On Idempotent Elements

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ABSTRACT

In this paper we study idempotent elements, we give some new properties of idempotent elements and provide some exam we also study central idempotent elements and orthogonal idempotent elements and give some new properties of such idempotent.

Finally we study special ring which satisfies the property \( x^n = x^{n+1} \) for all \( x \) in \( R \) and \( n \) is a positive integer, we represent such ring in terms of idempotent and nilpotent elements.

Keywords: Rings, idempotent elements, nilpotent elements.

1. Introduction:

Throughout this paper \( R \) denotes an associative rings with identity . Recall that:

1. A ring \( R \) is said to be reduced if \( R \) contains no non zero nilpotent elements.(2) For any element \( a \) of a ring \( R \) we define the right annihilator of \( a \) in \( R \) by, \( r(a) = \{ x \in R : ax = 0 \} \), and likewise the left annihilator of \( a \) in \( R \).

3. A ring \( R \) is regular provided that for every \( x \) in \( R \), there exists \( y \) in \( R \) such that \( x = xyx \) .see[2] (4) An elements \( e_1, e_2 \) of a ring \( R \) is said to be centeral idempotent elements if \( e_1e_2 = e_2e_1 \), and orthogonal idempotent elements if \( e_1e_2 = e_2e_1 = 0 \).
2. Properties of Centeral and Orthogonal Idempotent Elements:

In this section we study centeral and orthogonal idempotent elements and give some basic properties. Also we study special ring which satisfies the relation $x^n = x^{n-1}$, $x \in R$, $n$ is a positive integer.

Proposition 2-1: If $e_1, e_2$ are centeral idempotent elements of $R$, with $r(e_1 + e_2) = 0$, then $(1-e_1), (1-e_2)$ are orthogonal idempotent elements.

Proof: Consider $(e_1 + e_2)(1-e_1 - e_2 + e_1 e_2) = e_1 - e_1 e_2 + e_1 e_2 + e_2 - e_2 e_1 - e_2 + e_2 e_1 e_2 = 0,$ implies $(1-e_1 - e_2 + e_1 e_2) \in r(e_1 + e_2) = 0$, and $e_1 (1-e_2) = (1-e_2)$ Therefore $(1-e_1)(1-e_2) = 0.$

If $e_1, e_2$ are idempotent elements, then $e_1 e_2$ need not to be idempotent as the following example shows.

Example: let $R(Z_2)$ be the ring of all $2 \times 2$ matrices over the ring $Z_2$ (the ring of integers modulo 2) which are strictly upper triangular. Then the only idempotent matrices of $R(Z_2)$ are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not idempotent.

The following result gives the condition for $e_1 e_2$ to be idempotent

Lemma 2-2: If $R$ is a ring with every idempotent element is central, then $e_1 e_2$ is idempotent for every $e_1, e_2$ are idempotents.

Proof: Trivial.

Theorem 2-3: If $e_1, e_2$ are centeral idempotent elements of $R$, and $r(e_1 - e_2) = 0$, then $R = e_1 R \oplus e_2 R$.

Proof: Let $x \in e_1 R \cap e_2 R$. Then $x = e_1 r_1$ and $x = e_2 r_2$, for some $r_1, r_2$ in $R$.

Now, $e_1 r_1 = e_1 e_2 r_2$, but $x = e_1 r_1$, then $x = e_1 e_2 r_2$.

Multiplication two sided from left by $(e_1 - e_2)$ we get $(e_1 - e_2) x = 0$, so $x \in r(e_1 - e_2)$, then $e_1 R \cap e_2 R \subseteq r(e_1 - e_2)$, but $r(e_1 - e_2) = 0$. So $e_1 R \cap e_2 R = 0$. \hfill \ldots \ (1)$

Now, consider $(e_1 - e_2)(e_1 + e_2) = (e_1 - e_2)$ implies $(e_1 - e_2)(e_1 + e_2 - 1) = 0$, implies $(e_1 + e_2 - 1) \in r(e_1 - e_2) = 0$. Therefore $e_1 R + e_2 R = R$. \hfill \ldots \ (2)$
from (1) and (2) we get \( R = e_1R \oplus e_2R \). ■

**Proposition 2-4:** If \( e_1, e_2 \) are central idempotent elements of \( R \), then.

1. \( e_1R \cap e_2R = e_1e_2R \).
2. \( e_1R \cap e_2R = r(1-e_1) \cap r(1-e_2) \).
3. \( r(e_1 + e_2) = r(e_1) \cap r(e_2) \) if \( e_1R \cap e_2R = (0) \).

**Proof 1:** Let \( x \in e_1R \cap e_2R \), then \( x = e_1r_1 \) and \( x = e_2r_2 \), for some \( r_1, r_2 \in R \).

Since \( e_1x = e_1r_1 = x \), then \( e_1x = e_1e_2r_2 \) yields \( x = e_1e_2r_2 \in e_1e_2R \), so
\[
e_1R \cap e_2R \subseteq e_1e_2R.
\] ...
(1)

Now, let \( y \in e_1e_2R \), then \( y = e_1e_2r \), for some \( r \in R \) and this means \( y \in e_1R \).

Since \( e_1, e_2 \) are central idempotent elements, then \( y \in e_2R \)

implies \( y \in e_1R \cap e_2R \). Therefore
\[
e_1e_2R \subseteq e_1R \cap e_2R.
\] ...
(2)

From (1) and (2) we get \( e_1R \cap e_2R = e_1e_2R \).

**Proof 2:** let \( x \in e_1R \cap e_2R \), then \( x = e_1r_1 \) and \( x = e_2r_2 \), for some \( r_1, r_2 \in R \).

Since \( e_1r_1 = e_2r_2 = e_1e_2r_2 \), so \( x = e_1e_2r_2 \).

Multiplication two sided from left by \((1-e_1)\) we get \((1-e_1)x = 0 \), and \( x \in r(1-e_1) \).

Similarly we get \( x \in r(1-e_2) \), hence \( x \in r(1-e_1) \cap r(1-e_2) \).

Now, let \( y \in r(1-e_1) \cap r(1-e_2) \), then \( y \in r(1-e_1) \) and \( y = e_1y \in e_1R \) also \( y \in r(1-e_2) \) and \( y = e_2y \in e_2R \), so \( y \in e_1R \cap e_2R \) and hence \( e_1R \cap e_2R = r(1-e_1) \cap r(1-e_2) \).

**Proof 3:** let \( x \in r(e_1) \cap r(e_2) \), then \( x \in r(e_1) \) and \( e_1x = 0 \), \( x \in r(e_2) \) and \( e_2x = 0 \)

So \( (e_1 + e_2)x = 0 \) and \( x \in r(e_1 + e_2) \).

Now, let \( y \in r(e_1 + e_2) \). Then \( (e_1 + e_2)y = 0 \) and \( e_1y = -e_2y \) \( = -e_2e_1y \in e_2R \) (since every idempotent is central), then \( e_1y = e_1R \cap e_2R = (0) \) implies \( e_1y = 0 \) and \( y \in r(e_1) \).

Similarly we get \( y \in r(e_2) \), then \( y \in r(e_1) \cap r(e_2) \).

Hence \( r(e_1 + e_2) = r(e_1) \cap r(e_2) \). ■

Following [3] a ring \( R \) is said to be right semi-regular ring if for every \( a \) in \( R \), there exists \( b \) in \( R \) such that \( a = ab \), and \( r(a) = r(b) \).

**Proposition 2-5:** A ring \( R \) is a right semi-regular if and only if \( r(a) \) is generated by an idempotent.

**Proof:** see [1], theorem (1-1-12).
Theorem 2-6: If R is a right semi-regular ring with every idempotent is central, then for each a in R, there exists e in R such that aR ∩ eR = (0).
Proof: Let R be a right semi-regular ring, then r(a) = eR, where e is idempotent element, and let \( x \in aR \cap eR \), then \( x = ar \) and \( x = er' \), for some \( r, r' \) in R.
Now, \( x = er' = eer' = ex \), since \( e \in eR = r(a) \), then \( ea = ae = 0 \).
Since \( x = ar \), then \( e x = ear = 0 \) but \( ex = er' = x \), so \( x = 0 \).
Hence \( aR \cap eR = (0) \). ■

Proposition 2-7: If e is central idempotent element of R, then for each element \( x \) in R there exists \( y \) in R such that \( xy = ye = e \) if and only if \( x + (1 - e) \) invertibility of \( ye + (1 - e) \).
Proof: Let \( u = x + (1 - e) \) and \( v = ye + (1 - e) \).
Now, \( uv = (x + (1 - e))(y e + (1 - e)) = x ye + xe - xe + ye + 1 - e - ye - e + e = 1 \)
So, \( vu = 1 \).
Conversely, let \( (x + (1 - e))(y e + (1 - e)) = 1 \), implies \( x ye - e = 0 \) and \( x ye = e \).
Similarly we get \( ye = e \). ■

Theorem 2-8: If \( aR = eR \), then \( a = eu \), where e is central idempotent element and u is unit element of R.
Proof: Let \( aR = eR \), where e is central idempotent element of R,
Now, \( a = ea = ae \).
Also \( e = ax \), for some \( x \) in R
Put \( v = 1 - e + ex \) and \( u = 1 - e + a \), we find \( uv = vu = 1 \)
Now, \( eu = e(1 + a) = a \), then \( a = eu = ue \). ■

If \( e_1, e_2 \) are idempotent element of R, then \( (e_1 + e_2) \) need not to be idempotent as the following example shows.

Example: Let \( R(Z_2) \) be the ring of all \( 2 \times 2 \) matrices over the ring \( Z_2 \) (the ring of integers modulo 2) which are strictly upper triangular. Then the only idempotent matrices of \( R(Z_2) \) are:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Now, \( \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \) but \( \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \) is not idempotent.

Lemma 2-9: If \( e_1, e_2 \) are orthogonal idempotent elements of R, then \( (e_1 + e_2) \) is idempotent element.
Proof: Trivial.
If $e_1,e_2$ are idempotent elements of $R$, then $r(e_1+e_2) \neq r(e_1) \cap r(e_2)$ in general as the following example shows.

Example: let $R(Z_2)$ be the ring of all $2 \times 2$ matrices over the ring $Z_2$ (the ring of integers modulo 2) which are strictly upper triangular, then the element of $R(Z_2)$ are:

$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $r(I+E) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

The only idempotent elements of $R(Z_2)$ are: $\{I,O,A,C,E,F\}$

$r(I)=\{O\}$
$r(E) = \{O,F\}$
$r(I \cap r(E)=O\}$ while $r(I+E)=r(F) = \{O,D,E,A\}$

Proposition 2-10: If $e_1,e_2$ is orthogonal idempotent elements of $R$, then $r(e_1+e_2) = r(e_1) \cap r(e_2)$

**proof:** let $x \in r(e_1+e_2).$ Then $(e_1+e_2)x = 0$ and $e_1x = -e_2x$ multiplication two sided from left by $e_1$ we get $e_1x = 0$ and $x \in r(e_1)$. Also multiplication two sided from left by $e_2$ we get $e_2x = 0$ and $x \in r(e_2)$ so, $x \in r(e_1) \cap r(e_2)$.

Now, let $y \in r(e_1) \cap r(e_2)$, then $y \in r(e_1)$ and $e_1y = 0$, also $y \in r(e_2)$ and $e_2y = 0$, implies $(e_1+e_2)y = 0$ and $y \in r(e_1+e_2)$, so $r(e_1+e_2) = r(e_1) \cap r(e_2)$.

If $R$ is not commutative ring, then $e_1R + e_2R \neq (e_1+e_2)R$ as the following example shows:

Example: let $R(Z_2)$ be the ring of all $2 \times 2$ matrices over the ring $Z_2$ (the ring of integers modulo 2) which are strictly upper triangular, then the element of $R(Z_2)$ are:

$A=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $D=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $E=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $F=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $G=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $H=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$I=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $J=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $K=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $L=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $N=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $O=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $P=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

the idempotent matrices are: $\{A,C,D,E,F,G,H,M\}$

ER $= \{A,B,C,E\}$

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(E+M)R= \{A,G,H,N\}

Its clearly that ER+MR ≠ (E+M)R.

**Theorem 2-11:** If \(e_1,e_2\) are orthogonal idempotent elements of \(R\), then
\[e_1 R + e_2 R = (e_1 + e_2) R\]

**proof:** let \(x \in (e_1 + e_2) R\), then \(x= (e_1 + e_2) r\), for some \(r\) in \(R\), and this implies \(x = e_1 r + e_2 r \in e_1 R + e_2 R\).

Now, let \(y \in e_1 R + e_2 R\), then \(y = e_1 r_1 + e_2 r_2\), for some \(r_1, r_2\) in \(R\).

Multiplying two sided from left by \((e_1 + e_2)\) we get \((e_1 + e_2) y = y\) and \(y \in (e_1 + e_2) R\) and hence \(e_1 R + e_2 R = (e_1 + e_2) R\).

**Corollary 2-12:** If \(e_1, e_2, \ldots, e_r\) are orthogonal idempotent elements of \(R\), then \((e_1 + e_2 + \ldots + e_{r+1}) R = e_1 R + e_2 R + \ldots + e_{r+1} R\)

**Proof:** by induction

1. when \(n = 2\), the equality holds.
2. assume that equality holds when \(n = r\).
3. when \(n = r+1\)

\[
\left(\sum_{i=1}^{r+1} e_i\right) R = \left(\sum_{i=1}^{r} e_i + e_{r+1}\right) R = \left(\sum_{i=1}^{r} e_i\right) R + e_{r+1} R
\]

First we claim that \(x^n\) is idempotent element.

Now, \((x^n)^2 = x^n . x^n\) \(= x \cdot x \cdot \ldots \cdot (x \cdot x)\)

\(n\)-times

\(= x\cdot x \cdot \ldots \cdot (x \cdot x)\)

\((n-1)\)-times

\(= x^n\)

Now, let \(y = x - x^n\), then \(y = x - x^{n+1}\) and this implies \(y = x - x^n\), so \(y = x (1-x^n)\).
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Now, \( y^n = (x(1-x^n))^n \) so, \( y^n = x^n(1-x^n)^n \) (since every idempotent is central) and by [proposition 2-1], \((1-x^n)\) is also idempotent and by [if e is idempotent element of R, then e can not to be nilpotent] \((1-x^n)\) can not be nilpotent. So, \( y^n = x^n(1-x^n) = 0 \) and \( y \in N \), therefore \( x = x^n + x - x^n \in E + N \).

2) let \( y = x-x^2 \)
Now, \( x^{n-1}y = x^x + x^{n+1} = 0 \)
\[
0 = x^{n-2}xy = x^{n-2}y = x^{n-2}(x-x^2)y = x^{n-2}y^2
\]
\[
= x^{n-3}xy^2 = x^{n-3}y^2 = x^{n-3}(x-x^2)y^2 = x^{n-3}y^3
\]
\[0 = y^n \text{ this implies } (x-x^2)^n = 0, \text{ then } x-x^2 \in N \]
So, \( x + N = x^2 + N \). ■
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