Maximal Generalization of Pure Ideals

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ABSTRACT

The purpose of this paper is to study the class of the rings for which every maximal right ideal is left GP-ideal. Such rings are called MGP-rings and give some of their basic properties as well as the relation between MGP-rings, strongly regular ring, weakly regular ring and kasch ring.

Keywords: pure ideals, strongly regular, kasch ring.

1- Introduction:

Throughout this paper, R denotes as associative ring with identity. An ideal I of a ring R is said to be right(left) pure if for every $a \in I$, there exists $b \in I$ such that $a = ab$ ($a = ba$). This concept was introduced by Fieldhouse [6], [7], Al-Ezeh [2], [3] and Mahmood [9].

Recall that:

1- A ring R is regular if for every $a \in R$ there exists $b \in R$ such that $a = aba$, if $a = a^2 b$, R is called strongly regular.

2- A ring without non-zero nilpotent elements is called reduced.

3- For any element $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator and the left annihilator of a, respectively.

4- A ring R is said to be a left(right) uniform ring if and only if every non-zero left(right) ideals is essential.

5- Following [10], a ring R is said to be semi commutative if $xy = 0$ implies that $xRy = 0$, $x, y \in R$. Clearly every reduced ring is semi commutative.
is easy to see that $R$ is semi commutative if and only if every left(right) annihilator in $R$ is a two-sided ideal.

6-Y($R$), J($R$) are respectively the right singular ideal and the Jacobson radical of $R$.

2- MGP-rings

In this section, the concept of maximal GP-ideals is introduced and we use it to define MGP-rings. We study such rings and give some of their basic properties.

Following [8], an ideal $I$ of a ring $R$ is said to be right (left) GP-ideal (generalized pure ideal), if for every $a$ in $I$, there exists $b$ in $I$ and a positive integer $n$ such that $a^n = a^n b$ ($a^n = b a^n$).

**Definition 2.1:**

A ring $R$ is called a right (left) MGP-ring if and only if every maximal right (left) ideal is left (right) GP-ideal.

**Example:**

Let $Z_{12}$ be the ring of the integers modulo 12.

Then the maximal ideals, $I = \{0,3,6,9\}$, $J = \{0,2,4,6,8,10\}$ are GP-ideals.

The following theorem gives some interesting characteristic properties of right MGP-rings. Before that we need the next lemma in our proof.

**Lemma 2.2:**

Let $a$ be a non zero element of a ring $R$ and let $l(a) = 0$. Then for every positive integer $n$, $l(a^n) = 0$.

Proof: obvious #

**Theorem 2.3:**

If $R$ is a right MGP-ring and every ideal is principal, then any left regular element is right invertible.

Proof:

Let $0 \neq c \in R$, such that $l(c) = 0$. If $c R \neq R$, then there exists a maximal right ideal $M$ containing $cR$. Since $R$ is right MGP-ring, then $M$ is a left GP-ideal, there exists $d \in M$ and a positive integer $n$, such that $c^n = dc^n$ and $d = cx$, for some $x \in R$.

So $(1-cx) \in l(c^n)$, since $l(c) = 0$, then by Lemma 2.2 we have $l(c^n) = 0$, thus $cx = 1 \in M$, this contradicts $cR \neq R$. Therefore $cR = R$, and hence $c$ is a right invertible. #

**Lemma 2.4:**
Let $R$ be a reduced ring. Then for every $a \in R$, and every positive integer $n$, $a^n R \cap r(a^n) = 0$.

Proof: See [8]

**Proposition 2.5:**

Let $R$ be a reduced, MGP-ring. Then for every $a$ in $R$ and a positive integer $n$, $r(a^n)$ is a direct summand of $R$.

**Proof:**

To prove $r(a^n)$ is a direct summand, we claim that $a^n R + r(a^n) = R$. If this is not true, let $M$ be a maximal right ideal containing $a^n R + r(a^n)$. Since $R$ is MGP-ring, so $(a^n)^m = b (a^n)^m$ for some $b \in M$ and a positive integer $m$. This implies $(1-b) \in l(a^n) \subseteq M$ (R is reduced), and so $1 \in M$, a contradiction. Hence $a^n R + r(a^n) = R$.

Now, since $a^n R \cap r(a^n) = 0$, Lemma( 2.4), then $r(a^n)$ is a direct summand. #

Recall that, a ring $R$ is called a right (left) MP-ring if every maximal right (left) ideal is a left (right) pure.

We consider the condition (*): $R$ satisfies $l(b^n) \subseteq r(b)$ for any $b \in R$ and a positive integer $n$.

**Theorem 2.6:**

Let $R$ be a ring satisfying (*). Then $R$ is a right MGP-ring if and only if $R$ is strongly regular.

**Proof:**

If this is not true let $R$ be a right MGP-ring and let $b$ be any element in $R$. We shall prove that $bR + r(b) = R$.

If this is not true let $M$ be a maximal right ideal containing $bR + r(b)$. Since $R$ is an MGP-ring, then there exists $a \in M$ and a positive integer $n$ such that $b^n = ab^n$ which implies that $(1-a) \in l(b^n) \subseteq r(b)M$, thus $1 \in M$, a contradiction. Therefore $bR + r(b) = R$.

In particular, $b u + v = 1$, for some $u \in R$, $v \in r(b)$. So $b = b^2 u$, therefore $R$ is strongly regular.

Conversely; assume that $R$ is strongly regular, then by [1], $R$ is regular and reduced. Also by [9], $R$ is an MP-ring and semi commutative, then $R$ is an MGP-ring.

**Proposition 2.7:**

Let $R$ be a right MGP-ring satisfying (*). Then $Y(R) = 0$.

**Proof:**
If $Y(R) \neq 0$, then by a Lemma (7) of [10]; there exists $0 \neq a \in Y(R)$ with $a^2 = 0$. From Theorem (2.6) $R$ is strongly regular, that is $a = a^2 b$, for some $b \in R$. Hence $a = 0$, contradiction. Therefore $Y(R) = 0$. #

**Proposition 2.8:**

If $R$ is a right MGP – ring, then any reduced principal right ideal of $R$ is a direct summand.

**Proof:** Let $I = aR$ be a reduced principal right ideal of $R$. If $aR + r(a) \neq R$, then there exists a maximal right ideal $M$ of $R$ containing $aR + r(a)$.

Now, since $R$ is a right MGP-ring and $a \in M$, then there exists $b \in M$ and a positive integer $n$ such that $a^n = b a^n$, and hence $(1-b)a^n = 0$. Since $I$ is reduced then we have $(1-b) \in l(a^n) = r(a^n) \subseteq M$, this implies that $1 \in M$, which contradicts $M \neq R$. Therefore, $aR + r(a) = R$, thus $a = a^2 c$ for some $c \in R$. If we set $d = a^2 \in I$, then $a = a^2 d$. implies that $a = ada$ and hence $aR = eR$, where $e = ad$ is an idempotent element. Then by [6], $aR$ is a direct summand. #

**Proposition 2.9:**

Let $R$ be a right MGP-ring satisfying $(*)$. If $a^n b = 0$, for any $a, b \in R$ and a positive integer $n$, then $r(a^n) + r(b) = R$.

**Proof:** Assume that $r(a^n) + r(b) \neq R$. Let $M$ be a maximal right ideal containing $aR$. Since $R$ is a right MGP-ring and $a^n b = 0$ implies that $b \in r(a^n) \subseteq M$, there exists $c \in M$ and a positive integer $m$ such that $b^m = c b^m$, so $(1-c) \in l(b^m) \subseteq r(b) \subseteq M$, which implies that $1 \in M$, which is a contradiction. Therefore $r(a^n) + r(b) = R$.

**Theorem 2.10:**

Let $R$ be a uniform semi commutative, MGP-ring and every ideal is principal. Then $R$ is a division ring.

**Proof:** Let $0 \neq a \in R$ and $aR \neq R$, and let $M$ be a maximal right ideal containing $aR$. Since $R$ is an MGP-ring, then there exists $b \in aR \subseteq M$, and a positive integer $n$ such that $a^n = ba^n$. This implies that $a^n = aca^n$, for some $c \in R$. Since $R$ is uniform so every ideal is an essential ideal.

Let $x \in r(ar) \cap a^n R$. Then $acx = 0$ and $x = a^n z$ for some $z \in R$, so $aca^n z = 0$, yields $a^n z = 0 \Rightarrow x$. Therefore, $r(ac) \cap a^n R = 0$, since $R$ is a uniform ring and $a^n R \neq 0$, then $r(ac) = 0$. Since $R$ is semi commutative, $l(ac) = 0$, then by Theorem (2.3) $ac$ is a right invertible element, so there
exists $v \in R$ such that $acv = 1$. Hence $a(cv) = 1 \in M$, which is a contradiction. Therefore $aR = R$.

Now, since $ar = 1$ ($aR = R$), we have $ara = a$ which implies that $(1-ra) \in r(a) \subseteq l(ar) = r(ar) = 0$. Therefore, $(1-ra) = 0$, whence $ra = 1$, so $a$ is a left invertible. Thus $R$ is a division ring.

### 3-The relation between MGP-rings and other rings

In this section we give further properties of the MGP-rings and link between MGP-rings and other rings.

We shall begin this section with the following result, which gives the connection between MGP-rings and weakly regular rings.

Following [11], a ring $R$ is a right (left) weakly regular if $I^2 = I$ for each right (left) ideal $I$ of $R$. Equivalently, if $a \in aRa$ ($a \in RaR$) for every $a$ in $R$. Then $R$ is called weakly regular.

**Theorem 3.1**

Let $R$ be a right MGP-ring and satisfying (*). Then $R$ is a reduced weakly regular ring.

**Proof**: Let $a$ be a non-zero element in $R$ with $a^2 = 0$. Let $M$ be a maximal right ideal containing $r(a)$. Since $a \in r(a) \subseteq M$ and $R$ is an MGP-ring, then there exists $b \in M$ and a positive integer $n$ such that $a^n = ba^n$, which implies that $(1-b) \in l(an) \subseteq r(a) \subseteq M$, yielding $1 \in M$, which is a contradiction.

Therefore, $a = 0$, and hence $R$ is a reduced ring. We show that $RxR + r(x) = R$, for any $x \in R$.

Suppose that there exists $y \in R$ such that $Ry + r(y) \neq R$.

Then there exists a maximal right ideal $M$ of $R$ containing $Ry + r(y)$. Since $R$ is a right MGP-ring, there exists $a \in M$ and a positive integer $n$ such that $y^n = a y^n$ implying that $(1-a) \in l(y^n) \subseteq r(y) \subseteq M$, whence $(1-a) \in M$ and so $1 \in M$ implies that $M = R$, which is a contradiction. Therefore, $RxR + r(x) = R$, for any $x \in R$.

Hence $R$ is a right weakly regular ring. Since $R$ is reduced, it also can be easily verified that $R$ is a weakly regular ring.

**Definition 3.2**: [9]

A ring $R$ is said to be a right (left) Kasch ring if every maximal right (left) ideal is a right (left) annihilator.

**Theorem 3.3**
Every semi commutative right MGP-ring is a right Kasch ring.

**Proof:** Let $M$ be any maximal right ideal of $R$ and let $Y(R)$ be the right singular ideal of $R$.

If $M \cap Y(R) = 0$, then for any $y \in Y(R)$, $y \notin M$, this implies that $r(y)$ is an essential right ideal of $R$.

Let $x \in r(y) \cap r(1-y)$, then $yx = 0$ and $(1-y)x = 0$ yields $x=0$. Therefore $r(yy) = 0$. Since $R$ is semi-$y$ = 0, whence $r(1-r(I))$ commutative ring, then we have $l(1-y) = 0$.

By Theorem (2.3), $(1-y)$ is an invertible element of $R$. Hence $y \in J \subseteq M$, a contradiction.

Thus $M \cap Y(R) \neq 0$. Let $0 \cap M \neq Y(R)$.

Since $R$ is an MGP-ring, then there exists $b \in M$ and a positive integer $n$ such that $a^n = ba^n = ara^n$. We claim that $r(ar) \cap a^n R = 0$. If not, let $d \in r(ar) \cap a^n R$. Then $ar \cdot d = 0$ and $d = a^n x$ for some $x \in R$, so $ara^n x = 0$ implies that $a^n x = 0=d$. Therefore, $r(ar) \cap a^n R = 0$. But $r(ar)$ is essential, then $a^n R = 0$ and hence $a^n x = 0$, for all $x \in R$ implies that $a^n \in l(x) = r(x)$. Therefore, $M = r(x)$. Thus $R$ is a right Kasch ring.

**Corollary 3.4:**
Let $R$ be a reduced MGP-ring. Then $R$ is a Kasch ring.

**Proof:** Since $R$ is a reduced right MGP-ring. Then by Theorem (3.3) $R$ is a Kasch ring.
REFERENCES


