On \( \Pi – \) Pure Ideals

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Received on: 20/10/2013  
Accepted on: 12/2/2014

ABSTRACT

As a generalization of right pure ideals, we introduce the notion of right \( \Pi – \) pure ideals. A right ideal \( I \) of \( R \) is said to be \( \Pi – \) pure, if for every \( a \in I \) there exists \( b \in I \) and a positive integer \( n \) such that \( a^n \neq 0 \) and \( a^n b = a^n \). In this paper, we give some characterizations and properties of \( \Pi – \) pure ideals and it is proved that:

If every principal right ideal of a ring \( R \) is \( \Pi – \) pure then,

a). \( L (a^n) = L (a^{n+1}) \) for every \( a \in R \) and for some positive integer \( n \).

b). \( R \) is directly finite ring.

c). \( R \) is strongly \( \Pi – \) regular ring.

Keywords: Pure, strongly regular, \( \Pi – \) ring.

1. Introduction

Throughout this paper, a ring \( R \) denotes as associative ring with identity and all modules are unitary. We write \( J(R) \) for Jacobson radical of \( R \). \( L(x) \) (\( Y(x) \)) denotes the left (right) annihilator of \( x \) in \( R \).

Recall the following definitions and facts:

1- A ring \( R \) is called \( \Pi – \) regular \([3]\), if for any \( a \in R \), there exists \( b \in R \) and a positive integer \( n \) such that \( a^n = a^n ba^n \). A ring \( R \) is called strongly \( \Pi – \) regular if for any \( a \in R \), there exists \( b \in R \) and a positive integer \( n \) such that \( a^n = a^{2n} b \).
A ring $R$ is called a quasi ZI–ring \[8\], if for any non-zero elements $a, b \in R$, $ab = 0$ implies that there exists a positive integer $n$ such that $a^n \neq 0$ and $a^n R b^n = 0$.

A ring $R$ is said to be reduced \[9\], if it contains no non-zero nilpotent element.

$R$ is called right SXM if for each $0 \neq a \in R$, $r(a) = r(a^n)$ for a positive integer $n$ satisfying $a^n \neq 0$. For example, reduced rings are right SXM rings \[7\].

Pure ideals have been extensively studied for several years. Many authors studied some properties and connections between pure ideals and regular rings \[2\], \[4\],and \[5\].

2. **Pi – Pure Ideals**

In this section, we introduce the notion of a right Pi – pure ideals, with some of their basic properties. Also, we give a connection between Pi – pure ideals and pure ideals.

Following [1], an ideal $I$ of a ring $R$ is said to be right(left) pure ideal, if for any $a \in I$, there exists $b \in I$ such that $a = ab$. (a = ba).

Following [6], an ideal of a ring $R$ is said to be GP-ideal, if for every $a \in I$, there exists $b \in I$ and a positive integer $n$ such that $a^n = a^n b$.

**Definition (2.1):**

An ideal $I$ of a ring $R$ is said to be right Pi – pure ideal if for every $a \in I$, there exists a positive integer $n$ and $b \in I$, such that $a^n \neq 0$ and $a^n = a^n b$.

Clearly, every right pure ideal is a right Pi – pure ideal but the converse is not true.

**Example (1):**

Let $\mathbb{Z}_{12}$ be the ring of integers modulo 12 and $I = (3)$, $J = (4)$. Then, both $I$ and $J$ are Pi – pure ideals of $\mathbb{Z}_{12}$. Obviously, Pi – pure ideal implies GP-ideal.

It is clear that in the case of reduced rings, GP – ideals coincide. with Pi – pure.

**Example (2):**

Let $\mathbb{Z}_9$ be the ring of integers modulo 9 and the (3) is not Pi–pure, but GP – ideal.

We now consider a necessary and sufficient condition for Pi–pure to be pure ideal.

**Proposition (2.2):**

Let $R$ be right SXM ring. Then, every Pi – pure ideal is pure ideal.

**Proof:** Let $I$ be a right Pi – pure ideal, and let $a \in I$. Then, there exists $b \in I$ and a positive integer $n$ such that $a^n \neq 0$, and $a^n = a^n b$, this implies that $(1 – b) \in r(a^n) = r(a)$. ($R$ is right SXM). Therefore, $(1 – b) \in r(a)$ and $a = ab$. So, $I$ is pure ideal.

**Proposition (2.3):**

Let $R$ be a ring with every principal ideal is Pi – pure ideal. Then,
1- Every non-zero divisor element of $R$ is invertible.
2- $J(R)$ is a nil ideal.

**Proof:** It proved the same method as [5, Proposition. 3.2.6].

3. **The Connection Between Pi–Pure Ideals and Other Rings**

In this section, we study the connection between rings whose every principal ideal is Pi – pure and strongly Pi – regular rings and other rings.

**Proposition (3.1):**
Let $R$ be a ring such that every principal left ideal is right $\Pi$ – pure. Then, $L(a^n) = L(a^{n+1})$ for every $a \in R$ and for some a positive integer $n$.

**Proof :**

Let $a \in I$. Then, there exists $b \in I$, and a positive integer $n$, such that $a^n \neq 0$ and $a^n a^n b$ where, $b = ax$ for some $x \in R$.

Therefore $a^n = a^{n+1} x$. Let $y \in L(a^{n+1})$, $y a^{n+1} = 0$, then $y (a^{n+1} x) = 0$ So $ya^n = 0$ and $y \in L(a^n)$. Therefore, $L(a^{n+1}) \subseteq L(a^n)$. Clearly $L(a^n) \subseteq L(a^{n+1})$. So, $L(a^n) = L(a^{n+1})$.

Following [3], a ring $R$ is called directly finite if $ab = 1$ implies $ba = 1$ for all $a, b \in R$.

As a parallel result to [3, Proposition 2.1.13], the following result was obtained.

**Proposition (3.2):**

Let $R$ be a ring with every principal right ideal is $\Pi$ – pure. Then, $R$ is directly finite.

**Proof :**

Let $x, y \in R$ such that $xy = 1$. It is clear that $x^n y^n = 1$ and $x^{n+1} y^{n+1} = 1$ multiple by $y^{n+1}$. So $y^{n+1} x^{n+1} y^{n+1} = y^{n+1},$ and $(1 - y^{n+1} x^{n+1}) \in L(y^{n+1}) = L(y^n)$ (Proposition 3.1). Hence, $y^n = y^{n+1} x^{n+1} y^n = (y^{n+1} x) (x^n y^n) = y^{n+1} x$.

Now, $yx = (x^n y^n) yx = x^n (y^{n+1} x) = x^n y^n = 1$.

**Theorem (3.3):**

Let $R$ be a ring with every principal right ideal is right $\Pi$ – pure. Then, $R$ is strongly $\Pi$ – regular.

**Proof :**

For any $a \in R$, $aR$ is $\Pi$ – pure. Since $a \in aR$. There exists $b \in R$ and a positive integer $n$ such that $a^n \neq 0$,

and $a^n = a^{n+1} x$ for some $x \in R$

$= a^{n+1} x a^{n+2} x^2 = \ldots a^{2n} x^n = a^{2n} y$

Therefore, $R$ is strongly $\Pi$ – regular.

A right modulo $M$ is said to be $YJ$ – injective [9], if for any $0 \neq a \in R$, there exists an appositive integer $n$ such that $a^n \neq 0$ and any right $R$ – homomorphism. From $a^n R$ into $M$, extends to one from $R$ into $M$.

A ring $R$ is called a right $YJ$ – injective ring, if $R$ is $YJ$ – injective ring.

**Proposition (3.4):** [8]

Let $R$ be a quasi ZI ring. If every simple singular right $R$ – modulo is $YJ$ – injective. Then,

1- $R$ is reduced.

2- $I + r(a) = R$ for any non-zero ideal $I$ of $R$ and every $a \in I$.

**Theorem (3.5):**

Let $R$ be a quasi ZI – ring. If every simple singular right $R$ – modulo is $YJ$ – injective. Then, every ideal of $R$ is right $\Pi$ – pure.

**Proof :**

From Proposition (3.4), $I + r(a) = R$ for every non-zero ideal $I$ of $R$, and $a \in R$.

So, $b + d = 1$, $b \in I$, $d \in r(a)$, $ab + ad = a$. Therefore, $ab = a$, and $a^n b = a^n$ for some $a$ positive integer $n$ and $a^n \neq 0$. So, $I$ is $\Pi$ – pure.
REFERENCES


