Hosoya Polynomials of Steiner Distance of Complete m-partite Graphs and Straight Hexagonal Chains (*)

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ABSTRACT
The Hosoya polynomials of Steiner distance of complete m-partite graphs \( K(p_1, p_2, \ldots, p_m) \) and Straight hexagonal chains \( G_m \) are obtained in this paper. The Steiner n-diameter and Wiener index of Steiner n-distance of \( K(p_1, p_2, \ldots, p_m) \) and \( G_m \) are also obtained.

Keywords: Steiner distance, Hosoya polynomial, Steiner n-diameter, Wiener index.

1. Introduction
We follow the terminology of [2,3]. For a connected graph \( G = (V, E) \) of order \( p \), the Steiner distance [8,7] of a non-empty subset \( S \subseteq V(G) \) denoted by \( d_G(S) \) or simply \( d(S) \), is defined to be the size of the smallest connected subgraph \( T(S) \) of \( G \) that contains \( S \). \( T(S) \) is called a Steiner tree of \( S \). If \( |S|=2 \), then the definition of the Steiner distance of \( S \) yields the (ordinary) distance between the two vertices of \( S \). For \( 2 \leq n \leq p \) and \( |S|=n \), the Steiner distance of \( S \) is called Steiner n-distance of \( S \) in \( G \).
The Steiner n-diameter of a graph $G$ (or the diameter of the Steiner n-distance), denoted by $diam_n^* G$ or $\delta_n^*(G)$, is defined to be the maximum Steiner n-distance of all $n$-subsets of $V(G)$, that is

$$diam_n^* G = \max \{d_G(S) : S \subseteq V(G), |S| = n \}.$$  

**Remark 1.1.** It is clear that

1. If $n \geq m$, then $diam_n^* G \geq diam_m^* G$.
2. If $S' \subseteq S$, then $d_G(S') \leq d_G(S)$.

The **average Steiner n-distance** of a graph $G$, denoted by $\mu_n^*(G)$, or average $n$-distance of $G$ is the average of the Steiner $n$-distances of all $n$-subsets of $V(G)$, that is

$$\mu_n^*(G) = \left( \frac{p}{n} \right)^{-1} \sum_{S \subseteq V \atop |S| = n} d_G(S).$$

If $G$ represents a network, then the Steiner $n$-diameter of $G$ indicates the number of communication links needed to connect $n$ processors, and the average $n$-distance indicates the expected number of communication links needed to connect $n$ processors [8].

The **Steiner n-eccentricity** [7] of a vertex $v \in V(G)$, denoted by $e_n^*(v)$, is defined as the maximum of the Steiner $n$-distances of all $n$-subsets of $V(G)$ containing $v$. The **Steiner n-radius** of $G$, denoted by $rad_n^*(G)$, is the minimum of Steiner $n$-eccentricities of all vertices in $G$.

The **Steiner n-distance** of a vertex $v \in V(G)$, denoted by $W_n^*(v,G)$ is the sum of the Steiner $n$-distances of all $n$-subsets of $V(G)$ containing $v$.

The sum of Steiner $n$-distances of all $n$-subsets of $V(G)$ is denoted by $d_n^*(G)$ or $W_n^*(G)$. Notice that

$$W_n^*(G) = \sum_{S \subseteq V \atop |S| = n} d_G(S) = n^{-1} \sum_{v \in V(G)} W_n^*(v,G) = \left( \frac{p}{n} \right) \mu_n^*(G). \quad \ldots \quad (1.1)$$

The graph invariant $W_n^*(G)$ is called the Wiener index of the Steiner $n$-distance of the graph $G$.

Bounds for the average Steiner $n$-distance of a connected graph $G$ of order $p$ are given by Danklemann, Oellermann and Swart [4].

**Definition 1.2**[1] Let $C_n^*(G,k)$ be the number of $n$-subsets of distinct vertices of $G$ with Steiner $n$-distance $k$. The graph polynomial defined by
\[ H_n^*(G;x) = \sum_{k=0}^{n-1} \binom{k}{n} C_n^*(G,k)x^k, \quad \ldots \ldots (1.2) \]

where \( \delta_n^* \) is the Steiner \( n \)-diameter of \( G \); is called the \textbf{Hosoya polynomial of Steiner }\( n \)-\textbf{distance of }\( G \).[1].

Then the \textbf{ }\( n \)-\textbf{Wiener index of }\( G \), \( W_n^*(G) \) will be

\[ W_n^*(G) = \sum_{k=0}^{n-1} k C_n^*(G,k) \quad \ldots \ldots (1.3) \]

The following proposition summarizes some properties of \( H_n^*(G;x) \).

\textbf{Proposition 1.2.} For \( 2 \leq n \leq p(G) \),

1. \( \deg H_n^*(G;x) \) is equal to the Steiner \( n \)-diameter of \( G \).
2. \( H_n^*(G;x) = \sum_{k=0}^{n-1} \binom{k}{n} C_n^*(G,k) \left( \frac{p}{n} \right)^k \), \quad \ldots \ldots (1.4)
3. \( W_n^*(G) = \frac{d}{dx} H_n^*(G;x) \bigg|_{x=1} \), \quad \ldots \ldots (1.5)
4. For \( n=2 \), \( H_2^*(G;x) = H(G;x) - p \), \quad \ldots \ldots (1.6)

where \( H(G;x) \) is the ordinary Hosoya polynomial of \( G \).
5. Each end-vertex of a Steiner tree \( T(S) \) must be a vertex of \( S \).

For \( 1 \leq n \leq p \), let \( C_n^*(u,G,k) \) be the number of \( n \)-subsets \( S \) of distinct vertices of \( G \) containing \( u \) with Steiner \( n \)-distance \( k \). It is clear that \( C_1^*(u,G,0) = 1 \).

Define

\[ H_n^*(u,G;x) = \sum_{k=0}^{n-1} \binom{k}{n} C_n^*(u,G,k)x^k. \quad \ldots \ldots (1.7) \]

Obviously, for \( 2 \leq n \leq p \)

\[ H_n^*(G;x) = \frac{1}{n} \sum_{v \in V(G)} H_n^*(u,G;x). \quad \ldots \ldots (1.8) \]

Ali and Saeed [1] were first who studied this distance-based polynomial for Steiner \( n \)-distances, and established Hosoya polynomials of Steiner \( n \)-distance for some special graphs and graphs having some kind of regularity, and for Gutman’s compound graphs \( G_1 \cdot G_2 \) and \( G_1 : G_2 \) in terms of Hosoya polynomials of \( G_1 \) and \( G_2 \).

In this paper, we obtain the Hosoya polynomial of Steiner \( n \)-distance of a complete \( m \)-partite graph \( K(p_1,p_2,\ldots,p_m) \); and we determine the Hosoya polynomial of Steiner 3-distance of a straight hexagonal chain \( G_m \).

Moreover, \( \text{diam}_n^* K(p_1,p_2,\ldots,p_m) \) and \( \text{diam}_n^* G_m \) are also determined.
2. Complete m-partite Graphs

A graph \( G \) is \( m \)-partite graph \([3], m \geq 1\), if it is possible to partition \( V(G) \) into \( m \) subsets \( V_1, V_2, \ldots, V_m \) (called partite sets) such that every edge \( e \) of \( G \) joins a vertex of \( V_i \) to a vertex of \( V_j \), \( i \neq j \). A Complete \( m \)-partite graph \( G \) is an \( m \)-partite graph with partite sets \( V_1, V_2, \ldots, V_m \) having the added property that if \( u \in V_i \) and \( v \in V_j \), \( i \neq j \), then \( uv \in E(G) \). If \( |V_i| = p_i \), then this graph is denoted by \( K(p_1, p_2, \ldots, p_m) \).

It is clear that the order, the size and the diameter of \( K(p_1, p_2, \ldots, p_m) \) are \( \sum_{i=1}^{m} p_i \), \( \sum_{i<j} p_i p_j \), and \( 2 \), respectively.

The following proposition determines the diameter of Steiner distance of \( K(p_1, p_2, \ldots, p_m) \).

**Proposition 2.1.** For \( n \geq 2 \), \( m \geq 2 \), let \( p' = \max\{p_1, p_2, \ldots, p_m\} \), then \( \text{diam}^*_n K(p_1, p_2, \ldots, p_m) = \) \( \begin{cases} n, & \text{if } 2 \leq n \leq p', \\ n-1, & \text{if } p' < n \leq p. \end{cases} \)

**Proof.** Let \( S \) be any \( n \)-subset of the vertices of \( K(p_1, p_2, \ldots, p_m) \). If \( S \) contains \( u, v \) such that \( u \in V_i \) and \( v \in V_j \), \( i \neq j \), then \( \langle S \rangle \) is connected, and so \( d(S) = n-1 \).

If \( S \subseteq V_i \), for \( 1 \leq i \leq m \), then \( d(S) = n \), namely, the size of \( T(S) (\cong K(1, n)) \).

Therefore, taking \( S \subseteq V_{p'} \) and \( 2 \leq n \leq p' \), we get \( \text{diam}^*_n K(p_1, p_2, \ldots, p_m) = n \).

If \( n > p' \), then \( S \) must contain vertices from at least two different partite sets.

This completes the proof. \( T(S) (\cong K(1, n)) \)

**Theorem 2.2.** For \( n, m \geq 2 \),

\[
H_n^*(K(p_1, p_2, \ldots, p_m); x) = C_1 x^{n-1} + C_2 x^n,
\]

in which

\[
C_1 = \binom{p}{n} - \sum_{i=1}^{m} \binom{p_i}{n}, \quad C_2 = \sum_{i=1}^{m} \binom{p_i}{n}.
\]

**Proof.** From Proposition 2.1, for each \( n \)-subset \( S \), \( n-1 \leq d(S) \leq n \).

For each \( n \)-subset \( S \subseteq V_i \), \( 1 \leq i \leq m \), \( d(S) = n \), thus the numbers of such \( n \)-subset is \( C_2 \). Since, the number of all \( n \)-subsets is \( \binom{p}{n} \), then \( C_1 \) is as given in the statement of this theorem.

The next corollary follows directly from Theorem 2.2.

**Corollary 2.3.** For \( n, m \geq 2 \),
\[ W_n^*(K(p_1, p_2, \ldots, p_m)) = (n-1) \left( \frac{p}{n} \right) + \sum_{i=1}^{m} \left( \frac{p_i}{n} \right), \]

\[ \mu_n^*(K(p_1, p_2, \ldots, p_m)) = n - 1 + \frac{\sum_{i=1}^{m} p_i}{p/n}. \]

**Remark.** By combinatorial argument one can easily show that
\[ \sum_{i=1}^{m} p_i < \left( \frac{p}{n} \right), \quad m \geq 2. \]

Thus for \( m \geq 2 \),
\[ \mu_n^*(K(p_1, p_2, \ldots, p_m)) < n. \]

A complete \( m \)-partite graph is called a **regular complete \( m \)-partite graph** [3], if \( p_i = t \) for all \( i \), and it will be denoted by \( K_{m(t)} \). The Hosoya polynomial and the Wiener index of Steiner \( n \)-distance of \( K_{m(t)} \) are given in the following corollary. Its proof follows easily from Theorem 2.2.

**Corollary 2.4.** For \( 2 \leq n \leq p = mt \)

1. \( H_n^*(K_{m(t)}; x) = m \left( \frac{t}{n} \right)^n + \left[ \frac{mt}{n} - \frac{m}{n} \left( \frac{t}{n} \right) \right] x^{n-1}. \)
2. \( W_n^*(K_{m(t)}) = (n-1) \left( \frac{mt}{n} \right) + m \left( \frac{t}{n} \right). \)

**3. Straight Hexagonal Chains**

A cycle of length 6 can be drawn as a regular hexagon. A **Straight Hexagonal Chains** \( G_m, \quad m \geq 2, \) is a graph consisting of a chain of \( m \) hexagons such that every two successive hexagons have exactly one edge in common in the form shown in Fig. 3.1.

It is clear that
\[ p(G_m) = 4m + 2, \quad q(G_m) = 5m + 1. \]

One can easily show that
\[ \text{diam} G_m = 2m + 1. \]  

(3.1)

The graph \( G_m \) is known to Chemists [5,6] as benzenoid chain of \( m \) hexagonal rings.

We shall find a formula for the diameter of the Steiner \( n \)-distance of the graph \( G_m \) for some values of \( n \). The vertices of \( G_m \) are labeled as shown in Fig. 3.1.
**Proposition 3.1.** For \( m \geq 1, \ 2 \leq n \leq m + 2, \)
\[ \text{diam}_G \mathcal{G}_m = 2m + n - 1. \]

**Proof.** It is clear that for \( n=2, \)
\[ \text{diam}_G \mathcal{G}_m = d(u_1, u'_{2m+1}) = 2m + 1. \]
If \( n=3, \) we find that a 3-subset \( S' \) of maximum Steiner distance is
\[ S' = \{u_1, u_{2m+1}, u'_{2m}\}, \]
and so,
\[ \text{diam}_G \mathcal{G}_m = d_3 (S') = 2m + 2. \]
For \( n=4, \) we notice that a 4-subset \( S'' \) of maximum Steiner distance is
\[ S'' = \{u_1, u_{2m+1}, u'_{2m}, v\}, \]
in which \( v \in \{u'_2, u'_4, ..., u'_{2m-2}\}. \)
Thus
\[ \text{diam}_G \mathcal{G}_m = d_4 (S'') = 2m + 3 \]

Hence, in general for an \( n \)-subset \( S, \ 2 \leq n \leq m + 2, \) of maximum Steiner \( n \)-distance, we have the following cases:
(1) If \( n \) is even, then \( S \) consists of the first \( n \) vertices from the sequence:
\[ u_1, u'_2, u_{2m+1}, u'_{2m}, u_{2m-2}, u'_{2m-4}, u_{2m-6}, ..., \]
\[ u'_2, \text{ if } m \text{ is even}, \]
\[ u'_4, \text{ if } m \text{ is odd}. \]

When \( m \) is even, a Steiner tree, \( T(S) \) of such \( S \) consists of a \((2m+1)\)-path, say, \( u_1, u_2, u_3, ..., u_{2m+1}, u'_{2m+1} \) together with \( \frac{n-2}{2} \) paths each of length 2, namely
\[ (u_{2m-1}, u'_{2m-1}, u'_{2m-2}), (u_{2m-5}, u'_{2m-5}, u'_{2m-6}), ... \]
Therefore, the size of \( T(S) \) is
\[ (2m+1) + \frac{n-2}{2} = 2m + n - 1. \]
When \( m \) is odd \( T(S) \) has the same structure as for the case of even \( m \), and so have size \( 2m+n-1. \)
(2) If \( n \) is odd, then \( S \) consists of the first \( n \) vertices from sequence:
\[
\{ u_1, u_{2m+1}, u_{2m}, u_{2m-2}, u_{2m-4}, u_{2m-6}, \ldots, u'_2, \text{ if } m \text{ is odd,} \}
\]
\[
\{ u'_4, \text{ if } m \text{ is even.} \}
\]
When \( m \) is odd, a Steiner tree \( T(S) \) of such \( S \) consists of a \( 2m \)-path, say, \((u_1, u_2, \ldots, u_{2m}, u_{2m+1})\) together with \( \frac{n-1}{2} \) paths each of length 2, namely \((u_{2m+1}, u'_{2m+1}, u_{2m}), (u_{2m-3}, u'_{2m-3}, u_{2m-4}), \ldots. \) Therefore, the size of \( T(S) \) is
\[
2m + 2 \left( \frac{n-1}{2} \right) = 2m + n - 1.
\]
When \( m \) is even, \( T(S) \) has the same structure as for odd case of \( m \), and so has size \( 2m+n-1 \).

**Proposition 3.2.** For \( m \geq 3, m+3 \leq n \leq 2m \),
\[
diam^*_G(n) = 3m + \left\lceil \frac{n-m}{2} \right\rceil.
\]

**Proof.** An \( n \)-subset \( S \) of vertices, \( m+3 \leq n \leq 2m \) which has maximum Steiner \( n \)-distance consists of \( m+2 \) vertices described in the proof of Proposition 3.1 together with other \( m-2 \) vertices chosen in pairs, each pair consists of 2 vertices, belonging to a hexagon, one of degree 2 and the other of degree 3. For instance, when \( n \) and \( m \) are even, the added \( (m-2) \) vertices are \( u'_{2m}, u_{2m-1}, u_{2m-2}, u'_{2m-3}, \ldots. \) Each such pair of vertices gives one edge added to the size of \( T(S') \), \( |S'| = m+2 \). Therefore the Steiner \( n \)-distance of \( S \) is
\[
2m + (m+2-1) + \left\lceil \frac{n-m-2}{2} \right\rceil.
\]

**Remark.** For \( m \geq 2, n=p-2 \),
\[
diam^*_G(n) = n = 4m - 2.
\]
Thus, for \( 2m+1 \leq n \leq 4m \),
\[
3m + \left\lceil \frac{n-m}{2} \right\rceil \leq diam^*_G(n) \leq p-2,
\]
and
\[
diam^*_G(n) = p-1, \text{ for } n=p-1 \text{ or } p.
\]

We now find the Hosoya Polynomial of the Steiner 3-distance of \( G_m \).

**Theorem 3.3.** For \( m \geq 3 \), we have the following reduction formula for \( H_3^*(G_m;x) \),
\[
H_3^*(G_m;x) = 2H_3^*(G_{m-1};x) - H_3^*(G_{m-2};x) + F_m(x),
\]
where \( F_m(x) = 2x^{2m-1}(2m-3) + (9m-11)x + (13m-9)x^2 + (7m-1)x^3 + mx^4 \)

**Proof.** Let \( S \) be any 3-subset of \( V(G_m) \). We refer to Fig 3.1, and denote
Hosoya Polynomials of Steiner...

\[
A = \{u_1, u_2, u'_1, u'_2\}, \quad A' = \{u_{2m}, u_{2m+1}, u'_{2m}, u'_{2m+1}\},
\]
\[
B = \{u_3, u_5, \ldots, u_{2m-1}\}, \quad B' = \{u'_3, u'_5, \ldots, u'_{2m-1}\},
\]
\[
C = \{u_4, u_6, \ldots, u_{2m-2}\} \quad \text{and} \quad C' = \{u'_{4}, u'_{6}, \ldots, u'_{2m-2}\}.
\]

For all possibilities of \( S \subseteq V(G_m) - A \) (or \( S \subseteq V(G_m) - A' \) ), we have the corresponding polynomial \( H^*_3(G_{m-1}; x) \). And for all possibilities of \( S \subseteq V(G_m) - \{A \cup A'\} \), the corresponding polynomial is \( H^*_3(G_{m-2}; x) \).

Thus
\[
H^*_3(G_m; x) = 2H^*_3(G_{m-1}; x) - H^*_3(G_{m-2}; x) + F_m(x),
\]
in which \( F_m(x) \) is the Hosoya polynomial corresponding to all 3-subsets of vertices that each contains at least one vertex from \( A \) and at least one vertex from \( A' \). Therefore \( F_m(x) \) can be split into two polynomials \( F_1(x) \) and \( F_2(x) \), where \( F_1(x) \) is the Hosoya Polynomial of all 3-subsets \( S \) that each contains one vertex from \( A \), one vertex from \( A' \) and one vertex from \( W = B \cup B' \cup C \cup C' \), and \( F_2(x) \) is the Hosoya polynomial corresponding to all 3-subsets \( S \) such that \( S \subseteq A \cup A' \). \( S \cap A \neq \emptyset \) and \( S \cap A' \neq \emptyset \).

(I) Now, to find \( F_1(x) \), we consider the following subcases:

(a) If \( S = \{u_1, u_{2m}, y\} \) or \( \{u'_1, u'_{2m}, y\} \), then

(1) When \( y \in B \cup C \), there are \((2m-3)\) such subsets \( S \) each of 3-distance \((2m-1)\).

(2) When \( y \in B' \), there are \((m-1)\) such subsets \( S \), each of 3-distance \(2m\).

(3) When \( y \in C' \), there are \((m-2)\) such subsets \( S \), each of 3-distance \(2m+1\).

Therefore, for all such possibilities of \( S \), \( S = \{u_1, u_{2m}, y\} \) or \( \{u'_1, u'_{2m}, y\} \), \( y \in W \), the corresponding polynomial is

\[
P_1(x) = 2x^{m-1}[(2m-3) + (m-1)x + (m-2)x^2]
\]

(b) If \( S = \{u_1, u_{2m+1}, y\} \) or \( \{u'_1, u'_{2m+1}, y\} \), for all \( y \in W \), then the corresponding polynomial can be obtained by a similar way of (a) as given below

\[
P_2(x) = 2x^{m-1}[(2m-3) + (m-1)x + (m-2)x^2]
\]

(c) If \( S = \{u_1, u'_{2m}, y\} \) or \( \{u'_1, u_{2m}, y\} \), \( y \in W \), then the corresponding polynomial is

\[
P_3(x) = 4(2m-3)x^{2m}.
\]

(d) If \( S = \{u_1, u'_{2m+1}, y\} \) or \( \{u'_1, u_{2m+1}, y\} \), \( y \in W \), then the corresponding polynomial is

\[
P_4(x) = 4(2m-3)x^{2m+1}.
\]

(e) If \( S = \{u_2, u_{2m}, y\} \) or \( \{u'_2, u'_{2m}, y\} \), for all \( y \in W \), then the corresponding polynomial is

\[
P_5(x) = 2x^{2m-2}[(2m-3) + (m-1)x + (m-2)x^2] .
\]
(f) If \( S = \{u_1, u_2, u_{2m+1}, y\} \) or \( \{u_1', u_2', u_{2m+1}, y\} \), for all \( y \in W \), then the corresponding polynomial is
\[
P_6(x) = 2x^{2m-1}(2m - 3 + (m-1)x + (m-2)x^2).
\]

(g) If \( S = \{u_2, u_{2m}, y\} \) or \( \{u_2', u_{2m}, y\} \), for all \( y \in W \) then the corresponding polynomial is
\[
P_7(x) = 4(2m - 3)x^{2m-1}.
\]

(h) If \( S = \{u_2, u_{2m+1}, y\} \) or \( \{u_2', u_{2m+1}, y\} \), for all \( y \in W \), then the corresponding polynomial is
\[
P_8(x) = 4(2m - 3)x^{2m}.
\]

Therefore
\[
F_1(x) = \sum_{i=1}^{8} P_i(x)
\]
\[
= 2x^{2m-2}(2m - 3 + (2m - 13)x + (13m - 19)x^2 + (7m - 11)x^3 + (m - 2)x^4).
\]

(II) To find \( F_2(x) \), let \( S \) consists of two vertices from \( A \) and one vertex from \( A' \), or one vertex from \( A \) and two vertices from \( A' \). Thus we have \( 2 \binom{4}{2} = 2(24) \) possibilities for the 3-subsets \( S \), 24 of them give the same Hosoya polynomials for the other 24 cases. These 24 cases are listed in the following table with their Steiner 3-distances:

<table>
<thead>
<tr>
<th>no.</th>
<th>3-subsets ( S )</th>
<th>Steiner distances</th>
<th>no.</th>
<th>3-subsets ( S )</th>
<th>Steiner distances</th>
</tr>
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<tbody>
<tr>
<td>1.</td>
<td>( {u_1, u_2, u_{2m}} )</td>
<td>2m-1</td>
<td>13.</td>
<td>( {u_1', u_2', u_{2m}} )</td>
<td>2m-1</td>
</tr>
<tr>
<td>2.</td>
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<td>14.</td>
<td>( {u_1, u_2, u_{2m+1}} )</td>
<td>2m</td>
</tr>
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<td>3.</td>
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<td>2m</td>
<td>15.</td>
<td>( {u_1', u_2', u_{2m}'} )</td>
<td>2m</td>
</tr>
<tr>
<td>4.</td>
<td>( {u_1, u_2, u_{2m+1}'} )</td>
<td>2m+1</td>
<td>16.</td>
<td>( {u_1', u_2', u_{2m+1}'} )</td>
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</tr>
<tr>
<td>5.</td>
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<td>2m</td>
<td>17.</td>
<td>( {u_2, u_2', u_{2m}} )</td>
<td>2m</td>
</tr>
<tr>
<td>6.</td>
<td>( {u_1, u_1', u_{2m+1}} )</td>
<td>2m+1</td>
<td>18.</td>
<td>( {u_2, u_2', u_{2m+1}} )</td>
<td>2m+1</td>
</tr>
<tr>
<td>7.</td>
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<td>2m</td>
<td>19.</td>
<td>( {u_2, u_2', u_{2m}'} )</td>
<td>2m</td>
</tr>
<tr>
<td>8.</td>
<td>( {u_1, u_1', u_{2m+1}'} )</td>
<td>2m+1</td>
<td>20.</td>
<td>( {u_2, u_2', u_{2m+1}'} )</td>
<td>2m+1</td>
</tr>
<tr>
<td>9.</td>
<td>( {u_1, u_2, u_{2m}} )</td>
<td>2m+1</td>
<td>21.</td>
<td>( {u_2', u_2', u_{2m}} )</td>
<td>2m+1</td>
</tr>
<tr>
<td>10.</td>
<td>( {u_1, u_2, u_{2m+1}} )</td>
<td>2m+2</td>
<td>22.</td>
<td>( {u_2', u_2, u_{2m+1}} )</td>
<td>2m+2</td>
</tr>
<tr>
<td>11.</td>
<td>( {u_1, u_2, u_{2m}'} )</td>
<td>2m</td>
<td>23.</td>
<td>( {u_2', u_2, u_{2m}'} )</td>
<td>2m</td>
</tr>
<tr>
<td>12.</td>
<td>( {u_1, u_2, u_{2m+1}'} )</td>
<td>2m+1</td>
<td>24.</td>
<td>( {u_2', u_2, u_{2m+1}'} )</td>
<td>2m+1</td>
</tr>
</tbody>
</table>
Hosoya Polynomials of Steiner…

Therefore, there are 4 subsets \( S \) of 3-distance \((2m-1)\), 20 of 3-distance \( 2m \), 20 subsets of 3-distance \((2m+1)\) and 4 subsets of 3-distance \((2m+2)\). Thus,
\[
F_5(x) = 4x^{2m-3}(1 + 5x + 5x^2 + x^3).
\]
Adding \( F_5(x) \) to \( F_2(x) \) we get \( F_m(x) \) as given in the statement of the theorem.

**Remark.** Hosoya Polynomials of Steiner 3-distance of \( G_1 \) and \( G_2 \) are obtained by direct calculation as shown below:
\[
H_5(G_1;x) = 6x^2 + 12x^3 + 2x^4,
\]
and
\[
H_5(G_2;x) = 15x^2 + 36x^3 + 38x^4 + 27x^5 + 4x^6.
\]
The reduction formula given in Theorem 3.3 can be solved to obtain the following useful formula.

**Corollary 3.4.** For \( m \geq 3 \)
\[
H_5(G_m;x) = 3(3m-1)x^2 + 12(2m-1)x^3 + 2(18m-17)x^4
+ 27(m-1)x^5 + 4(m-1)x^6 + \sum_{k=0}^{m-3} (k+1)F_{m-k}(x),
\]
where
\[
F_{m-k}(x) = 2x^{2(m-k-1)}[(2m-2k-3) + (9m-9k-11)x + (13m-13k-9)x^2
+ (7m - 7k - 1)x^3 + (m - k)x^4].
\]

**Proof.** From Theorem 3.3,
\[
H_5(G_m;x) = 2H_5(G_{m-1};x) - H_5(G_{m-2};x) + F_m(x)
= 2[2H_5(G_{m-2};x) - H_5(G_{m-3};x) + F_{m-1}(x)] - H_5(G_{m-2};x) + F_m(x)
= 3H_5(G_{m-2};x) - 2H_5(G_{m-3};x) + F_m(x) + 2F_{m-1}(x)
= 3[2H_5(G_{m-3};x) - H_5(G_{m-4};x) + F_{m-2}(x)]
- 2H_5(G_{m-3};x) + F_m(x) + 2F_{m-1}(x)
= 4H_5(G_{m-3};x) - 3H_5(G_{m-4};x) + \sum_{k=0}^{2} (k+1)F_{m-k}(x)
= (m-1)H_5(G_2;x) - (m-2)H_5(G_1;x) + \sum_{k=0}^{m-3} (k+1)F_{m-k}(x) \quad \ldots (3.1)
\]
From the remark above, we have
\[
H_5(G_2;x) = 15x^2 + 36x^3 + 38x^4 + 27x^5 + 4x^6,
\]
and
\[
H_5(G_1;x) = 6x^2 + 12x^3 + 2x^4.
\]
Substituting in (3.1) and simplifying, we get the required result.
The 3-Wiener index of $G_m$ is given in the following corollary.

**Corollary 3.5.** For $m \geq 3$,

$$W^*_3(G_m) = \frac{4}{3} m(m-2)(8m^2 + 35m + 83) + 225m - 1$$

**Proof.** It is known that

$$W^*_3(G_m) = \frac{d}{dx} H^*_3(G_m; x)|_{x=1}$$

Hence $W^*_3(G_m) = 393m - 337 + 2 \sum_{k=0}^{m-3} [64k^3 + (116 - 128m)k^2 + (64m^2 - 180m + 68)k + 8(16m^2 - 13m + 4)]$

Now, using the fact that

$$\sum_{k=0}^{m-3} k = \frac{1}{2} (m-3)(m-2), \quad \sum_{k=0}^{m-3} k^2 = \frac{1}{6} (m-3)(m-2)(2m-5) \quad \sum_{k=0}^{m-3} k^3 = \left(\frac{1}{2} (m-3)(m-2)\right)^2,$$

and simplifying we get the required result. $\blacksquare$
REFERENCES


