Studying the Stability of a Non-linear Autoregressive Model (Polynomial with Hyperbolic Cosine Function)

Abdul Ghafoor Jasim Salim
drabdul_salim@uomosul.edu.iq
College of Computer Science and Mathematics, University of Mosul,

Anas Salim Youns Abdullah
College of Basic Education
University of Mosul,

Received on: 27/3/2013 **Accepted on: 24/6/2013**

ABSTRACT

In this paper we study the statistical properties of one of a non-linear autoregressive model with hyperbolic triangle function(polynomial with hyperbolic cosine function)by using the local linearization approximation method to find the stability of the model (singular point and its stability conditions and the stability of limit cycle).Where we started by the model of lower order (first and second and third order) and generalized the idea, and we tried to apply these theory results by using some of examples to explain one of important truth that says (if the model has unstable singular point, then it, maybe, has a stable limit cycle).

Keywords: *Non-linear time series model; Non-linear random vibration; Autoregressive model; Limit cycle; Singular point; Stability.*

دراسة ثبات نموذج الانحدار الذاتي غير الخطي (متعدد الحدود مع دالة جيب التمام الزائدي)

عبد الغفور جاسم العبيدي
كلية علوم الحاسبات والرياضيات
جامعة الموصل

أنس سالم يونس
كلية التربية الاساسية
جامعة الموصل

تاريخ استلام البحث: 2013/6/24 **تاريخ قبول البحث: 2013/3/27**

الملخص

تم في هذا البحث دراسة الصفات الإحصائية لأحد نماذج الانحدار الذاتي غير الخطي بدوال مثلثية زائدية (متعدد حدود بدالة الجيب تمام الزائدية) وباستخدام طريقة التقريب بالخطية المحلية (local linearization approximation method) لإيجاد استقرارية النموذج (النقطة المنفردة وشروط استقراريتها واستقرارية دورة النهاية) حيث بدأنا بالنموذج برتب دنيا(من الرتبة الأولى والثانية والثالثة)وعممنا الفكرة.وحاولنا تطبيق تلك النتائج النظرية باستخدام بعض الأمثلة لتوضيح إحدى الحقائق المهمة التي تقول(إذا كان النموذج يمتلك نقطة منفردة غير مستقرة فانه ربما يمتلك دورة نهاية مستقرة).

الكلمات المفتاحية : نموذج سلسلة زمنية غير خطية ؛ اهتزاز عشوائي غير خطي ؛ نموذج الانحدار التلقائي دورة الحد نقطة المفرد المزيد.

.

1. Introduction

In the field of discrete time non-linear time series modeling, there are many different types of a non-linear models which are considered by the researchers such as bilinear model (Priestley (1978), Rao (1977)) exponential autoregressive model (Ozaki and Oda (1977))[5] and threshold model (Tong (1990))[8].

In (1985) Ozaki proposed the method of local linearization approximation to find the stability of a non-linear exponential autoregressive models[7].

In (1986) Tsay R.S. studied the stability of non-linear time series[9]. In (1988) Priestley M.B. studied the non-stability and non-linear time series[9]. In (1990) Tong H. studied the dynamical system with stability of non-linear time series[8].

In this paper, we study the statistical properties of one of a non-linear autoregressive model with hyperbolic triangle function(polynomial with hyperbolic cosine function)by using the local linearization approximation method to find the stability of the model (singular point and its stability conditions and the stability of limit cycle) and we give some examples to explain this method.

2.Basic Concepts of Time Series

<u>Definition 2.1</u>: A difference equation of order n over the set of k-values 0, 1, 2, ... is an equation of the form $F(k, y_k, y_{k-1}, ...y_{k-n}) = 0$,

Where F is a given function, n is some positive integer, and k = 0, 1, 2, …... ., [4].

<u>Definition 2.2</u>: A time series is a set of observations measured sequentially through time. These measurements may be made continuously through time or be taken at a discrete set of time points. Then, a time series is a sequence of random variables defined on probability space multi variables refer by index (**t**) that back to index set T , and we refer to time series by $\{x(t); -\infty < t < \infty, t \in T\}$ if **t** takes continuous values, or $\{x_t; t = 0, \pm 1, \pm 2, ...\}$ if **t** takes a discrete values [3].

<u>Definition 2.3</u>: A time series $\{x_t\}$ represents a linear autoregressive model if it satisfies the following difference equation: $x_t + a_1 x_{t-1} + a_2 x_{t-2} + ... + a_p x_{t-p} = Z_t$

Where $\{Z_t\}$ is a white noise and $a_1, a_2, ..., a_p$ are real constants [2].

<u>Definition 2.4</u>: The exponential autoregressive model of order p, EXPAR(P) is defined by the following equation

$$x_t = \sum_{j=1}^{p} \left(\phi_j + \pi_j e^{-X_{t-1}^2} \right) x_{t-j} + Z_t$$

Where $\{Z_t\}$ is a white noise and $\phi_1, ..., \phi_p; \pi_1, ..., \pi_p$ are the parameters of the model [5].

<u>Definition 2.5</u>: The bilinear model of order (p,q,m,s) satisfies the equation

$$x_t = c + \sum_{i=1}^{p} \phi_i x_{t-i} - \sum_{j=1}^{q} \theta_j Z_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{s} \beta_{ij} x_{t-i} Z_{t-j} + Z_t \ ,$$

Where p , q , m and s are nonnegative, and $\{Z_t\}$ is a sequence of independent identically distributed random variables and $\phi_1, ..., \phi_p; \theta_1, ..., \theta_q; \beta_{ij}; \forall i = 1, ...m, \forall j = 1, ...s$ are the parameters of the model [6].

<u>Definition 2.6</u>: A singular point of $x_t = f\left(x_{t-1}, x_{t-2}, ..., x_{t-p}\right)$ is defined as a point ζ which every trajectory of $x_t = f\left(x_{t-1}, x_{t-2}, ..., x_{t-p}\right)$ beginning sufficiently close to it approaches either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable singular point and if it approaches it for $t \to -\infty$ we call it unstable singular point.

Obviously, a singular point ζ satisfies $\zeta = f(\zeta)$ [5].

<u>Definition 2.7</u>: A limit cycle of $x_t = f\left(x_{t-1}, x_{t-2}, ..., x_{t-p}\right)$ is defined as an isolated closed trajectory $x_{t+1}, x_{t+2}, ..., x_{t+q}$, where q is a positive integer. Isolated means that every trajectory beginning sufficiently near the limit cycle approaches either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable limit cycle and if it approaches it for $t \to -\infty$ we call it unstable limit cycle [5].

<u>Theorem 1:</u>

Let $\{x_t\}$ be expressed by the exponential autoregressive model

$$x_t = \left(\phi_1 + \pi_1 e^{-x_{t-1}^2}\right) x_{t-1} + Z_t$$

A limit cycle of period q , $x_{t+1}, x_{t+2}, ..., x_{t+q}$ of the model is orbitally stable if

$\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$. proof (see [5]).

The proposed model A non-linear autoregressive model (polynomial with hyperbolic cosine function) of order p is defined by $X_t = \sum_{i=1}^{p} [\phi_i \cosh(x_{t-1})]^i x_{t-i} + Z_t$,

Where $\{Z_t\}$ is a white noise process and $\phi_1, ..., \phi_p$ are the parameters(real constants) of the model(the proposed model).

3.The Stability of the Proposed Model

In this section, we shall study the stability of a non-linear autoregressive model with hyperbolic cosine function with low order such that p=1,2,3 and, then we generalized this idea to the general model of order p by using the local linear approximation method that consists of the following three steps:

Step(1):find the singular point of the model.

Step(2):study the stability condition of the singular point.

Step(3):find the stability condition of a limit cycle if it exists .

3.1 Singular Point

Consider the following model

$$x_t = \sum_{i=1}^{p} [\phi_i \cosh(x_{t-1})]^i x_{t-i} + Z_t \quad \text{...(1)}$$

Let p=1 , then we have

$$x_t = [\phi_1 \cosh x_{t-1}] x_{t-1} + Z_t \quad \text{...(2)}$$

Suppose that the white noise is not an effect (Z_t be minimum, i.e. $Z_t = 0$) to get a deterministic model which has a limit cycle, and by using $\zeta = f(\zeta)$ we get the singular point ζ as: $\zeta = [\phi_1 \cosh(\zeta)]\zeta$

or

$\zeta = \cosh^{-1}(\frac{1}{\phi_1}), \phi_1 \le 1, (\zeta \ne 0), (\phi_1 \ne 0)$

$or\, equivalently, \sin ce\ \cosh(\xi) = \frac{e^{\xi} - e^{-\xi}}{2}$ ..…………………………………(3)

$Then\ the\, non-zero \sin gular\ po\text{int}\ is\ given\, by:$

$\zeta = \ln\{\frac{1}{\phi_1}(1 \mp \sqrt{1-\phi_1^2})\}$

Therefore, the non-zero singular point exists if $\{\frac{1}{\phi_1}(1 \mp \sqrt{1-\phi_1^2})\} > 0$. Let p=2 , we have

$x_t = [\phi_1 \cosh x_{t-1}]x_{t-1} + [\phi_2^2 \cosh^2 x_{t-1}]x_{t-2} + Z_t$ ………………………………………(4)

Suppose that $Z_t = 0$, and $\zeta = f(\zeta)$, we get

$\zeta = [\phi_1 \cosh(\zeta)]\zeta + [\phi_2^2 \cosh^2(\zeta)]\zeta$

Since $\zeta \ne 0$, then we divide on it to get $[\phi_1 \cosh(\zeta)] + [\phi_2^2 \cosh^2(\zeta)] - 1 = 0$

$\cosh\zeta = (\frac{-\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2^2}}{2\phi_2^2})$

The singular points of the model in equation(4) are

$\zeta = \cosh^{-1}(\frac{-\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2^2}}{2\phi_2^2})$ …………………………………………………………......(5)

Let p=3 , then we have

$x_t = [\phi_1 \cosh x_{t-1}]x_{t-1} + [\phi_2^2 \cosh^2 x_{t-1}]x_{t-2} + [\phi_3^3 \cosh^3 x_{t-1}]x_{t-3} + Z_t$ ………………..(6)

Also, suppose that $Z_t = 0$, and $\zeta = f(\zeta)$, we get

$\zeta = [\phi_1 \cosh(\zeta)]\zeta + [\phi_2^2 \cosh^2(\zeta)]\zeta + [\phi_3^3 \cosh^3(\zeta)]\zeta\ , \zeta \ne 0$.

Therefore, we get a third order algebraic equation and by using reference [1], we have a, b and c are real constants such that $a = \frac{\phi_2^2}{\phi_3^3}, b = \frac{\phi_1}{\phi_3^3}, c = -\frac{1}{\phi_3^3}$.

$q = c - \frac{1}{3}ab + \frac{2}{27}a^3$

$\Delta = c^2 + \frac{4}{27}b^3 - \frac{2}{3}abc - \frac{1}{27}a^2b^2 + \frac{4}{27}a^3c$

Case one : $\Delta = 0$

Then, we get three real roots and we find it by

$x_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a}{3}, x_2 = x_3 = \sqrt[3]{\frac{q}{2}} - \frac{a}{3}$

Case two : $\Delta < 0$

Then, we get three different real roots and we find it by

$x_{k+1} = \sqrt[6]{16(q^2 - \Delta)} \cos\frac{\cos^{-1}\frac{-q}{\sqrt{q^2-\Delta}} + 2\pi k}{3} - \frac{a}{3}, k = 0,1,2$

Case three : $\Delta > 0$

Then, we get one real root and two complex conjugate roots and we find it by

$x_1 = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a}{3}$

$x_2 = -\frac{1}{2}(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}) - \frac{a}{3} + i\frac{\sqrt{3}}{2}(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}})$

$x_3 = -\frac{1}{2}(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}) - \frac{a}{3} - i\frac{\sqrt{3}}{2}(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}})$

The singular points of the model in equation (6) are as follows:

$\zeta = \cosh^{-1}(x_i), \forall i = 1,2,3$ ……………………………………………………...(7)

3.2 The Stability of Singular Point:

We will find the stability condition for the non-zero singular point as follows :

Put $x_s = \zeta + \zeta_s$ for all s=t,t-1 , in equation (2) (when p=1), and also suppose that the white noise is not an effect, then we have:

$$\zeta + \zeta_t = \phi_1[\cosh(\zeta + \zeta_{t-1})](\zeta + \zeta_{t-1}) \quad \ldots\ldots(8)$$

Then, $\zeta_t = \phi_1[\zeta \sinh(\zeta) + \cosh(\zeta)]\zeta_{t-1}$ ……(9)

Since, we have $\zeta = \cosh^{-1}(\frac{1}{\phi_1})$,then,

$$\zeta_t = [\phi_1 \cosh^{-1}(\tfrac{1}{\phi_1}) \sinh(\cosh^{-1}(\tfrac{1}{\phi_1})) + 1]\zeta_{t-1}$$

or $\zeta_t = h_1 \zeta_{t-1}$, where $h_1 = [\phi_1 \cosh^{-1}(\tfrac{1}{\phi_1}) \sinh(\cosh^{-1}(\tfrac{1}{\phi_1})) + 1]$ ……(10)

Equation (10) is a first order linear autoregressive model which is stable if the root λ_1 of the characteristic equation lies inside the unit circle, i.e. if $|\lambda_1| = |h_1| < 1$.

Note: The singular point of the proposed model of order one is not stable because of the amount $[\phi_1 \cosh^{-1}(\tfrac{1}{\phi_1}) \sinh(\cosh^{-1}(\tfrac{1}{\phi_1}))] > 0$, then the root must be bigger than one, that is meaning $|\lambda_1| = |h_1| > 1$.

We will find the stability condition for the non-zero singular points of equation(4) (when p=2) as follows :

$$\zeta_t = [\phi_1 \zeta \sinh(\zeta) + \phi_1 \cosh(\zeta) + 2\phi_2^2 \zeta \sinh(\zeta)\cosh(\zeta)]\zeta_{t-1} + [\phi_2^2 \cosh^2(\zeta)]\zeta_{t-2}$$

or

$$\zeta_t = h_1 \zeta_{t-1} + h_2 \zeta_{t-2} \quad \ldots\ldots(11)$$

where,

$$h_1 = [\phi_1 \zeta \sinh(\zeta) + \phi_1 \cosh(\zeta) + 2\phi_2^2 \zeta \sinh(\zeta)\cosh(\zeta)],$$

$$h_2 = [\phi_2^2 \cosh^2(\zeta)]$$

Then, from the compare between the roots of the equation (11) and it's coefficients we get

$$v^2 - h_1 v - h_2 = 0 = (v - \lambda_1)(v - \lambda_2) = v^2 - (\lambda_1 + \lambda_2)v + \lambda_1\lambda_2$$

Then, $h_1 = (\lambda_1 + \lambda_2), h_2 = -\lambda_1\lambda_2$

Where, λ_1, λ_2 are the roots of the characteristic equation of the model.

The stability condition is that $|\lambda_i| < 1$; for all i=1,2 .

We will find the stability condition for the non-zero singular points of equation (6) (when p=3) , as follows :

$$\zeta_t = [\phi_1 \zeta \sinh(\zeta) + \phi_1 \cosh(\zeta) + 2\phi_2^2 \zeta \sinh(\zeta)\cosh(\zeta) + 3\phi_3^3 \zeta \cosh^2(\zeta)\sinh\zeta]\zeta_{t-1} + [\phi_2^2 \cosh^2(\zeta)]\zeta_{t-2} + [\phi_3^3 \cosh^3(\zeta)]\zeta_{t-3} \quad \ldots\ldots(12)$$

Or $\zeta_t = h_1\zeta_{t-1} + h_2\zeta_{t-2} + h_3\zeta_{t-3}$ is a linear model of order three.

Where,

$$h_1 = [\phi_1 \zeta \sinh(\zeta) + \phi_1 \cosh(\zeta) + 2\phi_2^2 \zeta \sinh(\zeta)\cosh(\zeta) + 3\phi_3^3 \zeta \cosh^2(\zeta)\sinh\zeta]$$

$$h_2 = [\phi_2^2 \cosh^2(\zeta)]; h_3 = [\phi_3^3 \cosh^3(\zeta)]$$

The characteristic equation of linear model in equation (12) is

$$v^3 - h_1 v^2 - h_2 v - h_3 = 0$$

Then, $h_1 = \lambda_1 + \lambda_2 + \lambda_3, h_2 = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), h_3 = \lambda_1\lambda_2\lambda_3$

Where, $\lambda_1, \lambda_2, \lambda_3$ are the roots of the characteristic equation of the model.
The stability condition is that $|\lambda_i| < 1; \forall i = 1,2,3$.

The General Form:

Let the model in equation (1) be given, that is $x_t = \sum_{i=1}^{p} [\phi_i \cosh(x_{t-1})]^i x_{t-i} + Z_t$

we will find the stability condition for the non-zero singular points of equation (1) The characteristic equation of the given model is defined as:

$$v^p - h_1 v^{p-1} - h_2 v^{p-2} - h_3 v^{p-3} - \ldots - h_p = 0 \quad \ldots\ldots(13)$$

where,

$$h_1 = \phi_1[\zeta \sinh(\zeta) + \cosh(\zeta)] + 2\phi_2^2 \zeta \cosh(\zeta)\sinh(\zeta) + 3\phi_3^3 \zeta \cosh^2(\zeta)\sinh(\zeta) + \ldots$$
$$+ p\phi_p^p \zeta \cosh^{p-1}(\zeta)\sinh(\zeta)$$

$h_i = \phi_i^i \cosh^i(\zeta); \forall i = 2,3,\ldots\ldots\ldots p-1, p$.

The stability condition of singular point of equation (1) is the absolute values of the characteristic roots of equation (13) are all less than one, that means $|\lambda_i| < 1; \forall i = 1,2,3,\ldots, p$.

3.3 Limit cycle:

We find the stability condition for the limit cycle (if it exists) as follows:

Let the limit cycle of period q of the proposed model in the equation (2) has the form $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$. The points x_s close to the limit cycle is represented as $x_s = x_s + \zeta_s, \forall s = t, t-1$ and the same note on $\{Z_t\}$ when we find the singular point, then we have

$$x_t + \zeta_t = [\tfrac{\phi_1}{2}\left(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}\right)](x_{t-1} + \zeta_{t-1}) \quad \ldots\ldots(14)$$

therefore,

$$\zeta_t = \phi_1(\cosh x_{t-1} + x_{t-1}\sinh x_{t-1})\zeta_{t-1} \quad \ldots\ldots(15)$$

Equation (15) is a linear difference equation with a periodic coefficient, which is difficult to solve analytically what we want to know whether ζ_t of (15) converges to zero or not, and this can be checked by seeing whether $\left|\frac{\zeta_{t+q}}{\xi_t}\right|$ is less than one or not [7].

Let t=t+q in equation (15).

Then, $\zeta_{t+q} = \phi_1(\cosh x_{t+q-1} + x_{t+q-1}\sinh x_{t+q-1})\zeta_{t+q-1} \quad \ldots\ldots (16)$

Or $\zeta_{t+q} = \prod_{i=1}^{q} \phi_1(\cosh x_{t+i-1} + x_{t+i-1}\sinh x_{t+i-1})\zeta_t \quad \ldots\ldots(17)$

Then, equation (17) is orbitally stable if $\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$, (theorem1).

Therefore, the limit cycle of the proposed model (if it exists) is stable if

$$\left|\prod_{i=1}^{q} \phi_1(\cosh x_{t+i-1} + x_{t+i-1}\sinh x_{t+i-1})\right| < 1 \quad \ldots\ldots(18)$$

Let the limit cycle (for the 2nd-order)(equation (4)) has the form $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$. The points x_s near the limit cycle is represented as $x_s = x_s + \zeta_s, \forall s = t, t-1, t-2$ and also the same note on $\{Z_t\}$, then we have

$$x_t + \zeta_t = [\tfrac{\phi_1}{2}\left(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}\right)](x_{t-1} + \zeta_{t-1})$$
$$+[\tfrac{\phi_2^2}{4}\left((e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})})(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})})\right)](x_{t-2} + \zeta_{t-2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots..(19)$$

Then, by using maclaurin series expansion for the exponential function we get

$$\zeta_t = [\phi_1 x_{t-1}\sinh(x_{t-1}) + \phi_1\cosh(x_{t-1}) + 2\phi_2^2\sinh(x_{t-1})\cosh(x_{t-1})x_{t-2}]\zeta_{t-1}$$
$$+[\phi_2^2\cosh^2(x_{t-1})]\zeta_{t-2} \quad \ldots\ldots\ldots..\ldots.(20)$$

Then, we checked whether $\left|\frac{\zeta_{t+q}}{\xi_t}\right| < 1$ or not.

$$\zeta_{t+q} = [\phi_1 x_{t+q-1}\cosh(x_{t+q-1}) + \phi_1\sinh(x_{t+q-1}) + 2\phi_2^2\sinh(x_{t+q-1})\cosh(x_{t+q-1})x_{t+q-2}]\zeta_{t+q-1}$$
$$+[\phi_2^2\sinh^2(x_{t+q-1})]\zeta_{t+q-2} \quad \ldots\ldots..(21)$$

$$\zeta_{t+q} = \{\prod_{i=1}^{q}[\phi_1(\cosh x_{t+i-1} + x_{t+i-1}\sinh x_{t+i-1}) + 2\phi_2^2\sinh(x_{t+i-1})\cosh(x_{t+i-1})x_{t+i-2}]$$
$$+\prod_{i=2}^{q}[\phi_2^2\cosh^2(x_{t+i-1})]\}\zeta_t \quad \ldots\ldots\ldots\ldots..(22)$$

Therefore, (equation (22)) is orbitally stable if

$$\left|\{\prod_{i=1}^{q}[\phi_1(\cosh x_{t+i-1} + x_{t+i-1}\sinh x_{t+i-1}) + 2\phi_2^2\sinh(x_{t+i-1})\cosh(x_{t+i-1})x_{t+i-2}]\right.$$
$$\left.+\prod_{i=2}^{q}[\phi_2^2\cosh^2(x_{t+i-1})]\}\right| < 1 \quad \ldots\ldots\ldots\ldots\ldots.(23)$$

Let the limit cycle (for the 3rd-order) in equation (6) has the form $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$ .The points x_s near the limit cycle is represented as $x_s = x_s + \zeta_s, \forall s = t, t-1, t-2, t-3$ and the same note on $\{Z_t\}$, then we have

$$x_t + \zeta_t = [\tfrac{\phi_1}{2}\left(e^{x_{t-1}+\zeta_{t-1}} + e^{-x_{t-1}-\zeta_{t-1}}\right)](x_{t-1} + \zeta_{t-1}) +$$
$$[\tfrac{\phi_2^2}{4}\left((e^{x_{t-1}+\zeta_{t-1}} + e^{-x_{t-1}-\zeta_{t-1}})(e^{x_{t-1}+\zeta_{t-1}} + e^{-x_{t-1}-\zeta_{t-1}})\right)](x_{t-2} + \zeta_{t-2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots...(24)$$
$$+[\tfrac{\phi_3^3}{8}\left(e^{x_{t-1}+\zeta_{t-1}} + e^{-x_{t-1}-\zeta_{t-1}}\right)^3](x_{t-3} + \zeta_{t-3})$$

Then,

$$\zeta_t = [\phi_1 x_{t-1}\sinh x_{t-1} + \phi_1\cosh x_{t-1} + 2\phi_2^2 x_{t-2}\sinh x_{t-1}\cosh x_{t-1} +$$
$$3\phi_3^3 x_{t-3}\cosh^2 x_{t-1}\sinh x_{t-1}]\zeta_{t-1} + [\phi_2^2\cosh^2(x_{t-1})]\zeta_{t-2} + [\phi_3^3\cosh^3(x_{t-1})]\zeta_{t-3} \quad \ldots\ldots\ldots\ldots\ldots\ldots(25)$$

$$\zeta_{t+q} = [\phi_1 x_{t+q-1}\sinh x_{t+q-1} + \phi_1\cosh x_{t+q-1} + 2\phi_2^2 x_{t+q-2}\sinh x_{t+q-1}\cosh x_{t+q-1} +$$
$$3\phi_3^3 x_{t+q-3}\cosh^2 x_{t+q-1}\sinh x_{t+q-1}]\zeta_{t+q-1} + \quad \ldots\ldots\ldots\ldots..(26)$$
$$[\phi_2^2\cosh^2 x_{t+q-1}]\zeta_{t+q-2} + [\phi_3^3\cosh^3 x_{t+q-1}]\zeta_{t+q-3}$$

$$\zeta_{t+q} = \prod_{i=1}^{q}[\phi_1 x_{t+i-1}\sinh x_{t+i-1} + \phi_1\cosh x_{t+i-1} + 2\phi_2^2 x_{t+i-2}\sinh x_{t+i-1}\cosh x_{t+i-1} +$$
$$3\phi_3^3 x_{t+i-3}\cosh^2 x_{t+i-1}\sinh x_{t+i-1}]\ \zeta_t \quad \ldots\ldots\ldots\ldots.(27)$$
$$+\prod_{i=2}^{q}[\phi_2^2\cosh^2 x_{t+i-1}]\zeta_t + \prod_{i=3}^{q}[\phi_3^3\cosh^3 x_{t+i-1}]\ \zeta_t$$

Therefore, the model of equation (6) is orbitally stable if

$$\Big|\prod_{i=1}^{q}[\phi_1(\cosh x_{t+i-1}+x_{t+i-1}\sinh x_{t+i-1})+2\phi_2^2 x_{t+i-2}\sinh x_{t+i-1}\cosh x_{t+i-1}+$$
$$3\phi_3^3 x_{t+i-3}\cosh^2 x_{t+i-1}\sinh x_{t+i-1}]+\prod_{i=2}^{q}[\phi_2^2\cosh^2 x_{t+i-1}]+\prod_{i=3}^{q}[\phi_3^3\cosh^3 x_{t+i-1}]\Big|<1 \quad \ldots\ldots\ldots\ldots..(28)$$

Let the limit cycle (the proposed model in the general form) of equation (1) has the form $x_t, x_{t+1}, x_{t+2},..., x_{t+q} = x_t$ .The points x_s near the limit cycle is represented as $x_s = x_s + \zeta_s, \forall s = t, t-1, t-2, t-3,..., t-p$ and the same note on $\{Z_t\}$, then we have the model of equation (1) is orbitally stable if

$$\Big|\prod_{i=1}^{q}\{[\phi_1(\cosh x_{t+i-1}+x_{t+i-1}\sinh x_{t+i-1})+2\phi_2^2 x_{t+i-2}\cosh x_{t+i-1}\sinh x_{t+i-1}$$
$$+\sum_{j=3}^{p} j\phi_j^j x_{t+i-j}\cosh^{j-1} x_{t+i-1}\sinh x_{t+i-1}]\}+\prod\nolimits_{i=2}^{q}[\phi_2^2\cosh^2 x_{t+i-1}]+...+ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(29)$$
$$\prod_{i=p}^{q}[\phi_p^p\cosh^p x_{t+i-1}]\Big|<1$$

Theorem 2: A limit cycle of period q , x_{t+1} ,…….., x_{t+q} of the model in equation (1) is orbitally stable when all the eigen values of the matrix, $A = A_q \,.\, A_{q-1} \ldots A_1$, have absolute value less than one, where

$$A_i=\begin{pmatrix} \phi_1(\cosh x_{t+i-1}+x_{t+i-1}\sinh x_{t+i-1})+2\phi_2^2 x_{t+i-2}\cosh x_{t+i-1}\sinh x_{t+i-1}+\sum_{j=3}^{p} j\phi_j^j x_{t+i-j}\cosh^{j-1}x_{t+i-1}\sinh x_{t+i-1} & ;\phi_2^2\cosh^2 x_{t+i-1} & \ldots & . & \phi_p^p\cosh^p x_{t+i-1} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

4.Examples

In this section, we give two examples to explain how to find the singular points of the proposed model and the conditions of stability of singular points and limit cycle.

Example(1): If $\phi_1 = 0.1$ **,** then the model in equation (2) is $x_t = [0.1\cosh x_{t-1}]x_{t-1} + Z_t$

The Singular Point:

By using equation(3) we get the non-zero singular point, which is

$\zeta = \cosh^{-1}(\frac{1}{0.1}) = \cosh^{-1}(10) = 2.9932$ **.**

The stability of singular point:

Apply equation (10) we have that

$\zeta_t = 3.9781\zeta_{t-1}$ …………………………………………………………………….(*)

Then, equation(*) is a first order linear autoregressive process. Since, the root $(\lambda_1 = 3.9781)$ of the characteristic equation of equation (*) lies outside the unit circle . Then, the singular point is not stable.

The Limit Cycle:

Let the limit cycle of period q=4 which is $\{0.1005, 0.1009, 0.1380, 0.8463, 0.1005\}$ Then, from theorem (2) we get that

$A_1 = \begin{bmatrix} 0.1015 & 0 \\ 1 & 0 \end{bmatrix}$. $A_2 = \begin{bmatrix} 0.1029 & 0 \\ 1 & 0 \end{bmatrix}$ $A_3 = \begin{bmatrix} 0.2185 & 0 \\ 1 & 0 \end{bmatrix}$ $A_4 = \begin{bmatrix} 0.1015 & 0 \\ 1 & 0 \end{bmatrix}$

Since, $A = A_1.A_2.A_3.A_4$,then, $A = \begin{bmatrix} 0.0002 & 0 \\ 0.0023 & 0 \end{bmatrix}$ Therefore, all the eigen values of the matrix A have absolute value less than one, where $\lambda_1 = 0, \lambda_2 = 0.0002$. Then, the model has a stable limit cycle.This means that the model has unstable non-zero singular point but have a stable limit cycle.

Example(2):

If $\phi_1 = 1.333; \phi_2 = -0.444$ **,** then the model in equation (4) is

$x_t = [1.333\cosh x_{t-1}]x_{t-1} + [(-0.444)^2 \cosh^2 x_{t-1}]x_{t-2} + Z_t$

The Singular Point:

Apply equation(5) we get two non-zero singular points, which are $\zeta_1 = 0.8213\text{i}$, and $\zeta_2 = 2.6944 + 3.1416\text{i}$

The stability of singular point:

If $\zeta_1 = 0.8213\text{i}$, then apply equation (11) we have that

$\zeta_t = -0.055\zeta_{t-1} + 0.0917\zeta_{t-2}$ …………………………………………………...(**)

Then, equation (**) is a second order linear autoregressive model. The characteristic equation of equation (**) is $v^2 + 0.055v - 0.0917 = 0$ .Then, $\lambda_1 = 0.2764, \lambda_2 = -0.3316$ are the roots of the characteristic equation of the model in equation (**).Then, the singular point is stable because the roots of the characteristic equation of the model lie inside the unit circle. If $\zeta_2 = 2.6944 + 3.1416\text{i}$, then apply equation (11) we have that

$\zeta_t = (21.8885 + 37.0752\text{i})\zeta_{t-1} + (10.9097 + 0.00016\text{i})\zeta_{t-2}$ ………………………..(***)

Then, equation (***) is a third order linear autoregressive model.The characteristic equation of equation(***) is $v^2 - (21.8885 + 37.0752\text{i})v - (10.9097 + 0.00016\text{i}) = 0$

Then, $\lambda_1 = 22.0189 + 36.8570\text{i}, \lambda_2 = -0.1303 + 0.2181\text{i}$ are the roots of the characteristic equation of the model.Then, the singular point is not stable because one of the roots of the characteristic equation of the model lies outside the unit circle.

The Limit Cycle:

Let the limit cycle of period q=4 which is $\{9.34, 1.82, 0.84, 0.76, 9.34\}$

Then, from theorem (2) we get that

$A_{1`} = \begin{bmatrix} 6.51 & 0.33 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 3.3 & 0.37 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $A_3 = \begin{bmatrix} 14.68 & 1.98 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A_4 = \begin{bmatrix} 23076819.6 & 6386274.2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Since, $A = A_1.A_2.A_3.A_4$ Then,

$A = (10)^9 * \begin{bmatrix} 7.4451 & 2.0604 & 0 \\ 1.1265 & 0.3117 & 0 \\ 0.3388 & 0.0038 & 0 \end{bmatrix}$ .Therefore, one of the eigen values λ_3 of the matrix

A has absolute value more than one, where, $\lambda_1 = 0, \lambda_2 = 10^9 * 0.000199, \lambda_3 = 10^9 * 7.7569$,Then, the model has not a stable limit cycle.Therefore, the model has two non-zero singular points(one of them is a stable and the other is unstable)and also has unstable limit cycle.

5. Conclusion

The conclusions of this paper are as follows:

1- We find the non-zero singular point of the proposed model of order one and two and three.

2- We find the stability conditions of the non-zero singular point of the proposed model of order one and two and three and the general model.

3- We find the stability conditions of the limit cycle of the proposed model of order one and two and three and the general model.

4- We explain the stability conditions of a non-zero singular point and the stability conditions of the limit cycle in two examples and find that the model of order one example(1) is not stable singular point and a stable limit cycle and find that the model of order two example(2) have a two complex singular points ζ_1 , ζ_2 one of them ζ_1 is a stable and the other ζ_2 is unstable, and not a stable limit cycle.

REFERENCES

[1] Al-Azzawi, S.F.,(2012). Stability and Bifurcation of Pan Chaotic System by Using Routh-Hurwitz and Gardan Methods. Applied Mathematics and Computation, Vol.219, Issue3, pp 1144-1152.

[2] Chatfield, C.,(2004).sixth edition: The Analysis of Time Series: An introduction. Chapman and Hall, University of Bath, U.K.

[3] Chatfield, C.,(2000).Time-Series Forecasting. Chapman and Hall, the University of Bath, London, UK.

[4] Finizio, N. ,G. Ladas, (1982). An Introduction to Differential Equations with Difference Equations, Fourier Series, and Partial Differential Equations., Wadsworth Publishing Company Belmont, California A division of Wadsworth, Inc., University of Rhode Island.

[5] Ozaki, T.,(1982).The Statistical Analysis of Perturbed Limit Cycle Processes Using Nonlinear Time Series Models. Journal of Time Series Analysis,Vol.3,No.1,pp 29-41.

[6] Ozaki, T. , H. Oda, (1978).Non-linear Time Series Model Identification by Akaike's Information Criterion. In Information and System, Dubuisson, B. eds. Pergamon Press, pp 83-91.

[7] Ozaki, T.,(1985).Non linear Time Series Models and Dynamical System. E.J. Hannan, P.R. Krishnaiah, Rao M., eds., Handbook of statistics, Vol.5, pp25-83.

[8] Tong, H., (1990). Non-Linear Time Series: A Dynamical System Approach. Oxford University Press, Oxford, UK.

[9] Tsay, R.S., (2010). third edition: Analysis of Financial Time Series., John Wiley & Sons, Inc., Publication The University of Chicago Booth School of Business Chicago, IL.