On $S\pi$ – Weakly Regular Rings, II

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ABSTRACT

The main purpose of this paper is to study right(left) $S\pi$ – Weakly regular rings. Also we give some properties of $S\pi$ – Weakly regular rings, and the connection between such rings and CS-rings, MGP-rings and SSGP-rings.

Keywords: $S\pi$ – Weakly regular rings, CS-rings, MGP-rings, SSGP-rings.

1- Introduction

Throughout this paper, $R$ is an associative ring with identity. A ring $R$ is said to be right(left) $S$-weakly regular ring if for each $a \in R$, $a \in aRa^2R(a \in Ra^2Ra)$.
This concept was introduced by W.B. Vasantha kandasamy [7]. As a generalization of this concept the authors in [3] defined $S\pi -$ Weakly regular ring that is a ring such that for each $a \in R$, there exists a positive integer $n$, $a^n \in a^n Ra^{2n}R(a^n \in Ra^{2n}Ra^n)$. In the present work we develop further properties of $S\pi -$ Weakly regular rings, and we give the connection of $S\pi -$ Weakly regular rings with other rings.

Recall that:
1- An ideal $I$ of a ring $R$ is called right(left) GP-ideal if for every $a \in R$, there exists $b \in R$ and a positive integer $n$ such that $a^n = a^n b (a^n = ba^n)$. [6]
2- A ring $R$ is said to be a right MGP-ring if and only if every maximal right ideal is left GP-ideal.[6]
3- $R$ is called reduced if it has no non nilpotent elements.
4- According to cohn [1], a ring $R$ is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. It is easy to see that $R$ is reversible if and only if right (left) annihilator of $a$ in $R$ is two sided ideal.
5- A ring $R$ is said to be right SSGP-ring if every simple singular right $R$-module is GP-injective [4].
6- $J(R)$ denote the Jacobson radical.

2- $S\pi -$ Weakly regular rings

In this section we give some of basic properties, as well as a connection between CS-rings, MGP-rings, SSGP-rings and $S\pi -$ Weakly regular rings.

begins with the following definition .

**Definition 2.1:**[3]

$R$ is called right (left) $S\pi -$ Weakly regular ring if for each $a \in R$, there exists a positive integer $n = n(a)$ depending on $a$ such that $a^n \in a^nRa^{2n}R(a^n \in Ra^{2n}Ra^n)$. $R$ is called $S\pi -$ Weakly regular ring if it is both right and left $S\pi -$ Weakly regular ring.

**Theorem2.2:**

Let $R$ be $S\pi -$ Weakly regular ring and $a^n R = Ra^n$ with $r(a^n) \subseteq r(a)$ for every $a \in R$ and a positive integer $n$, then $J(R) = (0)$.

**Proof:**

Let $0 \neq a \in J(R)$, if $aR + r(a) \neq R$, then there exists a maximal right ideal $M$ containing $aR + r(a)$, since $R$ is $S\pi -$ Weakly regular ring,
then there exists \(b, c \in R\) and a positive integer \(n\) such that \(a^n = a^n ba^{2n} c\), so \(a^n (1 - ba^{2n} c) = 0\) implies that \((1 - ba^{2n} c) \in r(a^n) \subseteq r(a) \subseteq M\) so \(1 \in M\) a contradiction. Therefore \(aR + r(a) = R\), in particular \(ab + d = 1\) for some \(b \in R, d \in r(a)\), so \(a^2 b = a\) implies \(a(1 - ab) = 0\) since \(a \in J(R)\), so there exists an invertible element \(v\) in \(R\) such that \((1 - ab)v = 1\) multiply in the left by \(a\), \((a - a^2 b)v = a\) implies \(a = 0\). Therefore \(J(R) = (0)\).

**Theorem 2.3:**

Let \(R\) be a ring with out identity and with out divisors of zero. Then \(S\pi -\)Weakly regular ring if and only if for every \(a \in R\) there exists \(b, c \in R\) and a positive integer \(n\) with \(a^n = a^n ba^{2n} c\).

**Proof:**

Given a ring \(R\) with out identity and with out divisors of zero. Let \(R\) be \(S\pi -\)Weakly regular ring. Then for every \(a \in R\) we have \(a^n \in a^n Ra^{2n} R\), thus \(a^n = a^n ba^{2n} c\) for some \(b, c \in R\).

Conversely, if \(a^n = a^n ba^{2n} c\) for every \(a \in R\), and a positive integer \(n\) we have obviously \(R\) to the \(S\pi -\) Weakly regular ring and given \(R\) has no identity and zero divisors.

The following result is a connection between MGP- ring and \(S\pi -\)Weakly regular ring by adding reversible condition.

**Theorem 2.4:**

Let \(R\) be a right MGP-ring and reversible ring. Then \(R\) is a right \(S\pi -\)Weakly regular ring.

**Proof:**

Let \(R\) be a right MGP- ring, to prove \(R\) is a right \(S\pi -\)Weakly regular ring, let \(Ra^{2n} R + r(a^n) = R\), if not then there exists a maximal right ideal \(M\) containing \(Ra^{2n} R + r(a^n)\) such that \(Ra^{2n} R + r(a^n) \subseteq M\), since \(R\) is a right MGP- ring, then every maximal right ideal of \(R\) is left GP-ideal. Then for all \(a \in M\), there exists \(b \in M\) and a positive integer \(n\) such that \(a^n = ba^n\) then \(a^n - ba^n = 0\), implies \((1 - b)a^n = 0\) and \((1 - b) \in l(a^n) = r(a^n) \subseteq M\) (\(R\) is reversible ring), hence \(1 \in M\), a contradiction, hence \(Ra^{2n} R + r(a^n) = R\). Then \(ba^{2n} c + d = 1\), for some
$b,c \in R$ and $d \in r(a^n)$ implies that $a^nb^2c + a^nd = a^n$. Therefore $a^n \in a^nRa^2 R$, so $R$ is a right $S\pi$–Weakly regular ring.

**Example:**

Let $Z_{12}$ be the ring of integers module 12, then the maximal ideals, $I = \{0,3,6,9\}$, $J = \{0,2,4,6,8,10\}$ are GP-ideal, hence $Z_{12}$ is a right MGP-ring and a right $S\pi$–Weakly regular ring.

**Definition 2.5:**[2]

A ring $R$ is said to be right (left) CS- ring if every non zero right (left) ideal is essential in a direct summand.

The next result gives a sufficient condition for CS- ring to be $S\pi$–Weakly regular ring.

**Theorem 2.6:**

Let $R$ be a reversible right CS- ring and $r(a^{2n}) \subseteq r(a^n)$ for every $a \in R$ and a positive integer $n$. Then $R$ is a $S\pi$–Weakly regular ring.

**Proof:**

Let $0 \neq a \in R$ such that $Ra^{2n} R + r(a^n) = R$, if not then there exists a maximal right ideal $M$ containing $Ra^{2n} R + r(a^n)$ since $M$ is a direct summand, there exists $K$ right ideal such that $M \oplus K = R$, this meaning $Ra^{2n} R + r(a^n) \cap K = 0$, which implies that $Ra^{2n} K \subseteq MK \subseteq M \cap K = 0$, then $K \in r(a^{2n}) \subseteq r(a^n)$, hence $K \subseteq r(a^n) \subseteq M$ but $M \cap K = 0$, contradiction, if $Ka \neq 0$, $Ka \subseteq M \cap K = 0$ implies that $Ka = 0$, contradiction, hence $Ra^{2n} R + r(a^n) = R$, in particular $ra^{2n}s + t = 1$, where $r,s \in R$ and $t \in r(a^n)$, so $ra^{2n}s + ta^n = a^n$ hence $ta^n = 0$ (since $t \in r(a^n) = l(a^n)$ $R$ is reversible ring), implies $ra^{2n}s = a^n$, then $a^n \in Ra^{2n} Ra^n$ as well as $a^n ra^{2n}s + a^n t = a^n$, so $a^n \in a^n Ra^{2n} R$. Therefore $R$ is a $S\pi$–Weakly regular ring.

The following result is due to S.B. Nam [4].

**Lemma 2.7:**[4]

Let $R$ be a SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$. Then $R$ is a reduced ring.
Proposition 2.8:

Let $R$ be a SSGP ring and $l(a) \subseteq r(a)$ for every $a \in R$. Then $R$ is $S\pi -$Weakly regular ring.

Proof:

Assume that $R$ is SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$, then by Lemma 2.8, $R$ is reduced ring. We will show that $Ra^{2n}R + r(a^n) = R$, for any $a \in R$ and a positive integer $n$. Suppose that $Ra^{2n}R + r(a^n) \neq R$.

Then there exists a maximal right ideal $M$ of $R$ containing $Ra^{2n}R + r(a^n)$. Thus $R/M$ is GP-injective, so any $R$- homomorphism of $a^{2n}R \rightarrow R/M$ extends to one of $R$ into $R/M$. Let $f : a^{2n}R \rightarrow R/M$ be defined by $f(a^{2n}t) = t + M$, for all $t \in R$. Since $R$ is reduced ring, then $f$ is a well-defined $R$- homomorphism. Now, $R/M$ is GP-injective. So there exists $c \in R$ such that $1 + M = f(a^{2n}) = ca^{2n} + M$. Hence $1 - ca^{2n} \in M$ and so $1 \in M$, which is a contradiction. Therefore $Ra^{2n}R + r(a^n) = R$. In particular, $1 = ba^{2n}d + x$ for some $b, d \in R$ and $x \in r(a^n)$. Therefore $a^n = a^nba^{2n}d$. So $a^nR = a^nRa^{2n}R$ and hence $R$ is $S\pi -$Weakly regular ring.
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