The Finite Difference Methods for φ^4 Klein-Gordon Equation

Saad A. Manaa
College of Computer Sciences and Mathematics
University of Mosul

Laya Y. Hawell
College of Education
University of Duhok

Received on: 15/11/2006 **Accepted on: 24/12/2006**

ABSTRACT

We solved φ^4 Klein-Gordon equation numerically by using two finite difference methods: The first is the explicit method and the second is the implicit (Crank-Nicholson) method. Also, we studied the numerical stability of the two methods using Fourier (Von Neumann) method and it has been found that the first method is simpler and has faster convergence while the second method is more accurate, and the explicit method is conditionally stable while the implicit method is unconditionally stable.

Keywords: φ^4 Klein-Gordon equation, finite difference methods, explicit method, implicit (Crank-Nicholson) method, numerical stability, Fourier (Von Neumann) method, convergence.

طرائق الفروقات المنتهية لمعادلة φ^4 Klein-Gordon

سعد عبد الله مناع
كلية علوم الحاسوب والرياضيات، جامعة الموصل

ليا هاويل
كلية التربية، جامعة دهوك

تاريخ الاستلام: 2006/11/15 **تاريخ القبول: 2006/12/24**

الملخص

تم حل معادلة Klein-Gordon φ^4 عدديا باستخدام طريقتين من طرائق الفروقات المنتهية : الاولى هي الطريقة الصريحة والثانية هي الطريقة الضمنية (Crank-Nicholson) ثم تمت دراسة الاستقرارية العددية لكلتا الطريقتين وتبين من خلال دراسة الحل والاستقرارية بان الطريقة الصريحة هي الاسهل و الاسرع تقاربا من الطريقة الضمنية بينما الطريقة الضمنية هي الادق كذلك الطريقة الصريحة مستقرة بشروط بينما الطريقة الضمنية مستقرة من دون شرط.

الكلمات المفتاحية: معادلة Klein-Gordon φ^4، طرائق الفروقات المنتهية، الطريقة الصريحة، الطريقة الضمنية (Crank-Nicholson)، الاستقرارية العددية، التقارب.

1. Introduction

Partial differential equations are used to formulate and solve problems that involve unknown functions of several variables, such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, elasticity, or more generally any process that is distributed in space, or distributed in space and time. Very different physical problems may have identical mathematical formulations [16].

There is a long history in physics and mathematics of trying to find new nontrivial solutions to nonlinear wave equations. The literature on the subject goes back at least as far as 1845 when Russell published a paper about a surface wave he witnessed traveling for almost two miles in a shallow water channel [4].

2. The Mathematical Model

The Klein-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u - f(u) \qquad ...(1)$$

is one of the nonlinear extensions of the wave equation. For example, such an equation describes the vibration of a string that lies on an elastic foundation with nonlinear elastic forces. The elastic force density is describing by function $f(u)$ [9].

when $f(u) = \sin(u)$, then equation (1) becomes sine-Gordon equation, which is found by Zabusky and kruskal in 1965.

Fiore et al. (2005) gave arguments for the existence of exact travelling wave solutions of a perturbed sine Gordon equation on the real line or on the circle and classified them [5].

If $f(u) = mu - \varepsilon u^3$ then equation (1) becomes φ^4-nonlinear Klein Gordon equation(φ^4 equation)[2]:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^3 \qquad ...(2)$$

or[10]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^3 \qquad ...(3)$$

with initial and boundary conditions [12]

$$u(x,0) = f(x) \text{ and } \frac{\partial u(x,0)}{\partial t} = 0 \quad , \; t \geq 0 \;, \; 0 < x < 2\pi$$

$$u(0,t) = u(2\pi,t) = 0.$$

Equation (3) arises in quantum field theory with m denoting mass and ε is coupling constant [13].

The φ^4 equation was first proposed by Aubry , Krumhansl and Schrieffer in 1975 and 1976, to describe displacive and order-disorder transitions in solids, mainly magnetic compounds [15].

Sànchez et al. (1991) studied the propagation of topological solitons in a perturbed φ^4 equation, by means of numerical simulations tested on exact predictions for this system. [1].

Manna et al. (1997) proved that the antikink solution of φ^4 equation which was never obtained perturbatively can be obtained by perturbation expansion in the wave-number in the short-wave limit [10].

Nicolas in (2002) solved the global Cauchy problem for the φ^4 equation outside a black hole. Then using a Penrose compactificatin, he proved, in the mass less case, the existence of smooth asymptotic profiles and Sommerfeld radiation conditions, at the horizon and at null infinity, for smooth solutions [7].

Dmitriev et al. (2006) discussed some discrete φ^4 equations free of the Peierls-Nabarro barrier and identified for them the full space of available static solutions, including those derived recently in Physics but not limited to them [3].

3. Derivation of Explicit Method for φ^4 Equation

In this method we evaluate the unknown function $u_{p,q+1}$ at (t_{q+1}) depending on the known function $u_{p+1,q}$, $u_{p,q}$, $u_{p-1,q}$ at (t_q) and $u_{p,q-1}$ at (t_{q-1}) . We start by partition the rectangle $R =\{ (x,t) : 0 \le x \le 2\pi,\ 0 \le t \le b\}$ into a grid consisting of (n-1) by (m-1) rectangles with sides $\Delta x = h$ and $\Delta t = k$ [11]:

Start at the bottom row, where $t = t_1 = 0$ and the solution is known to be $u(x_p,t_1) = f(x_p)$.

we shall use a difference equation method to compute approximations

$\{u_{p,q} : p = 1,2,\ldots,n\}$ in successive rows for $q = 2,3,\ldots,m\}$

Now equation (3) becomes :

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{u_{p+1,q} - 2u_{p,q} + u_{p-1,q}}{h^2} - mu_{p,q} + \varepsilon u^3_{p,q}$$

$$\Rightarrow u_{p,q+1} - 2u_{p,q} + u_{p,q-1} = \frac{k^2}{h^2}[\, u_{p+1,q} - 2u_{p,q} + u_{p-1,q}\,] - k^2 m u_{p,q} + k^2 \varepsilon u^3_{p,q}$$

putting $r = k/h$

$\Rightarrow u_{p,q+1} - 2u_{p,q} + u_{p,q-1} = r^2(u_{p+1,q} - 2u_{p,q} + u_{p-1,q}) - k^2 m u_{p,q} + k^2 \varepsilon u^3_{p,q}$

$\Rightarrow u_{p,q+1} = r^2(u_{p+1,q} + u_{p-1,q}) + (2 - 2r^2 - k^2 m + \varepsilon k^2 u^2_{p,q})u_{p,q} - u_{p,q-1}$ ...(4)

Equation (4) represent the explicit difference approximation for φ^4 equation, where it is employed to find the row (q+1) across the grid where the approximation in both rows **(**q) and
(q–1) are known. The four known values on the right side $u_{p,q-1}$, $u_{p-1,q}$, $u_{p+1,q}$ and $u_{p,q}$ which are used to create the approximation $u_{p,q+1}$.

In order to find the computations we need to find the values of the second row as $t = t_2$, putting q = 1 in equation (4) yields

$$u_{p,2} = r^2(u_{p+1,1} + u_{p-1,1}) + (2 - 2r^2 - k^2 m + \varepsilon k^2 u^2_{p,1})u_{p,1} - u_{p,0} \quad ...(5)$$

A central difference approximation to the initial derivative condition gives that $\dfrac{u_{p,2} - u_{p,0}}{2k} = 0$

$$\Rightarrow u_{p,0} = u_{p,2} \quad ...(6)$$

substituting equation (6) in equation (5) yields

$$2u_{p,2} = r^2(u_{p+1,1} + u_{p-1,1}) + (2 - 2r^2 - k^2 m + \varepsilon k^2 u^2_{p,1})u_{p,1}$$

$$\Rightarrow u_{p,2} = \frac{r^2}{2}(u_{p+1,1} + u_{p-1,1}) + (1 - r^2 - \frac{k^2 m}{2} + \frac{k^2 \varepsilon}{2} u^2_{p,1})u_{p,1} \quad ...(7)$$

4. Derivation of Crank-Nicolson Method for φ^4 Equation

This method invented by John Crank and Phyllis Nicolson, in 1947, is based on numerical approximations for solutions. They replaced u_{xx} by the means of its finite difference representation on the (q-1)th and(q+1)th time rows [6,11].

Approximated the equation (1) by [6]

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{1}{2}\Big[\frac{u_{p+1,q-1} - 2u_{p,q-1} + u_{p-1,q-1}}{h^2} + \frac{u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}}{h^2}\Big] - m u_{p,q} + \varepsilon u^3_{p,q}$$

$$\Rightarrow 2u_{p,q+1} - 4u_{p,q} + 2u_{p,q-1} = r^2[(u_{p+1,q-1} - 2u_{p,q-1} + u_{p-1,q-1}) + (u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1})] - 2mk^2 u_{p,q} + 2\varepsilon k^2 u^3_{p,q}$$

where r = k/h

$$\Rightarrow 2u_{p,q+1} - r^2(u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}) = 4u_{p,q} - 2u_{p,q-1} + r^2(u_{p+1,q-1}$$

$$- 2u_{p,q-1} + u_{p-1,q-1}) - 2mk^2u_{p,q} + 2\varepsilon k^2u^3_{p,q}$$

$$\Rightarrow (2 + 2r^2)u_{p,q+1} - r^2(u_{p+1,q+1} + u_{p-1,q+1}) = (4 - 2mk^2 + 2\varepsilon k^2u^2_{p,q})u_{p,q} + r^2(u_{p+1,q-1} + u_{p-1,q-1}) - (2 + 2r^2)u_{p,q-1} \quad \ldots(8)$$

Equation (8) represents the implicit difference approximation for φ^4 equation where the left side of equation (8) contains three unknowns along (q+1)th time row and the right side four known values of u along the (q)th and (q-1)th time rows .

Equation (8) forms a tridaigonal linear system AX = B.

The boundary conditions are used in the first and last equations only,

$$u_{1,q-1} = u_{1,q+1} = 0 \quad \text{and} \quad u_{n,q-1} = u_{n,q+1} = 0$$

Crank-Nicolson equation (8) can be written in matrix form AX = B as follows:

$$\begin{bmatrix} 2+2r^2 & -r^2 & & & & \\ -r^2 & 2+2r^2 & -r^2 & & & \\ & -r^2 & 2+2r^2 & -r^2 & & \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & -r^2 & 2+2r^2 & -r^2 \\ & & & & -r^2 & 2+2r^2 \end{bmatrix} \begin{bmatrix} u_{2,q+1} \\ u_{3,q+1} \\ u_{4,q+1} \\ u_{5,q+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-3,q+1} \\ u_{n-2,q+1} \\ u_{n-1,q+1} \end{bmatrix} =$$

$$\begin{bmatrix} (4-2k^2m+2k^2\varepsilon u^2_{2,q})u_{2,q} - (2+2r^2)u_{2,q-1} + r^2u_{3,q-1} \\ r^2u_{2,q-1} + (4-2k^2m+2k^2\varepsilon u^2_{3,q})u_{3,q} - (2+2r^2)u_{3,q-1} + r^2u_{4,q-1} \\ r^2u_{3,q-1} + (4-2k^2m+2k^2\varepsilon u^2_{4,q})u_{4,q} - (2+2r^2)u_{4,q-1} + r^2u_{5,q-1} \\ \cdot \\ \cdot \\ \cdot \\ r^2u_{n-3,q-1} + (4-2k^2m+2k^2\varepsilon u^2_{n-2,q})u_{n-2,q} - (2+2r^2)u_{n-2,q-1} + r^2u_{n-1,q} \\ r^2u_{n-2,q-1} + (4-2k^2m+2k^2\varepsilon u^2_{n-1,q})u_{n-1,q} - (2+2r^2)u_{n-1,q-1} \end{bmatrix}$$

When the Crank-Nicolson method is implemented with a computer, the linear system AX = B can be solved by either direct methods or by iterative methods.

In order to start the computations we need to find the values in the second row when $t = t_2$, putting q = 1 in equation (8) yields

$$(2 + 2r^2)u_{p,2} - r^2(u_{p+1,2} + u_{p-1,2}) = (4 - 2mk^2 + 2\varepsilon k^2u^2_{p,1})u_{p,1}$$
$$+ r^2(u_{p+1,0} + u_{p-1,0}) - (2 + 2r^2)u_{p,0} \qquad \ldots(9)$$

substituting equation (6) in equation (9) yields

$$2(2 + 2r^2)u_{p,2} - 2r^2(u_{p+1,2} + u_{p-1,2}) = (4 - 2mk^2 + 2\varepsilon k^2u^2_{p,1})u_{p,1}$$

$$\Rightarrow (1+ r^2)u_{p,2} - \frac{r^2}{2}(u_{p+1,2} + u_{p-1,2}) = (1 - \frac{k^2m}{2} + \frac{k^2\varepsilon}{2}u^2_{p,1})\, u_{p,1} \qquad \ldots(10)$$

Equation (10) forms a tridaigonal linear system AX = B.

where the boundary conditions are used in the first and last equations only,

$u_{1,2} = 0$ and $u_{n,2} = 0$

Equation (10) can be written in matrix form AX = B as follows

$$\begin{bmatrix} 1+r^2 & -\frac{r^2}{2} & & & & \\ -\frac{r^2}{2} & 1+r^2 & -\frac{r^2}{2} & & & \\ & -\frac{r^2}{2} & 1+r^2 & -\frac{r^2}{2} & & \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & -\frac{r^2}{2} & 1+r^2 & -\frac{r^2}{2} \\ & & & & -\frac{r^2}{2} & 1+r^2 \end{bmatrix} \begin{bmatrix} u_{2,2} \\ u_{3,2} \\ u_{4,2} \\ u_{5,2} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-3,2} \\ u_{n-2,2} \\ u_{n-1,2} \end{bmatrix} = \begin{bmatrix} (1-\frac{k^2m}{2}+\frac{k^2\varepsilon}{2}u^2_{2,1})u_{2,1} \\ (1-\frac{k^2m}{2}+\frac{k^2\varepsilon}{2}u^2_{3,1})u_{3,1} \\ (1-\frac{k^2m}{2}+\frac{k^2\varepsilon}{2}u^2_{4,1})u_{4,1} \\ \cdot \\ \cdot \\ \cdot \\ (1-\frac{k^2m}{2}+\frac{k^2m}{2}u^2_{n-2,1})u_{n-2,1} \\ (1-\frac{k^2m}{2}+\frac{k^2\varepsilon}{2}u^2_{n-1,1})u_{n-1,1} \end{bmatrix}$$

5. Stability Analysis

Mathematically stable means small perturbations in the initial data (or small error at any time) remain small at later times. However, if small changes in the initial data produce larges in the final results.

6. Stability Analysis of Explicit Method by Fourier (Von Neumann) Method

To apply Von Neumann method on equation (1), we go to linearized stability analysis and we get after we eliminate the non linear term that [6,8,14]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - mu \qquad ...(11)$$

By using the explicit method for equation (11) we obtain

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{u_{p+1,q} - 2u_{p,q} + u_{p-1,q}}{h^2} - mu_{p,q} \qquad ...(12)$$

replacing $u_{p,q}$ by $\xi^q e^{i\beta ph}$ in equation (12) yields

$$\frac{\xi^{q+1}e^{i\beta ph} - 2\xi^q e^{i\beta ph} + \xi^{q-1}e^{i\beta ph}}{k^2} = \frac{\xi^q e^{i\beta(p+1)h} - 2\xi^q e^{i\beta ph} + \xi^q e^{i\beta(p-1)h}}{h^2}$$

$$- m\,\xi^q e^{i\beta ph}$$

$$\Rightarrow \xi^q e^{i\beta ph}[\xi - 2 + \xi^{-1}] = \xi^q e^{i\beta ph}[r^2(e^{i\beta h} - 2 + e^{-i\beta h}) - mk^2]$$

where $r = k/h$

dividing by $\xi^q e^{i\beta ph}$ leads to

$$\xi - 2 + \xi^{-1} = r^2(e^{i\beta h} - 2 + e^{-i\beta h}) - mk^2$$

$$\Rightarrow \xi - 2 + \xi^{-1} = r^2(-4\sin^2(\frac{\beta h}{2})) - mk^2$$

multiplying by ξ leads to

$$\Rightarrow \xi^2 - 2\xi + 1 = -4r^2\xi\sin^2(\frac{\beta h}{2}) - mk^2\xi$$

$$\Rightarrow \xi^2 - 2\xi + 1 + 4r^2\xi\sin^2(\frac{\beta h}{2}) + mk^2\xi = 0$$

$$\Rightarrow \xi^2 - 2A\xi + 1 = 0$$

where $A = 1 - 2r^2\sin^2(\frac{\beta h}{2}) - \frac{mk^2}{2}$ ...(13)

Hence the values of ξ are

$$\xi_{1,2} = A \mp \sqrt{A^2 - 1}$$

As r, β, k, h and m are positive real, A< 1 by equation(13).

Where ξ is Amplification factor, the necessary and sufficiently condition for numerical stability is $|\xi| \leq 1$ [6].

when $A < -1$, $|\xi_2| > 1$, giving instability .

when

$$-1 \le A < 1,\ A^2 \le 1 \Rightarrow \xi_{1,2} = A \mp i(1-A^2)^{1/2}$$

hence

$$\left|\xi_{1,2}\right| = \{A^2 + (1-A^2)\}^{1/2} = 1$$

proving that the equation (12) is stable for $-1 \le A < 1$. By equation (13), we then have

$$-1 \le 1 - 2r^2 \sin^2(\frac{\beta h}{2}) - \frac{mk^2}{2} < 1$$

The only useful inequality is

$$-1 \le 1 - 2r^2 \sin^2(\frac{\beta h}{2}) - \frac{mk^2}{2}$$

$$\Rightarrow -2 \le -2r^2 \sin^2(\frac{\beta h}{2}) - \frac{mk^2}{2}$$

$$\Rightarrow 2 \ge 2r^2 \sin^2(\frac{\beta h}{2}) + \frac{mk^2}{2} \qquad ...(14)$$

since $\sin^2(\frac{\beta h}{2}) = 1$ for some values of β hence equation (14) becomes

$$2 \ge 2r^2 + \frac{mk^2}{2}$$

$$\Rightarrow k^2 \le \frac{4h^2}{4 + h^2 m} \qquad ...(15)$$

Inequality (15) represents the imposed condition for explicit method for φ^4 equation to be stable.

7. Stability Analysis of Crank-Nicolson Method by Fourier (Von Neumann) Method

By using the Crank-Nicolson method for equation (11) we obtain

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{1}{2}\Big[\frac{u_{p+1,q-1} - 2u_{p,q-1} + u_{p-1,q-1}}{h^2} + \frac{u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}}{h^2}\Big] - mu_{p,q} \qquad ...(16)$$

replacing $u_{p,q}$ by $\xi^q e^{i\beta ph}$ in equation(16) yields

$$\frac{\xi^{q+1}e^{i\beta ph} - 2\xi^q e^{i\beta ph} + \xi^{q-1}e^{i\beta ph}}{k^2} = \frac{1}{2}\Big[\frac{\xi^{q-1}e^{i\beta(p+1)h} - 2\xi^{q-1}e^{i\beta ph} + \xi^{q-1}e^{i\beta(p-1)h}}{h^2} + \frac{\xi^{q+1}e^{i\beta(p+1)h} - 2\xi^{q+1}e^{i\beta ph} + \xi^{q+1}e^{i\beta(p-1)h}}{h^2}\Big] - m\,\xi^q e^{i\beta ph}$$

$$\Rightarrow \xi^q e^{i\beta ph}[\xi - 2 + \xi^{-1}] = \frac{r^2}{2}\xi^q e^{i\beta ph}[\xi^{-1}(e^{i\beta h} - 2 + e^{-i\beta h}) + \xi(e^{i\beta h} - 2 + e^{-i\beta h})] - \xi^q e^{i\beta ph}mk^2$$

where $r = k/h$

dividing by $\xi^q e^{i\beta ph}$ yields

$$\xi - 2 + \xi^{-1} = \frac{r^2}{2}[\xi^{-1}(e^{i\beta h} - 2 + e^{-i\beta h}) + \xi(e^{i\beta h} - 2 + e^{-i\beta h})] - mk^2$$

$$\Rightarrow \xi - 2 + \xi^{-1} = \frac{r^2}{2}[\xi^{-1}(-4\sin^2(\frac{\beta h}{2})) + \xi(-4\sin^2(\frac{\beta h}{2}))] - mk^2$$

$$\Rightarrow \xi - 2 + \xi^{-1} = -2r^2\xi^{-1}\sin^2(\frac{\beta h}{2}) - 2r^2\xi\sin^2(\frac{\beta h}{2}) - mk^2$$

multiplying by ξ leads to

$$\xi^2 - 2\xi + 1 = -2r^2\sin^2(\frac{\beta h}{2}) - 2r^2\xi^2\sin^2(\frac{\beta h}{2}) - mk^2\xi$$

$$\Rightarrow \xi^2 - 2\xi + 1 + 2r^2\sin^2(\frac{\beta h}{2}) + 2r^2\xi^2\sin^2(\frac{\beta h}{2}) + mk^2\xi = 0$$

$$\Rightarrow (1 + 2r^2\sin^2(\frac{\beta h}{2}))\xi^2 - 2(1 - \frac{mk^2}{2})\xi + 1 + 2r^2\xi\sin^2(\frac{\beta h}{2}) = 0$$

dividing by $(1 + 2r^2\sin^2(\frac{\beta h}{2}))$ leads to

$$\xi^2 - 2E\xi + 1 = 0$$

where

$$E = \frac{1-\frac{mk^2}{2}}{1+2r^2\sin^2(\frac{\beta h}{2})} \qquad ...(17)$$

Hence the values of ξ are $\xi_{1,2} = E \mp \sqrt{E^2-1}$

The necessary and sufficiently condition for $|\xi| \le 1$ is $|E| \le 1$, from equation (17) we have $\left|\frac{1-\frac{mk^2}{2}}{1+2r^2\sin^2(\frac{\beta h}{2})}\right| \le 1$

Hence the Crank-Nicolson method for φ^4 equation is unconditionally stable.

8. Practical Application

For numerical solution we take φ^4 equation which is represented in equation (1):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^3$$

with initial and boundary conditions

$u(x,0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = 0$, $t \ge 0$, $0 < x < 2\pi$

$u(0,t) = u(2\pi,t) = 0.$

For clearing we take the following practical example:

$u(x,0) = \sin(x)$ $\qquad 0 < x < 2\pi$

$m = 1$

$\varepsilon = 1/6$

Results we got are found in the following tables and figures:

Explicit x= 3.14159265 h=0.31415927 k=0.31035376 m =1, ε = 1/6	Crank-Nicolson x= 3.14159265 h=0.31415927 k=0.31035376 m = 1, ε = 1/6
0	0
-0.1215	-0.1490

-0.2319	-0.2845
-0.3204	-0.3936
-0.3776	-0.4647
-0.3974	-0.4895
-0.3776	-0.4647
-0.3204	-0.3936
-0.2319	-0.2845
-0.1215	-0.1490
0	0
0.1215	0.1490
0.2319	0.2845
0.3204	0.3936
0.3776	0.4647
0.3974	0.4895
0.3776	0.4647
0.3204	0.3936
0.2319	0.2845
0.1215	0.1490
0	0

Table (1)
Comparison of explicit and Crank-Nicolson methods

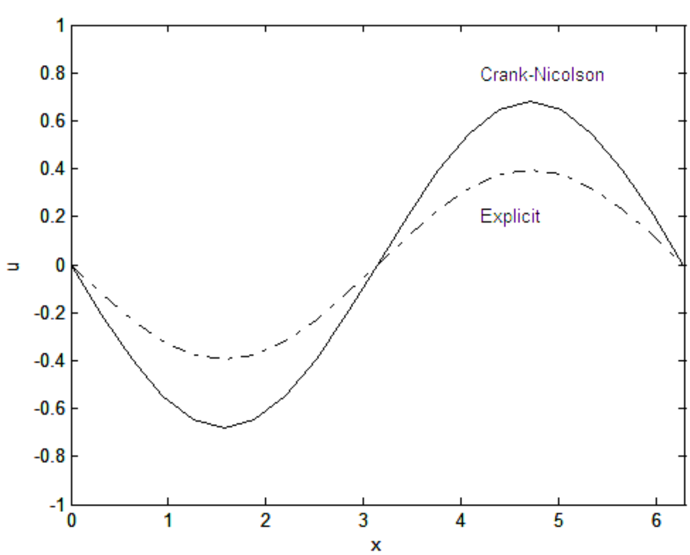

Figure (1) Comparison of explicit and Crank-Nicolson methods

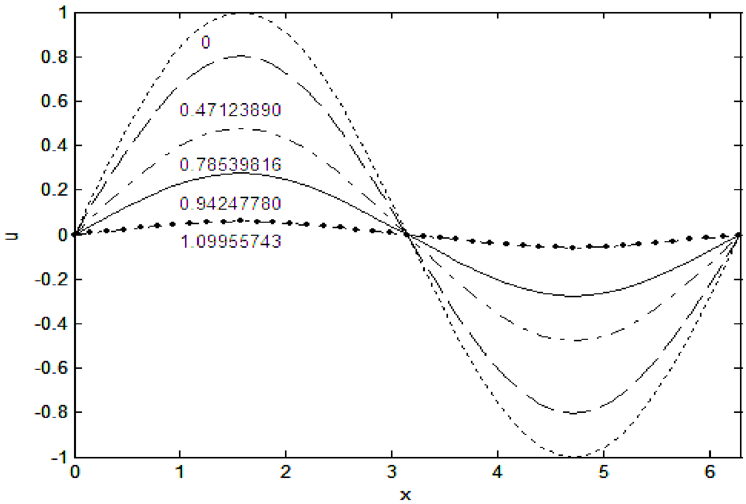

Figure (2) Explicit method with different x,
x=0, x=0.4712389, x=0.78539816, x=0.9424778, x=1.09955743

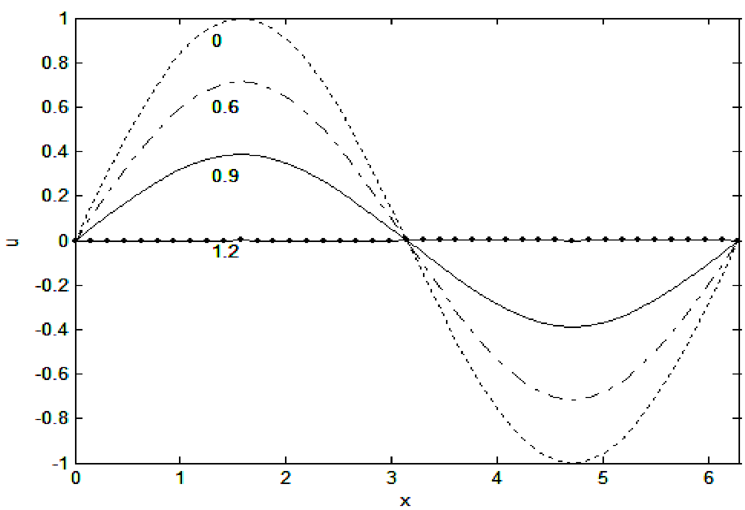

Figure (3)
Explicit method with different t,
t = 0, t = 0.6, t = 0.9, t = 1.2

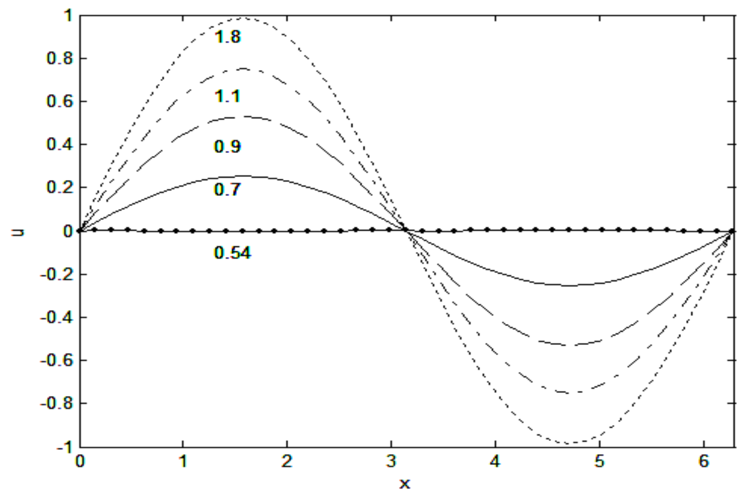

Figure (4)
Explicit method with different m,
m = 0.54, m = 0.7, m = 0.9, m = 1.1, m = 1.8

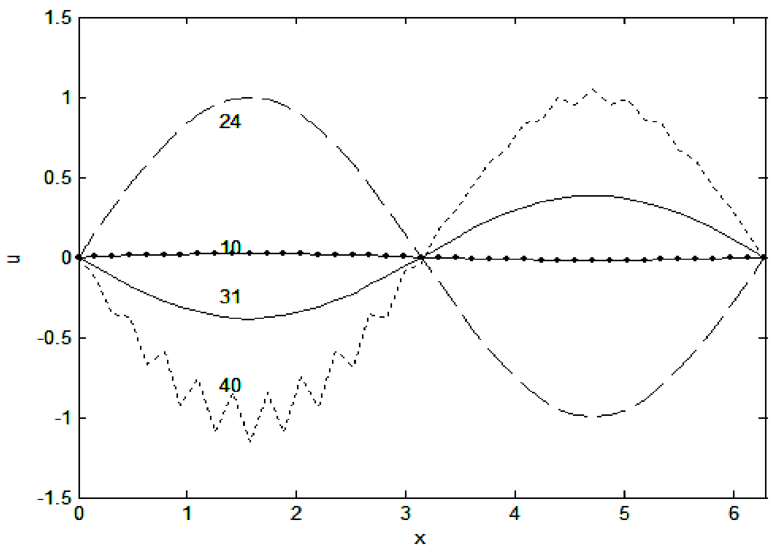

Figure (5)
Explicit method with different m,
m = 10, m = 24, m = 31, m = 40

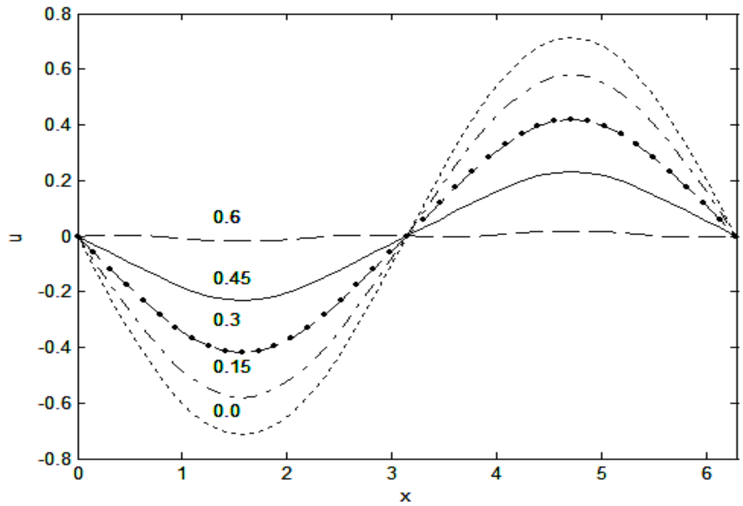

Figure (6)
Explicit method with different ε,
ε = 0, ε = 0.15, ε = 0.3, ε = 0.45, ε = 0.6

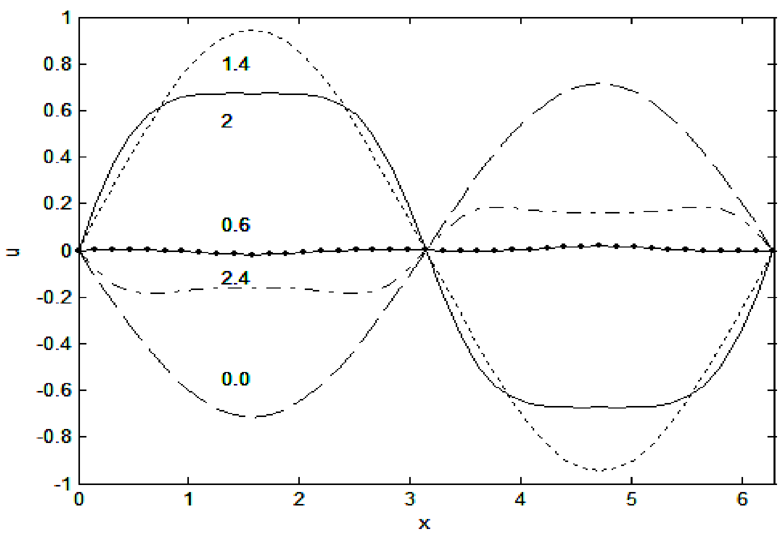

Figure (7)
Explicit method with different ε,
ε = 0.0, ε = 0.6, ε = 1.4, ε = 2, ε = 2.4

9. Conclusion

We saw that Crank-Nicolson method is more accurate than explicit method for one dimension problem

For the numerical stability the explicit method for φ^4 equation is stable under the condition

$$k^2 \leq \frac{4h^2}{4+h^2 m} \quad ,$$

and Crank-Nicolson method for φ^4 equation is unconditionally stable.

We saw from the tables and figures that Crank-Nicholson is more accurate than explicit method, see table (1) and figure (1). The numerical solution is symmetric and periodic, see figures (2), (3) which is useful i.e., the solution is the same for every period interval, so we need less time and less computations.

The value of m is affected because when we increase the value of m the solution increased too, see figure (4,5) , there is an upper bound of m (m=40) after this bound we can't get the true solution.

ε must be less than 2.4 see figure (6,7)

REFERENCES

[1] Angel, Sànchez and Luis Vàzquez (1991); " Topological Soliton Dynamics in a Stochastic φ^4 Model ", Phys. Lett. A, Vol. 152, No. 3,4 ,Pp. 184-190.

[2] Chen Liang, Xu Yong, Liu Zhongfei and Han Jiahua (2003); " The Extended Jacobian Elliptic Function Expansion Method and its Application to Nonlinear Wave Equation ", Fizika A Vol. 12, No.4, 161-170.

[3] Dmitriev, S.V., P. G. Kevrekidis, N. Yoshikawa, and D. J. Frantzeskakis (2006); " Exact Static Solutions for Discrete φ^4 Models Free of the Peierls-Nabarro Barrier : Discretized First Integral approach ", arXiv:nlin.PS/0603074v2 , 1-16.

[4] Ethan P. Honda and Matthew W. Choptuik (2002); " Fine Structure of Oscillons in the Spherically Symmetric φ^4 Klein-Gordon Model ", Phys. Rev. D, Vol. 65, 1-11.

[5] Fiore, G., A. D'Anna and M. De Angelis (2005); " Towards Soliton Solutions of a Perturbed Sine Gordon Equation ", arXiv:math-Ph/0507005 v1, 1-16.

[6] Gordon D. Smith (1965); " Numerical Solution of Partial Differential Equations : Finite Difference Methods ", second edition, Oxford University press.

[7] Jean-Philippe Nicolas (2002);" A Nonlinear Klein-Gordon Equation on Kerr Metrics ", J. Math Pures et Appliquèes, Vol. 81, No. 9, 885-914.

[8] Jürgen Pöschel . (1996); " Quasi-periodic Solutions for Nonlinear Wave Equations ", Comm. Math. Helv. Vol. 71,Pp. 269-296.

[9] Landa, P.S. (1996); " Nonlinear Oscillations and Waves in Dynamical Systems ", Kluwer Academic Publishers. Vol. 360.

[10] Manna, M.A. and V. Merle (1997); " Asymptotic Dynamics of Short-Waves in Nonlinear Dispersive Models ", arXiv:solv-int/9710013 v1, 1-4.

[11] Mathews, J.H. and K.D. Fink (2004); " Numerical Method Using Matlab ", Prentice-Hall, Inc. .

[12] Ole H. Hald (1999); " Optimal Prediction and the Klein-Gordon Equation ", arXiv:math.NA/9910048 v1, 1-16.

[13] Robert C. Mcowen (1995); " Partial Differential Equations : Methods and Applications ", Library of Congress Cataloging.

[14] Soffer, A. and M.I. Weinstein (1999); " Resonance, Radiation Damping and Instability in Hamiltonian Nonlinear Wave Equations ", Invent. math. Vol. 136, 9-74.

[15] Vladimir V. Konotop, Angel Sànchez and Luis Vàzquez (1991); " Kink Dynamics in the Weakly Stochastic φ^4 Model ", Phys. Rev. B, Vol. 44, No.6, 2554-2566.

[16] http://www.bookrags.com/Partial_differential_equation,WorldNet 1.7.1,2001 by Princeton University.