>Lemma(5-3)</u>: No thirteen points of weight zero can be collinear.
<u>Proof</u>:
Let L be 13- secant having on it 13 points of weight 0, then the remaining points of L is five, if all these points of weight 2 we get the weight of the line L is ten which is a contraduction . Since the weight of L is 12.#

<u>Corollary(5-3-1)</u>: The points of weight 0 form (157,12)-arc in PG(2,17).

6- <u>Monoidal arc</u>: In this section we consider (k,n;f)- arc which have only one point of weight greater than one , such arcs are called monoidal arcs, i.e a (k, n; f)- arc is called monoidal if Imf={0,1,v}and X_v=1.

<u>Example(6-1)</u>: An example of a monoidal arc can be obtained as follows:

From (6) when X_0 =272 and n=19, we get q=17, and from (4) & (5), we have X_1=34 & X_2=1 i.e there is a (35, 19; f)- arc of type (2,19) in PG(2,17) consists of the following points :
Pi for i = {3, 4, 12, 18, 89, 112, 122, 154, 178, 182, 194, 213, 235, 256, 261, 274, 281, 1, 19, 27, 33,104, 127, 137, 169, 193, 197, 209, 228, 250, 271, 276, 289, 296,16}
with $V(p_i)$=1 for i = {3, 4, 12, 89, 112, 122, 154, 178, 182, 194, 213, 235, 256, 261, 274, 281, 1, 19, 27, 33, 104, 127, 137, 169, 193, 197, 209, 228, 250, 271, 276, 289, 296,16}
and $V(p_i)$=2 ,for i=18
and the other points of the plane are of weight zero and form (272, 17)-arc
The lines of this arc are 17-secants, 16- secants and 0- secants and
1- The number of 17-secants are 16.
2- The number of 16-secants are 289.
3- The number of 0-secants are 2.
so the total number is 307.
see table-2.

Since table-2 contains 307 lines, so it is not suitable to write it completly.For this reason it is enough to write only the first twenty lines of the plane and the last twenty lines of it.

<u>Lemma (6-2)</u>: There is one point of weight 2 on each 17-secants of (272,17)- arc .

<u>Proof</u>:

Let L be 17-secants of (272, 17)-arc and having two points of weight 2, then the weight of L because 4 and this is a contraduction because the weight of all lines of the plane is either 2 or 19.#

<u>Corollary(6-2-1)</u>: There is no point of weight 1 on each 17- secants of (272,17)- arc.

<u>Lemma (6-3)</u>: There is no point of weight 2 on each 16- secant of (272,17)- arc.

<u>Proof:</u>

Let L is 16- secant line the remaining points of L is two .If one of these two points of weight 2 means that the weight of L is 3, which is a contraduction as in the previous lemma.#

<u>Corollary(6-3-1):</u> There are two points of weight 1 on each (16-secant) line of (272,17)- arc.

<u>Theorem(6-4)</u>: Each 0- secants line of (272, 17)- arc contains one point of weight 2 and seventeen points of weight 1.

Proof:From lemma (2-1), and type (2,19) of these lines, we know that the weight of each 0- secant line is 19 , this means that each 0- secant line must be contain one point of weight 2 and 17 points of weight 1 .#
see table -2.

In this table we have:

1- The **star** points are points of weight 2.
2- The **underlined** blodface points are points of weight 1.
3- The **other** points are of weight zero.

Table (1)

Lines	Points																		Type	$V(L_J)$
L_{288}	288	**289**	297	303	67	90	100	132	156	160	172	191	**213**	234	239	252	259	286	16-secant	2
L_{289}	**289**	290	298	304	68	91	101	133	157	161	173	192	214	**235**	240	253	260	287	16-secant	2
L_{290}	290	291	299	305	69	92	102	134	158	162	174	**193**	215	236	241	254	**261**	288	16-secant	2
L_{291}	291	292	300	306	70	93	103	135	159	163	175	**194**	216	237	242	255	262	**289**	16-secant	2
L_{292}	292	293	301	307	71	94	**104**	136	160	164	176	195	217	238	243	**256**	263	290	16-secant	2
L_{293}	293	294	302	**1**	72	95	105	**137**	161	165	177	196	218	239	244	257	264	291	16-secant	2
L_{294}	294	295	303	2	73	96	106	138	162	166	**178**	**197**	219	240	245	258	265	292	16-secant	2
L_{295}	295	**296**	304	**3**	74	97	107	139	163	167	179	198	220	241	246	259	266	293	16-secant	2
L_{296}	**296**	297	305	**4**	75	98	108	140	164	168	180	199	221	242	247	260	267	294	16-secant	2
L_{297}	297	298	306	5	76	99	109	141	165	**169**	181	200	222	243	248	**261**	268	295	16-secant	2
L_{298}	298	299	307	6	77	100	110	142	166	170	**182**	201	223	244	249	262	269	**296**	16-secant	2
L_{299}	299	300	**1**	7	78	101	111	143	167	171	183	202	224	245	**250**	263	270	297	16-secant	2
L_{300}	300	301	2	8	79	102	**112**	144	168	172	184	203	225	246	251	264	**271**	298	16-secant	2
L_{301}	301	302	**3**	9	80	103	113	145	**169**	173	185	204	226	247	252	265	272	299	16-secant	2
L_{302}	302	303	**4**	10	81	**104**	114	146	170	174	186	205	227	248	253	266	273	300	16-secant	2
L_{303}	303	304	5	11	82	105	115	147	171	175	187	206	**228**	249	254	267	**274**	301	16-secant	2
L_{304}	304	305	6	**12**	83	106	116	148	172	176	188	207	229	**250**	255	268	275	302	16-secant	2
L_{305}	305	306	7	13	84	107	117	149	173	177	189	208	230	251	**256**	269	**276**	303	16-secant	2
L_{306}	306	307	8	14	85	108	118	150	174	178	190	**209**	231	252	257	270	277	304	16-secant	2
L_{307}	307	**1**	9	15	86	109	119	151	175	179	191	210	232	253	258	**271**	278	305	16-secant	2

Table-2

Lines	Points																		Type	$V(L_J)$
L_1	**1**	2	10	**16**	87	110	120	152	176	180	192	211	233	254	259	272	279	306	16-secant	2
L_2	2	**3**	11	17	88	111	121	153	177	181	**193**	212	234	255	260	273	280	307	16-secant	2
L_3	**3**	**4**	**12**	**18***	**89**	**112**	**122**	**154**	**178**	**182**	**194**	**213**	**235**	**256**	**261**	**274**	**281**	**1**	0-secant	19
L_4	**4**	5	13	**19**	90	113	123	155	179	183	195	214	236	257	262	275	282	2	16-secant	2
L_5	5	6	14	20	91	114	124	156	180	184	196	215	237	258	263	**276**	283	**3**	16-secant	2
L_6	6	7	15	21	92	115	125	157	181	185	**197**	216	238	259	264	277	284	**4**	16-secant	2
L_7	7	8	**16**	22	93	116	126	158	**182**	186	198	217	239	260	265	278	285	5	16-secant	2
L_8	8	9	17	23	94	117	**127**	159	183	187	199	218	240	**261**	266	279	286	6	16-secant	2
L_9	9	10	**18***	24	95	118	128	160	184	188	200	219	241	262	267	280	287	7	17-secant	2
L_{10}	10	11	**19**	25	96	119	129	161	185	189	201	220	242	263	268	**281**	288	8	16-secant	2
L_{11}	11	**12**	20	26	97	120	130	162	186	190	202	221	243	264	269	282	**289**	9	16-secant	2
L_{12}	**12**	13	21	**27**	98	121	131	163	187	191	203	222	244	265	270	283	290	10	16-secant	2
L_{13}	13	14	22	28	99	**122**	132	164	188	192	204	223	245	266	**271**	284	291	11	16-secant	2
L_{14}	14	15	23	29	100	123	133	165	189	**193**	205	224	246	267	272	285	292	**12**	16-secant	2
L_{15}	15	**16**	24	30	101	124	134	166	190	**194**	206	225	247	268	273	286	293	13	16-secant	2
L_{16}	**16**	17	25	31	102	125	135	167	191	195	207	226	248	269	**274**	287	294	14	16-secant	2
L_{17}	17	**18***	26	32	103	126	136	168	192	196	208	227	249	270	275	288	295	15	17-secant	2
L_{18}	**18***	**19**	**27**	**33**	**104**	**127**	**137**	**169**	**193**	**197**	**209**	**228**	**250**	**271**	**276**	**289**	**296**	**16**	0-secant	19
L_{19}	**19**	20	28	34	105	128	138	170	**194**	198	210	229	251	272	277	290	297	17	16-secant	2
L_{20}	20	21	29	35	106	129	139	171	195	199	211	230	252	273	278	291	298	**18***	17-secant	2

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