The Existence, Uniqueness And Upper Bounds For Errors Of Six Degree Spline Interpolating The Lacunary Data (0,2,5)

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ABSTRACT

The object of this paper is to obtain the existence, uniqueness and upper bounds for errors of six degree spline interpolating the lacunary data (0,2,5). We also showed that the changes of the boundary conditions and the class of spline functions has a main role in minimizing the upper bounds for error in lacunary interpolation problem. For this reason, in the construction of our spline function which interpolates the lacunary data (0,2,5), we changed the boundary conditions and the class of spline functions which are given by [1] from first derivative to third derivative and the class of spline function from $C^2[0,1]$ to $C^4[0,1]$.

Keywords: spline function, boundary conditions, lacunary interpolation problem.

الحصول على الوجودِ والوحدانيةِ والحدودِ العلياِ مِنْ الأخطاءِ لدالة Spline من الدرجة السادسة الذي يندرج البيانات الفراغية (0,2,5)

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الملخص

إنّ هدف هذا البحث هو الحصول على الوجودِ والوحدانيةِ والحدودِ العلياِ مِنْ الأخطاءِ لدالة سبلاين من الدرجة السادسة الذي يندرج البيانات الفراغية (0,2,5)، تم الاستنتاج بان تغير الشروط الحدودية و صنف دالة سبلاين لَهُما دور رئيسي في تَقليل الحدودِ العلياِ للخطأِ في مشكلةِ الاندراج الفراغي. لهذا السبب، تم بناءِ دالة سبلاين الذي يندرج البيانات الفراغية (0,2,5)، غيّرنَا شروطَ الحَدَّ وصنفَ لدالة سبلاين الذي مُعطى مِن قِبل [1] مِنْ الاشتقاق الأولِ إلى الاشتقاق الثالثِ وصنفِ دالة سبلاين مِنْ $C^2[0,1]$ إلى $C^4[0,1]$.

الكلمات المفتاحية: دالة سبلاين ، الشروط الحدودية ، الاندراج الفراغي.

1. Introduction

The subject of lacunary interpolation by polynomials has a rich history from Ltd Stone. For complete historical background we refer to the survey article [6]. The aim of this paper is to continuous works [2,5] Jwamer, but for different lacunary interpolation case that is (0,2,5) lacunary interpolation by using spline function of degree six, also we can use the same idea but for different partitions and different degree of spline functions. Many of the authors and co-workers are working on the lacunary interpolation problem by spline functions, for examples [2, 4, 7] and their references, but on minimizing error bounds there is a few papers, for examples [2, 5].

In 2004, Jwamer [1], obtained the error bounds for (0, 2, 5) lacunary interpolation of certain classes of function by deficient six spline. In this paper we study the same lacunary interpolation data, but the essential difference here being in the boundary condition and the class of spline functions. The main object of this work is to show that the change of the boundary conditions and the class of spline functions affect on minimizing the upper bounds for error in lacunary interpolation problem.

For description our problem, let $\Delta: 0 = x_0 < x_2 < \ldots < x_{2m} = 1$ be a uniform partition of the interval [0, 1] with $x_i = \frac{i}{2m}$, i=0, 2, …, $2m$ and n=2m+1. We define the class of spline function $S_p(6,4,n)$ where $Sp(6,4,n)$ denotes the class of all splines of degree 6 which belongs to $C^4[0,1]$, and n is the number of knots, as follow:

Any element $S_\Delta(x) \in Sp(6,4,n)$ if the following two conditions are satisfied:

$$\left.\begin{array}{l} (i)\ S_\Delta(x) \in C^4[0,1] \\ (ii)\ S_\Delta(x) \text{ is a polynomial of degree six} \\ \quad \text{in each } [x_{2i}, x_{2i+2}],\ i = 0,1,\ldots,m-1 \end{array}\right\} \tag{1}$$

2. Construction of the Spline Function

If $P(x)$ is a polynomial of degree six on [0, 1], then we have

$$P(x)=P(0)A_0(x)+P(\tfrac{1}{3})A_1(x)+P(1)A_2(x)+ p''(\tfrac{1}{3})A_3(x)\, P'''(0)A_4(x)+$$

$$P'''(1)A_5(x)+ p^{(5)}(\tfrac{1}{3})A_6(x). \tag{2}$$

where

$$\left.\begin{array}{l} A_o(x) = \frac{1}{74}(-243x^6 + 486x^5 - 135x^2 - 182x + 74), \\ A_1(x) = \frac{1}{148}(729x^6 - 1458x^5 + 405x^2 + 324x), \\ A_2(x) = \frac{1}{148}(-243x^6 + 486x^5 - 135x^2 + 40x), \\ A_3(x) = \frac{1}{148}(81x^6 - 162x^5 + 119x^2 - 38x), \\ A_4(x) = \frac{1}{5328}(189x^6 - 270x^5 - 222x^3 - 665x^2 + 134x), \\ A_5(x) = \frac{1}{5328}(189x^6 - 378x^5 + 222x^4 + 3155x^2 - 806x), \\ A_6(x) = \frac{1}{479520}(-3645x^6 + 11286x^5 - 9990x^4 + 3155x^2 - 806x). \end{array}\right\} \tag{3}$$

In the subsequent section we need the following values:

For $f \in C^6[0,1]$ we have the following expansions

$$f(x_{2i+2}) = f(x_{2i}) + 2hf'(x_{2i}) + 2h^2 f''(x_{2i}) + \frac{4}{3}h^3 f'''(x_{2i}) + \frac{2}{3}h^4 f^{(4)}(x_{2i}) +$$

$$\frac{4}{15}h^5 f^{(5)}(x_{2i}) + \frac{4}{45}h^6 f^{(6)}(\lambda_{1,2i}), \quad x_{2i} < \lambda_{1,2i} < x_{2i+2}$$

$$
\left.
\begin{aligned}
&f(x_{2i-2}) = f(x_{2i}) - 2hf'(x_{2i}) + 2h''(x_{2i}) - \frac{4}{3}h^3 f'''(x_{2i}) + \frac{2}{3}h^4 f^{(4)}(x_{2i}) - \\
&\quad \frac{4}{15}h^5 f^{(5)}(x_{2i}) + \frac{4}{45}h^6 f^{(6)}(\lambda_{2,2i}),\ \ x_{2i-2} < \lambda_{2,2i} < x_{2i} \\
&f(t_{2i}) = f(x_{2i}) + \frac{2}{3}hf'(x_{2i}) + \frac{2}{9}h^2 f''(x_{2i}) + \frac{4}{81}h^3 f'''(x_{2i}) + \frac{2}{243}h^4 f^{(4)}(x_{2i}) + \\
&\quad \frac{4}{3645}h^5 f^{(5)}(x_{2i}) + \frac{4}{32805}h^6 f^{(6)}(\lambda_{3,2i}),\ x_{2i} < \lambda_{3,2i} < t_{2i} \\
&f(t_{2i-2}) = f(x_{2i}) - \frac{4}{3}hf'(x_{2i}) + \frac{8}{9}h^2 f''(x_{2i}) - \frac{32}{81}h^3 f'''(x_{2i}) + \frac{32}{243}h^4 f^{(4)}(x_{2i}) - \\
&\quad \frac{128}{3645}h^5 f^{(5)}(x_{2i}) + \frac{256}{32805}h^6 f^{(6)}(\lambda_{4,2i}),\ t_{2i-2} < \lambda_{4,2i} < x_{2i} \\
&f'(t_{2i}) = f'(x_{2i}) + \frac{2}{3}hf''(x_{2i}) + \frac{2}{9}h^2 f'''(x_{2i}) + \frac{4}{81}h^3 f^{(4)}(x_{2i}) + \frac{2}{243}h^4 f^{(5)}(x_{2i}) + \\
&\quad \frac{4}{3645}h^5 f^{(6)}(\lambda_{5,2i}),\ \ x_{2i} < \lambda_{5,2i} < t_{2i} \\
&f''(t_{2i}) = f''(x_{2i}) + \frac{2}{3}hf'''(x_{2i}) + \frac{2}{9}h^2 f^{(4)}(x_{2i}) + \frac{4}{81}h^3 f^{(5)}(x_{2i}) + \frac{2}{243}h^4 f^{(6)}(\lambda_{6,2i}), \\
&\qquad x_{2i} < \lambda_{6,2i} < t_{2i} \\
&f''(t_{2i-2}) = f''(x_{2i}) - \frac{4}{3}hf'''(x_{2i}) + \frac{8}{9}h^2 f^{(4)}(x_{2i}) - \frac{32}{81}h^3 f^{(5)}(x_{2i}) + \frac{32}{243}h^4 f^{(6)}(\lambda_{7,2i}) \\
&\qquad t_{2i-2} < \lambda_{7,2i} < x_{2i} \\
&f'''(x_{2i+2}) = f'''(x_{2i}) + 2hf^{(4)}(x_{2i}) + 2h^2 f^{(5)}(x_{2i}) + \frac{4}{3}h^3 f^{(6)}(\lambda_{8,2i}),\ \mathrm{x}_{2i} < \lambda_{8,2i} < x_{2i} \\
&f'''(x_{2i-2}) = f'''(x_{2i}) - 2hf'''(x_{2i}) + 2h^2 f^{(4)}(x_{2i}) - 2h^3 f^{(5)}(x_{2i}) + \frac{4}{3}h^4 f^{(5)}(\lambda_{9,2i}), \\
&\quad x_{2i-2} < \lambda_{9,2i} < x_{2i} \\
&f'''(t_{2i}) = f'''(x_{2i}) + \frac{2}{3}h\ f^{(4)}(x_{2i}) + \frac{2}{9}h^2 f^{(5)}(x_{2i}) + \frac{4}{81}h^3 f^{(6)}(\lambda_{10,2i}),\ \mathrm{x}_{2i} < \lambda_{10,2i} < t_{2i} \\
&f^{(4)}(t_{2i}) = f^{(4)}(x_{2i}) + \frac{2}{3}h\ f^{(5)}(x_{2i}) + \frac{2}{9}h^2 f^{(6)}(\lambda_{11,2i}),\ x_{2i} < \lambda_{11,2i} < t_{2i} \\
&f^{(5)}(t_{2i}) = f^{(5)}(x_{2i}) + \frac{2}{3}h\ f^{(6)}(\lambda_{12,2i}),\ x_{2i} < \lambda_{12,2i} < t_{2i} \\
&f^{(5)}(t_{2i-2}) = f^{(5)}(x_{2i}) - \frac{2}{3}h\ f^{(6)}(\lambda_{13,2i}),\ t_{2i-2} < \lambda_{13,2i} < x_{2i}\,.
\end{aligned}
\right\rbrace \quad (4)
$$

3. Existence and Uniqueness:

In this section we prove the following theorem about the existence and uniqueness of the spline function $S_p(6,4,n)$:

Theorem 1:

Given arbitrary numbers $f(x_{2i})$, $f^{(r)}(t_{2i})$, i=0,1,…,m-1; r=0, 2,5 and $f'''(x_0)$, $f'''(x_{2m})$, there exits a unique spline $S_n(x) \in S_p(6,4,n)$ such that

$$\left.\begin{aligned} &S_n(x_{2i}) = f(x_{2i}) \ , i = 0,1,\ldots,m \\ &S_n^{(r)}(t_{2i}) = f^{(r)}(t_{2i}), i = 0,1,\ldots,m-1; r = 0,2,5 \\ &S_n'''(x_0) = f'''(x_0), S_n'''(x_{2m}) = f'''(x_{2m}) \end{aligned}\right\} \tag{5}$$

Poof:

The proof depends on the following representation of $S_n(x)$ for $2ih \le x \le (2i+2)h$, $i=0,1,\ldots,m-1$. We have

$$S_n(x) = f(x_{2i})A_0(\frac{x-2ih}{2h}) + f(t_{2i})A_1(\frac{x-2ih}{2h}) + f(x_{2i+2})A_2(\frac{x-2ih}{2h}) +$$

$$+4h^2 f''(t_{2i})A_3(\frac{x-2ih}{2h}) + 8h^3 S_n'''(x_{2i})A_4(\frac{x-2ih}{2h}) + 8h^3 S_n'''(x_{2i+2})A_5(\frac{x-2ih}{2h}) + 32h^5 f^{(5)}(t_{2i})A_6(\frac{x-2ih}{2h}). \tag{6}$$

On using equ.(6) and conditions

$$S_n'''(0) = f'''(0)\ ,\ S_n'''(1) = f'''(1). \tag{7}$$

We see that $S_n(x)$ as given by (6) satisfies (1) and is sextic in $[x_{2i}, x_{2i+2}]$, $i=0,1,\ldots,m-1$. We also need to show that whether it is possible to determine $S_n'''(x_{2i})\ , i = 1,2,\ldots,m-1$ uniquely. For this purpose we use the fact that

$S_n^{(4)}(x_{2i+}) = S_n^{(4)}(x_{2i-})\ , i = 1, 2, \ldots, m-1;$

where $S_n^{(4)}(x_{2i+}) = \lim_{x \to x_{2i}^+} S_n^{(4)}(x)$ and $S_n^{(4)}(x_{2i-}) = \lim_{x \to x_{2i}^-} S_n^{(4)}(x)$, with the help of (6) and (7) reduced to

$$\frac{151}{148}h^3 S_n'''(x_{2i-2}) + \frac{463}{148}h^3 S_n'''(x_{2i}) - \frac{1}{2}h^3 S_n'''(x_{2i+2}) =$$

$$\frac{3645}{296}f(x_{2i}) + \frac{3645}{148}f(x_{2i-2}) - \frac{10935}{296}f(t_{2i-2}) - \frac{1215}{74}h^2 f''(t_{2i-2}) - h^5 f^{(5)}(t_{2i}) + \frac{61}{74}h^5 f^{(5)}(t_{2i-2}), \tag{8}$$

$$i = 1, 2, \ldots, m-1.$$

Equation (8) is a strictly tri-diagonal dominant system which has a unique solution (see [3]). Thus $S_n'''(x_{2i})$, $i=1, 2\ldots, m-1$ can be obtained uniquely by the system (8) which establishes Theorem 1.

4. Error Bounds:

In this section, the upper bounds for errors studied in the following results:

Theorem 2:

Let $f \in C^6[0,1]$ and $S_n(x) \in S_p(6,4,n)$ be a unique spline satisfying the conditions of Theorem 1, then

$$\left\| S_n^{(r)}(x) - f^{(r)}(x) \right\| \le 27.179859 m^{r-6} w(f^{(6)}; \frac{1}{m}) + 2\, m^{r-6} \left\| f^{(6)} \right\|, r = 0,1,2,3,4,5. \tag{9}$$

Where $w(f^{(6)}; \frac{1}{m})$ denotes the modulus of continuity of $f^{(6)}$ and $\| f^{(6)} \| = \max\{ | f^{(6)}(x) |; 0 \le x \le 1 \}$.

***In order to prove Theorem 2 we need the following*:**

Lemma 1:

Let us write $E_{2i}=\left|S_n'''(x_{2i})-f'''(x_{2i})\right|$, then for $f\in C^6[0,1]$, we have

$$\max E_{2i}\le \frac{312}{119}h^3 w(f^{(6)};\frac{1}{m}), \text{ for } i=1, 2, \dots, m\text{-}1. \tag{10}$$

Proof:

From (8) we have

$$\frac{151}{148}h^3(S_n'''(x_{2i-2})-f'''(x_{2i-2}))+\frac{463}{148}h^3(S_n'''(x_{2i})-f'''(x_{2i}))-\frac{1}{2}h^3(S_n'''(x_{2i+2})-f'''(x_{2i+2}))=$$
$$\frac{3645}{296}f(x_{2i})+\frac{3645}{148}f(x_{2i-2})-\frac{10935}{296}f(t_{2i-2})+\frac{1215}{74}h^2f''(t_{2i-2})-h^5f^{(5)}(t_{2i})+\frac{61}{74}h^5f^{(5)}(t_{2i-2})-\frac{151}{148}h^3f'''(x_{2i-2})-$$
$$\frac{463}{148}h^3f'''(x_{2i})-\frac{1}{2}h^3f'''(x_{2i+2})=\frac{81}{37}h^6f^{(6)}(\lambda_{2,2i})-\frac{32}{111}h^6f^{(6)}(\lambda_{4,2i})-\frac{80}{37}h^6f^{(6)}(\lambda_{7,2i})+\frac{151}{111}h^6f^{(6)}(\lambda_{9,2i})+\frac{2}{3}h^6f^{(6)}(\lambda_{8,2i})-$$
$$\frac{2}{3}h^6f^{(6)}(\lambda_{12,2i})-\frac{122}{111}h^6f^{(6)}(\lambda_{13,2i})=\frac{468}{111}h^5\alpha_1 w(f^{(6)};\frac{1}{m}),\quad |\alpha_1|\le 1$$

The result (10) follows on using the property of diagonal dominant [3].

Lemma 2:

Let $f\in C^6[0.1]$ then

$$\left|S_n^{(5)}(x_{2i+})-f^{(5)}(x_{2i})\right|\le\frac{329975}{26418}hw(f^{(6)};\frac{1}{m}), \tag{11}$$

$$\left|S_n^{(5)}(x_{2i-})-f^{(5)}(x_{2i})\right|\le\frac{584099}{26418}hw(f^{(6)};\frac{1}{m}), \tag{12}$$

$$\left|S_n^{(4)}(t_{2i})-f^{(4)}(t_{2i})\right|\le\frac{296155}{79254}h^2w(f^{(6)};\frac{1}{m}), \tag{13}$$

$$\left|S_n^{(3)}(t_{2i})-f^{(3)}(t_{2i})\right|\le\frac{55114}{39627}h^3\, w(f^{(6)};\frac{1}{m}), \tag{14}$$

and

$$\left|S_n'(t_{2i})-f'(t_{2i})\right|\le\frac{223607}{16048935}h^5\, w(f^{(6)};\frac{1}{m}). \tag{15}$$

Proof:

From (6) we have

$$h^5S_n^{(5)}(x_{2i+})=\frac{3645}{148}f(x_{2i})-\frac{10935}{296}f(t_{2i})+\frac{3645}{296}f(x_{2i+2})-\frac{1215}{74}h^2f''(t_{2i})-\frac{225}{148}h^3S_n'''(x_{2i})-$$
$$\frac{315}{148}h^3S_n'''(x_{2i-2})+\frac{209}{74}h^5f^{(5)}(t_{2i})$$

Hence

$$h^5(S_n^{(5)}(x_{2i+})-f^{(5)}(x_{2i}))=-\frac{1}{222}h^6f^{(6)}(\lambda_{3,2i})+\frac{81}{74}h^6f^{(6)}(\lambda_{1,2i})-\frac{5}{37}h^6f^{(6)}(\lambda_{6,2i})-\frac{105}{37}h^6f^{(6)}(\lambda_{8,2i})+$$
$$\frac{209}{111}h^6f^{(6)}(\lambda_{12,2i})-\frac{225}{148}h^3(S_n'''(x_{2i})-f(x_{2i}))-\frac{315}{148}h^3(S_n'''(x_{2i-2})-f'''(x_{2i+2}))$$
$$=\frac{661}{222}h^6\alpha_2 w(f^{(6)};\frac{1}{m})-\frac{225}{148}h^3(S_n'''(x_{2i})-f'''(x_{2i}))-\frac{315}{148}h^3(S_n'''(x_{2i-2})-f'''(x_{2i+2})),\ |\alpha_2|\le 1$$

By using (10), we get (11). The proofs of (12)-(15) are similar, and we only mention that

$$h^5 S_n^{(5)}(x_{2i-}) = -\frac{3645}{148} f(x_{2i}) + \frac{10935}{148} f(t_{2i-2}) - \frac{3645}{74} f(x_{2i-2}) + \frac{1215}{37} h^2 f''(t_{2i-2}) + \frac{225}{74} h^3 S_n'''(x_{2i-2}) + \frac{315}{74} h^3 S_n'''(x_{2i}) - \frac{98}{37} h^5 f^{(5)}(t_{2i-2}), \quad (16)$$

$$h^4 S_n^{(4)}(t_{2i}) = \frac{1215}{148} f(x_{2i}) - \frac{3645}{296} f(t_{2i}) + \frac{1215}{296} f(x_{2i+2}) - \frac{405}{74} h^2 f''(t_{2i}) - \frac{149}{148} h^3 S_n'''(x_{2i}) - \frac{31}{148} h^3 S_n'''(x_{2i-2}) + \frac{61}{222} h^5 f^{(5)}(t_{2i}),$$

$$h^3 S_n^{(3(}(t_{2i}) = \frac{135}{37} f(x_{2i}) - \frac{405}{74} f(t_{2i}) + \frac{135}{74} f(x_{2i+2}) - \frac{90}{37} h^2 f''(t_{2i}) + \frac{49}{111} h^3 S_n'''(x_{2i}) - \frac{2}{111} h^3 S_n'''(x_{2i-2}) - \frac{58}{333} h^5 f^{(5)}(t_{2i}),$$

and

$$hS_n'(t_{2i}) = -\frac{62}{37} f(x_{2i}) + \frac{261}{148} f(t_{2i}) - \frac{13}{148} f(x_{2i+2}) + \frac{50}{111} h^2 f''(t_{2i}) - \frac{134}{2997} h^3 S_n'''(x_{2i}) - \frac{10}{2997} h^3 S_n'''(x_{2i-2}) + \frac{212}{14985} h^5 f^{(5)}(t_{2i}).$$

Proof of Theorem 2:

For $0 \le y \le 1$, we obtain $A_0(y)+A_1(y)+A_2(y)=1$.

Let $x_{2i} \le x \le x_{2i+2}$. On using (16) and (7),we obtain

$$S_n^{(5)}(x) - f^{(5)}(x) = (S_n^{(5)}(x_{2i+}) - f^{(5)}(x))A_0(\frac{x-2ih}{2h}) + (S_n^{(5)}(x_{2i+2}) - f^{(5)}(x))A_2(\frac{x-2ih}{2h}) + (S_n^{(5)}(t_{2i}) - f^{(5)}(x)A_1(\frac{x-2ih}{2h}) = L_1+L_2+L_3 \quad (17)$$

From (3) it follows that:

$|A_0(x)| \le 1$, $|A_1(x)| \le 1$ *and* $|A_2(x)| \le 1$

Since $f^{(5)}(x) = f^{(5)}(x_{2i}) + (x - x_{2i}) f^{(6)}(\lambda)$, for $x_{2i}<\lambda<x$,

$$L_1 = (S_n^{(5)}(x_{2i+}) - f^{(5)}(x))A_0(\frac{x-2ih}{2h}) = (S_n^{(5)}(x_{2i}) - f^{(5)}(x_{2i}) - (x - x_{2i}) f^{(6)}(\lambda))A_0(\frac{x-2ih}{2h}).$$

On using (2) and $|x - x_{2i}| \le 2h$, we obtain

$$|L_1| \le \frac{329975}{26418} hw(f^{(6)}; \frac{1}{m}) + 2h\|f^{(6)}\| \quad (18)$$

Similarly,

$$|L_2| \le \frac{58409}{26418} hw(f^{(6)}; \frac{1}{m}) + 2h\|f^{(6)}\|, \quad (19)$$

and

$$L_3 = (S_n^{(5)}(t_{2i}) - f^{(5)}(x)A_1(\frac{x-2ih}{2h}) \ = (S_n^{(5)}(t_{2i}) - f^{(5)}(t_{2i}) + f^{(5)}(t_{2i}) - f^{(5)}(x_{2i}) - (x - x_{2i}) f^{(6)}(\lambda)). \quad (20)$$

From (4) we obtain

$f^{(4)}(t_{2i}) - f^{(4)}(x_{2i}) = \frac{2}{3} hf^{(5)}(\lambda_{10,2i})$, and $f^{(5)}(t_{2i}) - f^{(5)}(t_{2i}) = 0$.

Substitute this result and (12) in (19), we obtain

$$|L_3| \le 2hw(f^{(6)};\frac{1}{m}) . \quad (21)$$

Putting (18) – (21) in (17) we obtain

$$\left|S_n^{(5)}(x) - f^{(5)}(x)\right| \le \frac{457037}{13209} hw(f^{(6)};\frac{1}{m}) + 4h\left\|f^{(6)}\right\|. \quad (22)$$

This proves Theorem 2 for r=5. To prove Theorem 1.2 for r=4, since

$$S_n^{(4)}(x) - f^{(4)}(x) = \int_{t_{2i}}^{x} (S_n^{(5)}(t) - f^{(5)}(t))dt + S_n^{(4)}(t_{2i}) - f^{(4)}\ (t_{2i}).$$

On using (14) and (22) we obtain

$$\left|S_n^{(4)}(x) - f^{(4)}(x)\right| \le \frac{5780599}{79254} l^2 w(f^{(6)};\frac{1}{m}) + 8\mathrm{h}^2\left\|f^{(6)}\right\|. \quad (23)$$

This proves Theorem 2 for r=4. To prove Theorem 2 for r=*3*, since

$$S_n^{(3)}(x) - f^{(3)}(x) = \int_{t_{2i}}^{x} (S_n^{(4)}(t) - f^{(4)}(t))dt + S_n^{(3)}(t_{2i}) - f^{(3)}(t_{2i}).$$

On using (15) and (23) we obtain

$$\left|S_n^{(3)}(x) - f^{(3)}(x)\right| \le \frac{957368}{4403} h^3 w(f^{(6)};\frac{1}{m}) + 16\mathrm{h}^3\left\|f^{(6)}\right\|. \quad (24)$$

Which proves Theorem 2 for r=3. To prove Theorem 2 for r= *2*, since

$$S_n''\ (x) - f''(x) = \int_{t_{2i}}^{x} (S_n^{(3)}(t) - f^{(3)}(t))dt + S_n''\ (t_{2i}) - f''(t_{2i}) .$$

Since $S_n''\ (t_{2i}) - f''\ (t_{2i}) = 0$, and using (23) we obtain

$$\left|S_n''\ (x) - f''(x)\right| \le \frac{1914736}{4403}\ h^4 w(f^{(6)};\frac{1}{m}) + 32\mathrm{h}^4\left\|f^{(6)}\right\|. \quad (25)$$

Which proves Theorem 2 for r=2. The proof of Theorem 2 for r= 1, since

$$S_n'\ (x) - f'(x) = \int_{t_{2i}}^{x} (S_n^{(2)}(t) - f^{(2)}(t))dt + S_n'\ (t_{2i}) - f'(t_{2i}) .$$

Since $S_n(t_{2i}) - f'(t_{2i}) = 0$, and on using (16) and (25), we obtain

$$\left|S_n'(x) - f'(x)\right| \le \frac{13958649047}{16048935} h^5 w(f^{(6)};\frac{1}{m}) + 64h^5\left\|f^{(6)}\right\|. \quad (26)$$

This proves Theorem 2 for r=1.To prove Theorem 2 for r=0 , since

$$S_n(x) - f(x) = \int_{t_{2i}}^{x} (S_n'(t) - f'(t))dt + S_n(t_{2i}) - f(t_{2i}) \,,$$

$$S_n(t_{2i}) - f(t_{2i}) = 0 \,,$$

using (26) we obtain

$$|S_n(x) - f(x)| \le \frac{27917298094}{16048935} h^6 \; w(f^{(6)}; \frac{1}{m}) + 128 h^6 \left\| f^{(6)} \right\|.$$

Put $h = \dfrac{1}{2m}$ in the above result we obtain

$$|S_n(x) - f(x)| \le 27.179859 \; m^{-6} \; w(f^{(6)}; \frac{1}{m}) + 2m^{-6} \left\| f^{(6)} \right\|.$$

This completes the proof of theorem 1.2.

5. Conclusions

In this paper, we conclude that the changes of the boundary conditions and the class of spline functions together affect on minimizing error bounds in the subject of lacunary interpolation by spline functions.

REFERENCES

[1] Jwamer, K. H., (2004), Lacunary interpolation by spline function (0,2,5) case ,Journal of Pure and Applied Science, Salahaddin university,16(1), pp.61-66.

[2] Jwamer, K. H., (2007), Minimizing error bounds in (0,2,3) lacunary interpolation by sextic spline function, Journal of Mathematics and Statistics ,USA, 3(4), pp. 249-256 .

[3] Kincaid, D and Cheney, W., (2002), Numerical Analysis, Mathematics of scientific computing, third edition, printed in USA.

[4] Mishra, R. S. and Muthur, K. K., (1980), Lacunary interpolation by splines (0,2,3) and (0,2,4) cases, Acta Math. Acad. Sci. Hungar.,36(3-4), pp.251-260.

[5] Saeed, R. K. and Jwamer, K. H., (2005), Minimizing error bounds in lacunary interpolation by spline function, (0,1,4)case. Journal of Al-Nahrain University, 8 (2), pp. 114-119.

[6] Swarts, B. K. and Varga, R. S., (1972), Error bounds for spline and L.spline interpolation, J. Approx Theory .6, pp.6-49.

[7] Varma,A. S., (1973), (0,2) Lacunary interpolation by splines. I, Acata Math. Acad. Sci. Hungar. 31(3-4), pp. 443-447.