On GP- Ideals

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ABSTRACT

In this work we give some new properties of GP- ideals as well as the relation between GP- ideals, π- regular and simple ring. Also we consider rings with every principal ideal are GP- ideals and establish relation between such rings with strongly $\pi -$ regular and local rings.

Keywords: pure ideals, GP-ideals, strongly $\pi -$ regular ,local rings.

المثاليات من النمط–GP

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الملخص

في هذا البحث أعطينا بعض الخواص الجديدة للمثاليات التي من النمط –GP. ثم وجدنا العلاقة بين المثاليات من النمط –GP والحلقات المنتظمة من النمط – π والحلقات البسيطة. كذلك وجدنا العلاقة بين الحلقات التي فيها كل مثالي رئيسي من النمط –GP، والحلقات المنتظمة بقوة من النمط – π، والحلقات المحلية .

الكلمات المفتاحية: المثاليات النقية ،المثاليات من النمط– GP ، الحلقات المنتظمة بقوة من النمط – π، والحلقات المحلية .

1- Introduction

Throughout this paper, R is an associative rings with identity. An ideal I of a ring R is said to be right (left) pure if for each $a \in I$ there exists $b \in I$ such that
$a = ab$ $(a = ba)$. This concept was introduced by Fieldhouse [6] and study by AL-Ezeh [1],[2],[3]. As a generalization of this concept Shuker and Mahmood [11] defined right (left) GP- (generalized pure) ideals that is an ideal I of a ring R such that for every $a \in I$ there exists $b \in I$ and a positive integer n such that $a^n = a^n b$ $(a^n = ba^n)$.

We recall some concepts and notations which will be used in this paper. The
Jacobson radical of R, and the set of all nilpotent elements are denoteds by $J(R)$ and $N(R)$, respectively. According to Cohn [5], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A ring R is called periodic

ring, if for every $x \in R$ there exist two positive integers n, m and $n \neq m$ such that $x^n = x^m$ [4]. R is called reduced if R has no non-zero nilpotent elements. A ring R is π- regular if for every $a \in R$ there exists $b \in R$ and a positive integer n such that $a^n = a^n b a^n$ [7]. A ring R is called strongly π- regular [7], if for every $a \in R$, there exists a positive integer n, depending on a, and $x \in R$ such that $a^n = a^{n+1}x$. A ring R is called weakly right (left) duo(W.R.D)(W.L.D) if for each $a \in R$ there exists a positive integer n, depending on a, such that $a^n R$ (Ra^n) is two sided ideal.

2- Generalized pure Ideals (General properties)

In this section, some of basic properties and characterization of generalized pure ideals are given. Also we consider, connections between generalized pure ideals, π- regular rings, simple rings and periodic ring.

Lemma 2.1.[9]

For a ring R the following statements are equivalent:
1-R is reversible ring.
2-For each $a \in R$, $l(a) = r(a)$.

Lemma 2.2.

Let R be a reversible ring. Then for every $a \in R$ and a positive integer n, $r(a^n) = l(a^n)$.

Proof:

Let a be a non–zero element of R and let $b \in r(a^n)$. Then $a^n b = 0 = aa^{n-1}b$. This means $a^{n-1}b \in r(a) = l(a)$ (by Lemma 2.1), that is $a^{n-1}ba = 0 = aa^{n-2}ba$ this mean $a^{n-2}ba \in r(a) = l(a)$ (by Lemma 2.1), that is $a^{n-2}ba^2 = 0, \ldots\ldots,$ and so on. So $aba^{n-1} = 0$ this means $ba^{n-1} \in r(a) = l(a)$, that is $ba^n = 0$ and $b \in l(a^n)$. Therefore $r(a^n) \subseteq l(a^n)$. By similar method we prove that $l(a^n) = r(a^n)$. So $r(a^n) = l(a^n)$. #

The following proposition gives the relation between right and left GP- ideals.

Proposition 2.3.

Let R be a reversible ring and let I be any ideal of R. Then I is right GP- ideal if and only if I is left GP- ideal.

Proof:

Suppose that I is right GP- ideal and let $a \in I$. Then there exists $b \in I$ and a positive integer n such that $a^n = a^n b$. Now, $a^n(1-b) = 0$ implies $(1-b) \in r(a^n) = l(a^n)$ (by Lemma 2.2), so $a^n = ba^n$. Therefore I is a left GP-ideal. The converse holds by similar method. #

The following theorem gives the condition on left GP- ideals to be $\pi -$ regular.

Proposition 2.4.

Let R be W.R.D ring. Then the following are equivalent:
1-R is a $\pi -$ regular ring.
2- Every ideal of R is a left GP- ideal.

Proof:

$(1) \Rightarrow (2)$: It is clear .
(2) $\Rightarrow$(1): Assume (2). Then every ideal of R is a left GP- ideal. Let $r \in R$ since R is W.R.D. So $r^n R$ is an ideal of R, hence $r^n R$ is a left GP- ideal. Since $r^n \in r^n R$ so there exists $x \in r^n R$ and a positive integer n such that $r^n = xr^n$, but $x \in r^n R$, so $x = r^n z$ for some $z \in R$, hence $r^n = r^n z r^n$. Therefore R is a $\pi -$ regular ring. #
Recall that, an element $0 \neq a \in R$ is left (right) regular if and only if $l(a) = 0$ $(r(a) = 0)$.

Following [11], a ring R is called bounded index of nilpotency, if there exists a positive integer n such that $a^n = 0$ for all nilpotent elements a of R.

Lemma 2.5.[11]

Let R be a prime ring of bounded index of nilpotency. Then every non-zero two sided ideal of R contains a regular element .

Proposition 2.6.

Let R be a prime ring of bounded index of nilpotency. If every ideal of R is a right GP- ideal, then R is a simple ring.

Proof:

Let I be a non-zero two sided ideal of R. Since R is a prime ring of bounded index of nilpotency, then by Lemma 2.5, I contains a regular element $a \in I$. Since every ideal of R is GP- ideal, then there exists $b \in I$ and a positive integer n such that $a^n = a^n b$. Thus $a^n(1-b) = 0$. Since a is regular so a^n is also regular element and $r(a^n)=0$. Yielding $b=1$, whence $I=R$. Therefore R is simple ring. #

Now, a necessary and sufficient condition for GP- ideal to be periodic ring is considered in the following result:

Proposition 2.7.

Let R be a periodic ring and $r(a^n)\subseteq r(a)$ for every $a\in R$ and a positive integer n. Then every ideal of R is GP- ideal.

Proof:

Since R is periodic ring, then every ideal is a periodic. Let $x\in I$, then there exists $n,m\in z^+$, $m>n$, such that $x^n=x^m$. So $x^n(1-x^{m-n})=0$ hence $(1-x^{m-n})\in r(x^n)\subseteq r(x)$ implies that $x=xx^{m-n}$. Thus $x=xy$, $y=x^{m-n}$. Therefore I is GP- ideal . #

Corollary 2.8.

Let R be a periodic and reduced ring. Then every ideal of R is left (right) GP-ideal.

3- Rings with every principal ideals are generalized pure.

In this section we study rings with every principal ideals are right GP- ideal, and we give some of their basic properties, as well as the connection between GP- ideals and other rings.

Lemma 3.1. [8]

A ring R is local if and only if for any two elements r and s, $r+s=1$ implies that either r or s is a unit.

Proposition 3.2.

If R is a local ring and every principal right ideal is a GP- ideal, then every element in R is either a unit or a nilpotent.

Proof:

Let $a\in R$. Since every principal right ideal is a GP- ideal, then there exists $x\in aR$ and a positive integer n such that $a^n=a^nx=a^nay$ for some $y\in R$, So $a^n(1-ay)=0$

If $a^n=0$, then a is nilpotent.

If $(1-ay)\neq 0$ and $a^n\neq 0$, then $1-ay$ is zero divisor that is $1-ay$ is non unit. Since $1-ay+ay=1$, by Lemma 3.1, ay is a unit, this implies that a is a unit .

If $1-ay=0$, then a is a unit. Thus a is either a unit or a nilpotent. #

Proposition 3.3.

If every principal ideal is a right GP- ideal, then each of its elements is a unit or a zero divisor.

Proof :

Let a be a non–zero divisor in R. Since every principal ideal is a right GP- ideal, then there exists an element $x \in aR$ and a positive integer n such that $a^n = a^n x = a^n ay$ for some $y \in R$. So $a^n(1-ay)=0$. Since a is non-zero divisor, a^n is non–zero divisor. Therefore $1-ay=0$, which implies $ay=1$, thus a is a unit. #

Theorem 3.4.

Let R be a reversible ring. If every principal ideal is a right GP- ideal, then for any $a \in R$, $a^n = ea^n$ and $l(a^n) = l(e)$, where e is an idempotent element of R .

Proof :

Let a be a non–zero element in R. Then aR is a right GP- ideal and there exists $b \in aR$ and a positive integer n such that $a^n = a^n b = a^n az = a^{n+1}z = aa^n z = aa^{n+1}zz = a^2a^n z^2 = a^2a^{n+1}z^3 = \ldots\ldots\ldots\ldots$ $= a^{2n}z^n = a^{2n}x$
for some $x \in R$. Which implies $(1-a^n x) \in r(a^n) = l(a^n)$ (Lemma 2.3). Thus $a^n = a^n xa^n$, and let $e = a^n x$ then $a^n = ea^n$. Let $r \in l(a^n)$. Then $ra^n = 0$ this implies $ra^n x = 0 = re$, thus $r \in l(e)$. So $l(a^n) \subseteq l(e)$. Now, let $y \in l(e)$ then $ye = 0$ this implies $ya^n x = 0$ and $ya^n xa^n = 0 = ya^n$. Thus $y \in l(a^n)$. So $l(e) \subseteq l(a^n)$. Therefore $l(a^n) = l(e)$. #

Next the Jacobson radical and the set of all nilpotent elements $N(R)$ of a GP- ideal is considered:

Proposition 3.5.

Let R be a ring such that every principal ideal is a right GP- ideal. Then $J(R) = N(R)$.

Poof :

Let $0 \neq a \in J(R)$. Then aR is a right GP- ideal and hence there exists $b \in aR$ and a positive integer n such that $a^n = a^n b = a^n ar$ for some $r \in R$.

Therefore $(1-ar)$ is invertible $(a \in J(R))$. Thus there exists an invertible element u such that $(1-ar)u = 1$. Multiply from the left by a^n we obtain $(a^n - a^n ar)u = a^n$ whence it follows that $a^n = 0$. So $a \in N(R)$ and hence $J(R) \subseteq N(R)$. Since $N(R) \subseteq J(R)$. Thus $J(R) = N(R)$. #

Lemma 3.6.

Let R be a duo ring. Then R is $\pi -$ regular if and only if $a^n R$ is generated by an idempotent for every $a \in R$ and a positive integer n.

The following theorem extends Lemma 3.6 and Theorem 2.4:

Corollary 3.7.

Let R be duo ring. Then every principal ideal is a right GP- ideal if and only if $a^n R$ is generated by an idempotent for every $a \in R$ and a positive integer n.

Proof:

Assume that every principal ideal is GP- ideal, then by Theorem 2.4 and by Lemma 3.6, $a^n R$ is generated by an idempotent .

Conversely, assume that $a^n R$ generated by an idempotent, then by Lemma 3.6, R is $\pi -$ regular ring and by Theorem 2.4, every principal ideal is GP- ideal. #

The following result shows that every reversible ring is strongly $\pi -$ regular when every principal right ideal of R is a left GP- ideal .

Theorem 3.8.

Let R be a reversible ring. Then the following statements are equivalent:

1- R is a strongly $\pi -$ regular ring.

2- Every principal right ideal of R is a left GP- ideal.

Proof:

(1) $\Rightarrow$ (2): It follows from Theorem 2.4.

(2) $\Rightarrow$ (1): Assume every principal right ideal of R is a left GP- ideal. Then for every $a \in aR$ there exists $b \in aR$ and a positive integer n such that $a^n = ba^n = axa^n$ for some $x \in R$. So $(1-ax)a^n = 0$ implies that $1 - ax \in l(a^n) = r(a^n)$ (since R is reversible ring and by Lemma 2.2), $a^n = a^{n+1}x$. Therefore R is a strongly $\pi -$ regular ring. #

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