MGP and EGP Rings

Raida D. Mahammod
raida.1961@uomosul.edu.iq

Shahla M. Khalil
moayadshahla@gmail.com

College of Computer Sciences and Mathematics
University of Mosul, Iraq

Received on: 12/3/2008
Accepted on: 12/6/2008

ABSTRACT

The purpose of this paper is to study the rings in which every maximal (essential) right ideal is a left GP-ideal. Such rings will be called right MGP-rings (EGP-rings). We give the basic properties of such rings and their connection with strongly $\pi$-regular rings, fully left idempotent rings, and $S$-weakly regular rings.

Keywords: right MGP-rings, right EGP-rings, strongly $\pi$-regular rings, fully left idempotent rings, $S$-weakly regular rings.

1. Introduction:

In this paper, all rings are assumed to be associative ring with identity. An ideal $I$ of a ring $R$ is said to be right (left) pure ideal if for every $a \in I$ there exists $b \in I$ such that $a = ab$ ($a = ba$). This concept was introduced by Fieldhouse [3],[4].

Recall that:

1- R is called reduced if R has no non-zero nilpotent element.
2- According to Cohn [2], a ring R is called reversible if $ab = 0$ implies $ba = 0$, for $a,b \in R$. It is easy to see that R is reversible if and only if right (left) annihilator of $a$ in R is two sided ideal [2].
3- A ring R is said to be a right (left) Kasch ring if every maximal right (left) ideal is a right (left) annihilator [8].
4- According to [5], a ring $R$ is called strongly $\pi$- regular if for every $x \in R$, there exists a positive integer $n$ such that $x^n R = x^{2n} R$.

5- A ring $R$ is called right (left) quasi-duo if every maximal right (left) ideal of $R$ is two sided ideal [1].

6- A ring $R$ is said to be ERT if every essential right ideal is two-sided [1].

2- Maximal Generalized Pure Rings (MGP- rings)

In this section, some basic properties of MGP- rings are given. Also the relations between such rings and strongly $\pi$- regular, weakly $\pi$- regular rings are given.

**Following** [10], an ideal $I$ of a ring $R$ is said to be right (left) GP- ideal if for every $a \in I$, there exists $b \in I$ and a positive integer $n$ such that $a^n = a^n b$ ($a^n = b a^n$).

**Definition 2.1.**[10]: A ring $R$ is called a right (left) MGP- rings if and only if every maximal right (left) ideal is left (right) GP- ideal.

**Theorem 2.2:** Let $R$ be a ring with $l(a^n) \subseteq r(a)$, for any $a \in R$ and a positive integer $n$. If $R$ is a right MGP- ring, then the Jacobson radical of $R$ is zero $(J(R) = (0))$.

**Proof:** Let $0 \neq a \in J(R)$. If $aR + r(a) \neq R$, then there exists a maximal right ideal $M$ containing $aR + r(a)$. Since $R$ is right MGP- ring, then $M$ is left GP- ideal, so there exists $b \in M$ and a positive integer $n$ such that $a^n = ba^n$, $(1-b)a^n = 0$ hence $(1-b) \in l(a^n) \subseteq r(a) \subseteq M$. So $1 \in M$, a contradiction. Therefore $aR + r(a) = R$, and so $ar + d = 1$ for some $r \in R$ and $d \in r(a)$, this implies that $a = a^2 r$. Since $a \in J(R)$, then there exists an invertible element $v$ in $R$ such that $(1-ar)v = 1$, so $(a-a^2 r)v = a$ yields $a = 0$. This proves that $J(R) = 0$. #

**Theorem 2.3:** Let $M$ be a maximal right ideal of $R$. Then $R$ is MGP- ring if and only if for every $a \in M$, $M + l(a^n) = R$ for some positive integer $n$.

**Proof:** Let $M$ be a maximal right ideal of $R$ and $a \in M$, since $R$ is MGP- ring, then $M$ is a left GP- ideal so there exists $b \in M$ such that $a^n = ba^n$ for some positive integer $n$. This implies that $(1-b) \in l(a^n)$. Therefore $R = M + l(a^n)$.

Conversely, assume that $M + l(a^n) = R$ for every $a \in M$ and a positive integer $n$, then $t + s = 1$ for some $t \in M$ and $s \in l(a^n)$. So $ta^n + sa^n = a^n$ and this implies $a^n = ta^n$. Whence $R$ is MGP- ring. #

**Lemma 2.4.**[6]: Every strongly $\pi$- regular is $\pi$- regular ring.

**Recall that**, $R$ is said to be W.R.D (weakly right duo), if for each $a \in R$ there exists a positive integer $n$ such that $a^n R = Ra^n R$.

**Theorem 2.5.**[7]: Let $R$ be W.R.D ring. Then the following are equivalent:
1- $R$ is a $\pi$- regular.
2- Every ideal of $R$ is a left GP- ideal.

**Theorem 2.6:** If $R$ is a right Kasch ring and reversible, then the following statements are equivalent:
1- $R$ is a strongly $\pi$- regular ring.
2- $R$ is a right MGP- ring.
Proof: (1) $\Rightarrow$ (2):
Assume that $R$ is strongly $\pi$-regular ring then by Lemma 2.4 and Theorem 2.5, $R$ is a right MGP-ring.

(2) $\Rightarrow$ (1):
Assume that $R$ is a right MGP-ring. Let $M$ be any maximal right ideal of $R$. Since $R$ is a right Kasch ring, then $M = r(a^n)$ for some $a \in R$ and a positive integer $n$. For any $x \in M$, we have $a^n x = 0$, and so $a^n Rx = 0$. This implies that $Rx \subseteq r(a^n) = M$, which proves that $M$ is a two sided ideal of $R$. We claim that $b^n R + r(b^n) = R$. If not, there is a maximal right ideal $N$ of $R$ such that $b^n R + r(b^n) \subseteq N$. Since $R$ is MGP-ring, then $N$ is a left GP-ideal and $b^n \in N$. Then there exists $y \in N$ such that $b^n = yb^n$. Hence $(1 - y)b^n = 0$ and so $(1 - y) \in l(b^n) = r(b^n) \subseteq N$ (because $R$ is reversible). Thus $1 \in N$, a contradiction. Therefore $b^n R + r(b^n) = R$. In particular $b^n u + v = 1$ for some $u \in R$ and $v \in r(b^n)$, so $b^n = b^n u$. Thus $R$ is strongly $\pi$-regular.

Theorem 2.7: Let $R$ be a right Kasch ring and reversible. If $R$ is a right MGP-ring. Then for each completely prime ideal $P$ of $R$, $P = \bigcup_{x \in P} r(x)$.

Proof: By Theorem 2.6, $R$ is a strongly $\pi$-regular ring. Let $Q = \bigcup_{x \in P} r(x)$ and show that $Q = P$. If $x \in Q$, then $x \in r(y)$ for some $y \notin P$, thus $yx = 0 \in P$ and hence $x \in P$. Therefore $Q \subseteq P$.

On the other hand, by the Lemma 2.4, $P$ is $\pi$-regular ideal, thus for each $x \in P$, there exists $u \in P$ and a positive integer $n$ such that $x^n = x^n u x^n$, which implies that $x^n (1 - ux^n) = 0 \in P$. Then $x^n \in l(1 - ux^n) = r(1 - ux^n)$ (because $R$ is reversible ring). Since $(1 - ux^n) \notin P$ for otherwise $1 \in P$, which is impossible. Thus $x^n \in Q$, so that $P \subseteq Q$, whence $Q = P$. #

Following [9], a ring $R$ is called right (left) weakly $\pi$-regular if, for every $x \in R$ there exists a positive integer $n$ such that $x^n \in x^n Rx^R$ ($x^n \in Rx^n Rx^R$). $R$ is weakly $\pi$-regular if it is both right and left weakly $\pi$-regular.

The next result gives us a sufficient condition for MGP-ring to be weakly $\pi$-regular.

Theorem 2.8: Let $R$ be a semi prime ring with each non-zero right ideal contains a non-zero two sided ideal. If $R$ is an MGP-ring, then it is weakly $\pi$-regular ring.

Proof: Assume that $0 \neq a \in R$ such that $a^2 = 0$, then by assumption there is a non-zero two sided ideal of $R$ with $I \subseteq aR$. We claim that $l(a) \cap I \neq \emptyset$, for if $la = 0$ then $l \subseteq l(a)$ and we are done. If $la \neq 0$, then $la \subseteq l \cap l(a) \neq (0)$. Now, $(I \cap l(a))^2 \subseteq l(a)I \subseteq l(a) a R = (0)$. Since $R$ is semi prime ring, then $l \cap l(a) = (0)$ which is a contradiction, consequently $R$ is reduced. We show that $Rx^n R + r(x^n) = R$ for any $x \in R$ and a positive integer $n$. Suppose that there exists $y \in R$ such that $Ry^n R + r(y^n) \neq R$. Then there exists a maximal right ideal $M$ of $R$ containing
Raida D. Mahammod and Shahla M. Khalil

Since $R$ is a right MGP- ring, then $M$ is a left GP- ideal. So there exists $a \in M$ and a positive integer $m$ such that $(y^n)^m = a(y^n)^m$ implies $(1 - a) \in l((y^n)^m) = r((y^n)^m) = r(y^n) \subseteq M$. Hence $(1 - a) \in M$ and so $1 \in M$ implies that $M = R$ which is a contradiction. Therefore $Rx^n + r(x^n) = R$ for any $x \in R$. In particular $cx^d + u = 1$ for some $u \in r(x^n)$ and $c, d \in R$, hence $x^n = x^c x^d$. So $R$ is a right weakly $\pi$ - regular ring. Since $R$ is reduced it is also can be easily verified that $R$ is weakly $\pi$ - regular ring. #

**Theorem 2.9.[6]** Let $R$ be duo ring. Then $R$ is $\pi$ - regular if and only if $R$ is strongly $\pi$ - regular.

**Lemma 2.10.[13]** If $R$ is left or right quasi- duo and $J(R) = (0)$. Then $R$ is reduced ring.

**Proposition 2.11.[6]** Let $R$ be W.R.D. Then the following statement are equivalent:
1- $R$ is a weakly $\pi$ - regular ring.
2- $R$ is a strongly $\pi$ - regular ring.

The following theorem extends Theorem 2.6.

**Proposition 2.12:** Let $R$ be a right duo ring. Then the following statements are equivalent:
1- $R$ is a strongly $\pi$ - regular ring.
2- $R$ is a $\pi$ - regular ring.
3- $R$ is a right weakly $\pi$ - regular ring.
4- $R$ is a weakly $\pi$ - regular ring.
5- $R$ is a right MGP- ring.
6- Every maximal right ideal is a left pure.

**Proof:** (1) $\Rightarrow$ (2): Follows from Theorem 2.9.
(2) $\Rightarrow$ (3): Obvious.
(3) $\Rightarrow$ (4): Follows from Proposition 2.11 and Theorem 2.5. Thus every maximal right ideal is GP- ideal. Now, we show that $J(R)$ is nil.

Let $r \in J(R)$. Then $r^n R = r^n R r^n R$ for some positive integer $n$, and so $r^n (1 - s) = 0$ for some $s \in R r^n R$. But since $r \in J(R)$, $(1 - s)$ is invertible and thus we have $r^n = 0$, showing that $J(R)$ is nil.

(4) $\Rightarrow$ (1): Assume (4), then from Theorems 2.5 and 2.9, $R$ is strongly $\pi$ - regular. #

**Lemma 2.13.[10]** Let $R$ be a reduced ring. Then every GP- ideal is pure ideal.

By the Proposition 2.12, Lemmas 2.10 and 2.13, we have the following theorem.

**Theorem 2.14:** Let $R$ be a right quasi-duo and $J(R) = 0$. Then the following statements are equivalent:
1- $R$ is a strongly $\pi$ - regular ring.
2- $R$ is a $\pi$ - regular ring.
3- $R$ is a right (left) weakly $\pi$ - regular ring.
4- $R$ is a weakly $\pi$ - regular ring.
5- $R$ is a right MGP- ring.
6- Every maximal right ideal is a left pure.

3- EGP- rings
In this section, we introduce a new essential GP- ideals which are called EGP- rings. We give some of their basic properties, as well as a connection between EGP- rings and \(-S\) - weakly regular rings, strongly regular rings.

**Definition 3.1:** A ring \(R\) is said to be right EGP- rings, if every essential right ideal of \(R\) is a left GP- ideals.

**Recall that** [11], a ring \(R\) is a right (left) \(-S\) - weakly regular if for each \(a \in R\), \(a \in aRaR \) (\(a \in RaR\)). A ring \(R\) is called \(-S\) - weakly regular ring if it is both right and left \(-S\) - weakly regular ring. We start this section by recalling the following propositions:

**Proposition 3.2.**[12]: A ring \(R\) is \(-S\) - weakly regular ring if and only if \(R\) is a reduced weakly regular ring.

**Proposition 3.3.**[10]: Let \(R\) be a duo ring. Then \(R\) is regular if and only if every ideal \(I\) of \(R\) is left pure.

Now, the following result is given:

**Theorem 3.4:** Let \(R\) be a ring with \(aR = Ra\). Then \(R\) is a reduced, right EGP- ring if and only if \(R\) is an \(-S\) - weakly regular ring.

**Proof:** Assume that \(R\) is a reduced right EGP- ring. Let \(a \in R\) and \(I = RaR + r(a)\). We claim that \(I\) is an essential right ideal of \(R\). Suppose this is not true, then there exists a non-zero ideal \(J\) of \(R\) such that \(I \cap J = (0)\). Then \((RaR)J \subseteq IJ \subseteq I \cap J = (0)\).

Since \(aR \subseteq RaR\), then \(aR \cap J = (0)\). But \((aR)J \subseteq aR \cap J = (0)\) implies \(J = (0)\), a contradiction; hence \(I\) is an essential right ideal. Since \(R\) is right EGP- ring, then \(I\) is a left GP- ideal, for every \(a \in I\) there exists \(b \in I\) and a positive integer \(n\) such that \(a^n = ba^n\). Since \(b \in I\) and \(I = RaR + r(a)\), then \(b = ca^2d + h\) for some \(c, d \in R\) and \(h \in r(a)\) hence \(a^n = ba^n = ca^2da^n + ha^n\). Since \(R\) is reduced then \(r(a) = l(a) = l(a^n)\). So \(a^n = ca^2da^n + 0\) implies that \((1-ca^2d)a^n = 0\) and \((1-ca^2d) \in l(a^n) = r(a^n) = r(a)\) hence \(a = aca^2d\). Therefore \(R\) is an \(-S\) - weakly regular ring.

Conversely, assume that \(R\) is an \(-S\) - weakly regular ring. Then by Proposition 3.2, \(R\) is reduced weakly regular ring. Let \(a \in R\) and \(aR = aRa = aRRa = aRa\) since \((aR = Ra)\). So \(R\) is regular and by Proposition 3.3, \(R\) is EGP- ring. 

**Lemma 3.5.**[9]: Let \(R\) be a weakly regular ring. Then \(J(R) = (0)\).

However, we have the following proposition:

**Proposition 3.6:** Let \(R\) be a right duo ring. Then the following statements are equivalent:
1- \(R\) is a strongly regular ring.
2- \(R\) is a regular ring.
3- \(R\) is a right weakly regular ring.
4- \(R\) is an \(-S\) - weakly regular ring.
5- \(R\) is a right EGP- ring and \(l(a^n) \subseteq r(a)\) for every \(a \in R\) and a positive integer \(n\).

**Proof:** (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3): They are obvious.
(3) ⇒ (4): Assume (3), then by Lemmas 3.5, 2.10 and Proposition 3.2, R is an S – weakly regular ring.

(4) ⇒ (5): It follows from Theorem 3.4.

(5) ⇒ (1): Assume that R is right EGP- ring. For any \( a \in T \), let \( T = aR + r(a) \) be a right ideal, by a similar method of the proof used in Theorem 3.4, \( T \) is an essential ideal. Since R is a right EGP- ring, then \( T \) is a left GP- ideal. For every \( a \in T \), there exists \( b \in T \) and a positive integer \( n \) such that \( a^n = ba^n \), which implies \( (1 - b) \in l(a^n) \subseteq r(a) \subseteq T \), so \( 1 \in T \) and \( T = R \). Therefore \( aR + r(a) = R \).

In particular \( ar + d = 1 \) for some \( r \in R \) and \( d \in r(a) \). Then \( a = a^2r \). Thus R is strongly regular ring. 

**Definition 3.7,[8]:** A ring R is called fully right (left) idempotent if every right (left) ideal of R is idempotent.

**Theorem 3.8:** If R is ERT, then the following condition are equivalent:

1- R is a fully left idempotent ring.

2- R is a right EGP- ring.

**Proof:** (1) ⇒ (2): Assume (1), and let \( E \) be an essential right ideal of R then it is an ideal of R (since R is ERT). Since R is a fully left idempotent ring, then for any \( x \in E \), \( Rx = (Rx)^2 \) which implies that \( x = ax \) for some \( a \in RxR \subseteq E \). Therefore \( x \in Ex \) for each \( x \in E \). So \( E \) is left pure implies \( E \) is a left GP- ideal. Thus R is EGP- ring.

(2) ⇒ (1): Assume (2), for any \( a \in R \), set \( L = Ra^nR + l(Ra^nR) \), and let \( K \) be a complement right ideal of R such that \( L \oplus K \) is an essential right ideal of R. Now, \( KRa^nR \subseteq K \cap Ra^nR \subseteq K \cap L = 0 \) implies that \( K \subseteq l(Ra^nR) \). Whence \( K \subseteq K \cap L = 0 \). This shows that \( L \) is an essential right ideal of R which is an ideal of R, by hypothesis R is a right EGP- ring. Therefore \( L \) is left GP- ideal and \( a \in L \) implies that \( a^n = da^n \) for some \( d \in L \) and a positive integer \( n \). If \( d = u + v \), \( u \in Ra^nR \) and \( v \in l(Ra^nR) \), then \( a^n = ua^n + va^n = ua^n \in Ra^nRa^n \) which implies that \( a^n \in (Ra^n)^2 \); whence \( Ra^n = (Ra^n)^2 \). Therefore R is a fully left idempotent ring. 

**REFERENCES**

64


