# On The Basis Number Of Semi-Strong Product Of $K_2$ With Some Special Graphs

**Ghassan T. Marougi**
*College of Computer Sciences and Mathematics*
*University of Mosul, Iraq*

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**ABSTRACT**

The basis number, *b(G)* ,of a graph *G* is defined to be the smallest positive integer *k* such that *G* has a *k*-fold basis for its cycle space. We investigate the basis number of semi-strong product of $K_2$ with a path, a cycle, a star, a wheel and a complete graph.

**Keywords:** Basis number, Cycle space.

**حول العدد الأساس للجداء شبه المتين لبيان K2 مع بعض البيانات الخاصة**

**غسان طوبيا مروكي**
*كلية علوم الحاسوب والرياضيات، جامعة الموصل*

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**الملخص**

يعرف العدد الاساس، b(G) لبيان G على انه العدد الصحيح الموجب الاصغر k بحيث ان G له قاعدة ذات ثنية–k لفضاء داراته .في هذا البحث قمنا بحساب العدد الاساس للجداء شبه المتين لبيان $K_2$ مع كل من الدرب والدارة والنجمة والعجلة والبيان التام.

**الكلمات المفتاحية:** العدد الأساس، فضاء الدارات.

## 1-Introduction.

In recent years, there was a grawing literature on the basis number of graphs. We refer the readers to the papers [1],[2],[3],[4],[5] and [8]. Throughout this paper, we consider only finite, undirected and simple graphs. Our terminology and notations will be standard except as indicated. For undefined terms, see [7] and [11].

Let *G* be a connected graph, and let $e_1, e_2, \ldots, e_q$ be an ordering of the edges. Then any subset *S* of edges corresponds to a (0,1)-vector $(a_1, a_2, \ldots, a_q)$ in the usual way, with $a_i = 1$ if $e_i \in S$ and $a_i = 0$ otherwise, for $i=1,2, \ldots,q$. These vectors form a q-dimensional vector space, denoted by $(Z_2)^q$ over the field $Z_2$.

The vectors in $(Z_2)^q$ which correspond to the cycles in *G* generate a subspace called the cycle space of *G*, and denoted by $\xi(G)$. It is well known that

$$\dim \xi(G) = \gamma(G) = q - p + k,$$

where p is the number of vertices, *k* is the number of connected components and $\gamma(G)$ is the cyclomatic number of *G*. A basis for $\xi(G)$ is called <u>h-fold</u> if each edge of *G* occurs in at most *h* of the cycles in the basis. The basis number of $G$, denoted by $b(G)$, is the smallest positive integer $h$ such that $\xi(G)$ has an h-fold basis, and such a basis is called a required basis of $G$ and denoted by $B_r(G)$. If $B$ is a basis for $\xi(G)$ and e is an edge of $G$, then the fold of $e$ in $B$, denoted by $f_B(e)$ is defined to the number of cycles in $B$ containing $e$. The first important result of the basis number occured in 1937 when MacLane [9] proved that a graph $G$ is planar if and only if $b(G)\leq 2$.

**Definition:** The semi-strong product of two disjoint graphs G=($V_1,E_1$) and H=($V_2,E_2$) is the graph G*H with vertex set $V_1 \times V_2$ in which ($v_1,v_2$) is joined to ($u_1,u_2$) whenever [$v_1u_1 \in E_1$ *and* $v_2u_2 \in E_2$] *or* [$v_1u_1 \in E_1$ *and* $v_2=u_2$].Note that the semi-strong product of graphs is neither associative nor commutative; so G*H and H*G are not isomorphic in general. It is clear that

$$\deg_{G*H}(u,v) = \deg_G(u) \,.\, \deg_H(v) + \deg_G(u)$$

where $\deg_G(u)$ is the degree of vertex u in G. Thus the number of edges in G*H is $2q_1q_2+p_2q_1$, where $p_i$ and $q_i, i=1,2$ are the number of vertices and edges respectively in G and H. Moreover G*H contains as subgraphs $V_2$ copies of G; for each vertex v$\in V_2$ there is a v-copy $G_v$ of G with vertex set{(x,v):x$\in V_1$}. It is clear that $\bigcup_{v\in V_2} G_v$ is a subgraph [7] of G*H.

The basis number of the complete graphs, complete bipartite graphs and n-cube are determined in[10] and [6]. The basis number of the cartesian product of some graphs is determined in [2].

The purpose of this paper is to determine the basis number of the semi-strong product of $K_2$ with some special graphs. It is proved that

$$b(K_2*P_n) = b(P_n*K_2) = 2; \quad n\geq 3,$$

$$b(K_2*C_n) = \begin{cases} 2, & \text{for even } n\geq 4, \\ 3, & \text{for odd } n\geq 3, \end{cases}$$

$$b(C_n*K_2) = 3, \quad n\geq 4,$$

$$b(K_2*S_n) = b(S_n*K_2) = 2; \quad n\geq 3,$$

$$b(W_n*K_2) = 3, \quad n\geq 4,$$

and

$$b(K_2*K_n) = \begin{cases} 3, & \text{for } n = 3,4,5 \text{ and } 6 \\ 4, & \text{for } n\geq 7 \end{cases}$$

## 2- The basis number of $K_2 * P_n$ and $P_n * K_2$.

Let the vertex sets of path $P_n$ and the cycle $C_n$ be the addition group $Z_n$ of positive integers residue modulo n. Let the path $P_n$ be *0,1,2…,n-1* and the cycle $C_n$ be *0,1,2,…,(n-1)0*.It is clear that if n=2,then $K_2 * P_2$ is the 4-cycle (x,u)(y,u)(x,v)(y,v)(x,u), therefore b($K_2 * P_2$)=1.

It is not difficult to see that $K_2 * P_n$ , $n \geq 3$ can be embedded in a plane[7].Therefore, b($K_2 * P_n$) = 2, for $n \geq 3$.

**Theorem 1**. For every positive integers $n \geq 3$, b($K_2 * P_n$)= b($P_n * K_2$ )=2.

**Proof:** One can observe that the graph $K_2 * P_n$ , $n \geq 3$ can be embedded in the plane, therefore by MacLanes theorem[9],b($K_2 * P_n$)=2.Similarly,the graph $P_n * K_2$, $n \geq 3$ is planar graph (observe that $P_n * K_2$ is not isomorphic to $K_2 * P_n$ ),therefore by MacLanes Theorem[9], b($P_n * K_2$ ) = 2.

## 3-The basis number of $K_2 * C_n$ and $C_n * K_2$ .

It can be shown that for every even integer $n \geq 4$,the graph $K_2 * C_n$ is cubic having 2n vertices and can be embedded in a plane, therefore

b($K_2 * C_n$) = 2,for every even $n \geq 4$.

**Theorem 2.** For every even integer $n \geq 4$,we have b($K_2 * C_n$) = 2.

**Proof.** Since the graph $K_2 * C_n$, $n \geq 4$ is planar, therefore by MacLanes theorem[9], we have b($K_2 * C_n$) = 2.

**Theorem 3.** For every odd integer $n \geq 3$,we have b($K_2 * C_n$)=3.

**Proof.** One can easily show that the graph $K_2 * C_n$, for odd $n \geq 3$ contains subgraph homeomorphic to $K_{3,3}$. Thus the graph $K_2 * C_n$, is non planar and so by MacLanes theorem [9], b($K_2 * C_n$) $\geq$ 3. To complete the proof we show a 3-fold basis for $\xi$( $K_2 * C_n$). Consider the following set of cycles:

$$B(K_2 * C_n) = S \cup T$$

Where,

$$S = \{(0,j)(1,j+1)(0,j+1)(1,j)(0,j)\}: \ j=0,1,2,\ldots,n-1 \bmod(n)\},$$

and

$$T = \{(0,0)(1,1)(0,2)(1,3)\ldots(0,n-1)(1,n-1)(0,0)\}.$$

It is clear that the cycles $S \cup T - \{ C \}$,where

$C$ ={(0,n-1)(1,0)(0,0)(1,n-1)(0,n-1)} forms boundaries of planar subgraph F of $K_2 * C_n$(see Figure).Therefore

S$\cup$T-{ $C$} is independent set of cycles. On the other hand the cycle $C$ contains the edge (0,n-1)(1,0) which is not present in any cycle of S$\cup$T-{ $C$}.Therefore, S$\cup$T is independent set of cycles. Since

$$B(K_2 * C_n) = n+1 = \gamma\,(K_2 * C_n),$$

then B($K_2 * C_n$) is a basis for $\xi(K_2 * C_n)$.

To find the fold of the basis B($K_2 * C_n$). It is clear that

$f_S(e) \leq 2$, $f_T(e) \leq 1$, for each edge e$\in$ E($K_2 * C_n$)- { $C$},

$f_S(e) = 1$, $f_T(e) = 1$, for each edge e$\in$ { $C$},

where

$C$ = {(0,n-1)(1,0)(0,0)(1,n-1)(0,n-1)}.

Thus, the fold in B($K_2 * C_n$) of every edge of $K_2 * C_n$ is not more than 3.

Hence B($K_2 * C_n$) is a 3-fold basis. This completes the proof of the theorem.

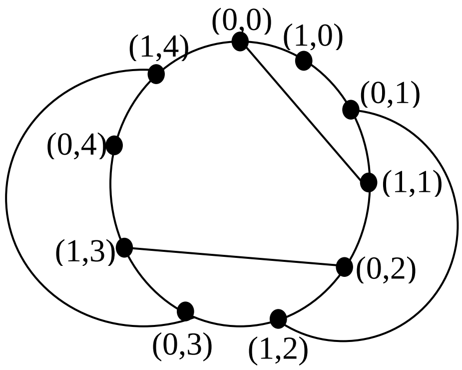


Figure : The planar subgraph F of $K_2 * C_5$

Now, we consider the semi-strong product $C_n * K_2$ .It is easy to show that $C_3 * K_2$ , is planar graph ,therefore b($C_3 * K_2$) = 2.

**Theorem 4**. For every integers n$\geq$4 ,we have b($C_n * K_2$) = 3.

**Proof**. One can easily show that the graph $C_n * K_2$ , n$\geq$4 contains subgraph homeomorphic to complete bipartite graph $K_{3,3}$ [7].Thus

$C_n * K_2$ is nonplanar and so by MacLanes theorem[9], b($C_n * K_2$) $\geq$3 for n$\geq$4 .

To complete the proof of the theorem we show a 3-fold basis for $\xi(C_n * K_2)$.

Consider the set of cycles in $C_n * K_2$ , B($C_n * K_2$)=A$\cup$D$\cup$V

Where

A = { $a_i$ =(i,0)(i+1,1)(i,1)(i+1,0)(i,0): i=0,1,2,…,n-1(mod n) },

D = { $d_i$ = (i,0)(i+1,0)(i+2,0)(i+1,1)(i,0): i=0,1,2,…,n-2(mod n) },

And

$$V=\begin{cases} (0,0)(1,1)(2,0)(3,1)(4,0)\ldots(n-1,1)(0,0), \\ (0,1)(1,0)(2,1)(3,0)(4,1)\ldots(n-1,0)(0,1): \text{ if n odd} \\ \\ (0,0)(1,1)(2,0)(3,1)\ldots(n-1,0)(0,0), \\ (0,1)(1,0)(2,1)(3,0)\ldots(n-1,1)(0,1): \text{ if n even} \end{cases}$$

Since

$|B(C_n * K_2)|$ = n+(n-1)+2

= 2n+1

= $\gamma$ ( $C_n * K_2$)

It is clear that the cycles A,D and V are independent since they are boundaries of planar graph. Also, A⋃D is independent set of cycles since if $a_i$ is any cycle generated from cycles of A, then $a_i$ contains an edge (i+1,1)(i,1) For each i=0,1,2,…,n-1(mod n) which is not present in any cycle of D. Moreover if $c_i$ is any cycle generated from cycles of A⋃D, then $c_i$ contains an edge of the form (i,0)(i+1,0) for each i=0,1,2,…,n-2 which is not present in any cycle of V, therefore A⋃D⋃V is independent set of cycles and so it is a basis for $\xi$( $C_n * K_2$).

To find the fold of B($C_n * K_2$),partition the edge set E($C_n * K_2$) into L⋃M⋃N, where

L = { i,0)(i+1,1),(i,1)(i+1,0): i=0,1,2,…,n-1(mod n) },

M = {(0,j)(n-1,j):j=0,1},

and

N = { i,j)(i+1,j): i=0,1,2,…,n-2(mod n) and j=0,1}.

Then one may verify that

$f_A$(e) = 1, $f_D$(e) $\leq 1$, $f_V$(e) $\leq 1$, for each edge e $\in$ L;

$f_A$(e) = 1, $f_D$(e) $\leq 1$, $f_V$(e) $\leq 1$, for each edge e $\in$ M;

$f_A$(e) = 1, $f_D$(e) $\leq 2$ $f_V$(e)= 0, for each edge e $\in$ N.

Thus the fold in B($C_n * K_2$) of every edge of $C_n * K_2$ is not more than 3. Hence B($C_n * K_2$) is a 3-fold basis for $\xi$( $C_n * K_2$). The proof is complete.

## 4. The basis number of $K_2 * S_n$ and $S_n * K_2$ .

In this section we consider the semi-strong product of $K_2$ with a star $S_n$ which is isomorphic to complete bipartite graph $K_{1,n-1}$ .Denote the vertex set of the star $S_n$ by 0123…(n-1), where $\deg_{S_n}(0) = n-1$,and all other vertices are of degree 1.Since $S_2 = P_2$ ,therefore the graph $K_2 * S_2$ is the cycle {(0,0)(1,1)(0,1)(1,0)(0,0)},therefore b($K_2 * S_2$)=1.

Similarly , b($S_2 * K_2$)=1. On the other hand, for n $\geq$ 3,the graph $K_2 * S_n$ is planar graph ,therefore b($K_2 * S_n$)=2.Similarly, for n $\geq$ 3,the graph $S_n * K_2$ is planar graph ,therefore b($S_n * K_2$)=2.

## 5. The basis number of $W_n * K_2$ and $K_2 * K_n$

In this section we consider the semi-strong product of a wheel with $K_2$, where $W_n$ is the join of the cycle 123…(n-1)1 with the vertex 0. That is, $W_n = C_{n-1} + K_1$ .

**Theorem 5**. For every integers $n \geq 4$ ,we have b($W_n * K_2$) = 3.
**Proof**. One can easily show that the graph $W_n * K_2$ , $n \geq 4$ contains subgraph homeomorphic to complete bipartite graph $K_{3,3}$ .Thus $W_n * K_2$ is nonplanar and so by MacLanes theorem[9], b($W_n * K_2$) $\geq 3$. To complete the proof of the theorem we show a 3-fold basis for $\xi(W_n * K_2)$.Consider the set of cycles in $W_n * K_2$ :

$$B(W_n * K_2) = \bigcup_{j=0}^{1} B_r(W_n^j) \cup A \cup D \cup E \cup C ,$$

Where $B_r(W_n^j)$ is a required basis for a j-copy, $W_n^j$ .That is,

$B_r(W_n^j)$ = {(0,j)(i,j)(i+1,j)(0,j): i=1,2,…,n-1 mod(n-1) and j=0,1},
A = {(i,0)(i+1,1)(i,1)(i+1,0)(i,0): i=1,2,…,n-1 mod(n-1) },
D = {(0,0)(i,1)(0,1)(i,0)(0,0): i=1,2,…,n-1 },
E = {(i,1)(i+1,1)(i+2,0)(i+1,0)(i,1), (i,0)(i+1,0)(i+2,1)(i+1,1)(i,0): i=0,1,2,…,n-3 },

and

C = {(0,0)(1,1)(n-1,1)(0,0)}.

It is clear that

$$|B(W_n * L_m)| = 2(n-1)+(n-1)+(n-1)+2(n-2)+1$$
$$= 6n-7 = \gamma(W_n * K_2).$$

It is clear that $\bigcup_{j=0}^{1} B_r(W_n^j)$ is a 2-fold required basis of $W_n^j$. Also, A,D,E and C are independent set of cycles because they are boundaries of planar subgraph of $W_n * K_2$. Moreover, $A \cup D$ is independent since it is edge-disjoint cycles. On the other hand, if $c_i$ is any cycle generated from cycles in $A \cup D$, then $c_i$ belong to A or D since $A \cup D$ is edge-disjoint cycles, hence if $c_i \in A$, then cycle $c_i$ contains an edge of the form (i,0)(i+1,1), for each i=1,2,…,n-1 mod(n-1) ,which is not present in any cycle of E; if $c_i \in D$, then there is no edge in common with the cycles of E. Therefore, $A \cup D \cup E$ is independent set of cycles. Furthermore if $a_i$ is any cycle generated from cycles in $A \cup D \cup E$, then $a_i$ contains an edge of the form (i,0)(i+1,1), (i,1)(i+1,0), (0,0)(i+1,1) or (0,1)(i+1,0) for each i = 0,1,2,…,n-2 which is not present in any cycle of $\bigcup_{j=0}^{1} B_r(W_n^j)$ ,therefore $\bigcup_{j=0}^{1} B_r(W_n^j)$ $\cup$A$\cup$D$\cup$E is independent set of cycles. To prove that C is independent of $\bigcup_{j=0}^{1} B_r(W_n^j)$ $\cup$A$\cup$D$\cup$E . Suppose that C is a sum modulo 2 of cycles in $\bigcup_{j=0}^{1} B_r(W_n^j)$ $\cup$A$\cup$D$\cup$E. Then C= $\sum_{j=1}^{m} d_j \pmod 2$ ,where $d_j$ is a linear combination of cycles in $\bigcup_{j=0}^{1} B_r(W_n^j)$ $\cup$A$\cup$D$\cup$E . Thus $d_1 = C \oplus \sum_{i=2}^{m} d_i$ (mod 2).Therefore

$$d_1 = C \oplus d_2 \oplus d_3 \oplus \ldots \oplus d_m \subseteq E(A \bigcup D),$$

where $\oplus$ is the ring sum. But

E(A$\cup$D)= {(i,0)(i+1,1)} $\cup$ {(i,0)(i+1,0)} $\cup$ {(0,0)(i,1)}

which is an edge set of a forest. This contradicts the fact that $d_1$ is a cycle or edge disjoint union cycles. Thus ($\bigcup_{j=0}^{1} B_r(W_n^j)$ $\cup$A$\cup$D$\cup$E) $\cup$C, is a basis for $\xi$( ***Wn∗K2***).

To find the fold of B(***Wn∗K2***), partition the edge set of ***Wn∗K2*** into

$$Q_1 = E(\bigcup_{j=0}^{1} C_{n-1}^j), \quad Q_2 = E(\bigcup_{j=0}^{1} S_n^j),$$

$Q_3$= {(0,0)(i,1), (0,1)(i,1): i=1,2,…,n-1 },

and

$Q_4$ = E(***Wn∗K2***)-{ $Q_1 \bigcup Q_2 \bigcup Q_3$ }.

Therefore, if G= $\bigcup_{j=0}^{1} B_r(W_n^j)$ , then

$f_G$(e) = 1, $f_A$(e) = 1, $f_D$(e) = 0, $f_{EUC}$ (e) $\le$ 1, for each edge e $\in Q_1$ ,
$f_G$(e) = 1, $f_A$(e) = 0, $f_D$(e) = 1, $f_{EUC}$ (e) = 0, for each edge e $\in Q2$ ,
$f_G$(e) = 0, $f_A$(e) = 0, $f_D$(e) = 1, $f_{EUC}$ (e) $\le$ 1, for each edge e $\in Q3$ ,
$f_G$(e) = 0, $f_A$(e) $\le$ 1, $f_D$(e) = 0, $f_{EUC}$ (e) $\le$ 2, for each edge e $\in Q4$ .

Thus B(***Wn∗K2***) is a 3-fold basis for $\xi$( ***Wn∗K2***). The proof is complete.

Now, consider the basis number of ***K2∗Kn*** .

It is clear that the graph $K_2 * K_n$ is a complete bipartite graph $K_{n,n}$ . Schmeichel [10] proved that b($K_{m,n}$ ) = 4 for m,n $\geq$ 5 except for the following: $K_{5,r}$ and $K_{6,s}$ where r=5,6,7,8 and s=6,7,8,10 . Also, Alsardary and Ali [4] proved that b($K_{5,r}$ ) = b($K_{6,s}$ ) = 3 for r=5,6,7,8 and s=6,7,8,10 . Therefore the following proposition follows from [4] and [10].

**Proposition**. $b(K_2 * K_n) = \begin{cases} 3, & \text{for } n=3,4,5 \text{ and } 6 \\ 4, & \text{for } n \geq 7. \end{cases}$

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