Classification of Zero Divisor Graphs of Commutative Rings of Degrees 11,12 and 13

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ABSTRACT

In 2005 Wang investigated the zero divisor graphs of degrees 5,6,9 and 10. In 2012 Shuker and Mohammad investigated the zero divisor graphs of degrees 7 and 8. In this paper, we consider zero divisor graphs of commutative rings of degrees 11, 12 and 13.

Key word: Zero-divisor, Ring, Zero-divisor graph.

تصنيف بيانات قواسم الصفر للحلقات الابدالية ذات الدرجات 11, 12 و 13

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الملخص

في عام 2005 درس Wang بيانات قواسم الصفر للحلقات الابدالية من الدرجات9,6,5 و10. في عام 2012 درسShuker and Mohammad بيانات قواسم الصفر للدرجتين 7 و8 .في هذا البحث درسنا بيانات قواسم الصفر للحلقات الابدالية من الدرجات 12,11 و 13.

الكلمات المفتاحية : قواسم الصفر , حلقة , بيان قواسم الصفر.

1. Introduction

The concept of zero divisor graph of a commutative ring was introduced by Beck in [3], he let all elements of the ring be vertices of a graph. In [1] Anderson and Livingston introduced and studied the zero divisor graph whose vertices are the non-zero zero divisors.

Throughout this paper, all rings are assumed to be commutative rings with identity, and Z(R) be the set of zero divisors. We associate a simple graph $\Gamma(R)$ to a ring R with vertices $Z(R)^*= Z(R)-\{0\}$, the set of all non-zero zero divisors of R. For all distinct $x,y\in Z(R)^*$, the vertices x and y are adjacent if and only if xy=0. In [1] Anderson and Livingston proved that for any commutative ring R, $\Gamma(R)$ is connected.

In [6], Wang investigated the zero divisor graphs of degree 5, 6, 9 and 10. In [5], we consider the zero divisor graphs of degree 7 and 8. In this paper, we extend these results to consider the zero divisor graphs of commutative rings of degrees 11,12 and 13.

2. Rings with |Z(R)*|=11

The main aim of this section is to find all possible zero divisor graphs of 11 vertices and rings correspond to them.

Recall that if R is a finite ring, then every element of R either unit or zero divisor [2]. In [6] Wang proved the following result.

Lemma 2.1: Let (R_1,m_1) and (R_2,m_2) are local rings, then $|Z(R_1xR_2)^*|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1||m_2|-1$. ■

In [5] we extended Wang's result.

Lemma 2.2: If (R_1,m_1), (R_2,m_2) and (R_3,m_3) are finite local rings, then $|Z(R_1xR_2xR_3)^*|=|R_1|x|R_2|x|m_3|+|Z(R_1xR_2)|x(|R_3|-|m_3|)-1$ where

$|Z(R_1xR_2)|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1|x|m_2|$.■

As a direct consequence to Lemma 2.2, we obtain the following:

Corollary 2.3: If R_1, R_2 and R_3 are finite fields, then

$|Z(R_1xR_2xR_3)^*|= |R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3|$. ■

Corollary 2.4 : If R is a finite ring and $R\cong R_1xR_2xR_3$, then $|Z(R)^*|\geq 13$ for some local ring R_i but not field. ■

Corollary 2.5: If R_i is local not field for some $1\leq i_1,i_2\leq 3$,then $|Z(R)^*|\geq 27$.■

Lemma 2.6: [6] Let R be a ring and $R \cong R_1xR_2xR_3$, where R_i is local for i=1,2,3, then

1- If $|R_i|\geq 3$ for some i_1, i_2, then $|Z(R)^*|\geq 13$.
2- If $|R_i|\geq 4$ for some i, then $|Z(R)^*|\geq 12$.

Lemma 2.7: [6] Let R $R_1xR_2xR_3xR_4$, where R_i is local for every i. Then $|Z(R)^*|\geq 14$.

Next, we prove two fundamental lemmas

Lemma 2.8 : Let R be a ring with $|Z(R)^*|=11$, then $R\cong R_1xR_2$, where R_1 and R_2 are local rings.

Proof: Let $R \cong R_1$ x R_2 …x R_n, where each R_i is a local ring. If $n \geq 4$ or n=3 with R_i not field for some i=1,2 and 3, then we have a contradiction , by Lemma 2.7 and Corollary 2.4 respectively. It is clear that if n=1, then $|Z(R)|=12$ and hence, it also a contradiction so that we can investigate the case when n=3 and R_i are fields for each i=1,2,3 .By Corollary 2.3 , $|Z(R_1xR_2xR_3)^*|= |R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3|=11$. If $|R_1|=|R_2|=2$, then $|R_3|=11/3$ which is a contradiction. If $|R_1|=2$, $|R_2|=3$, then $|R_3|=5/2$, which is a contradiction. If $|R_1|=2$ and $|R_2|\geq 4$, then by Lemma 2.6(2) $|Z(R)^*|\geq 12$, which is a contradiction. If $|R_1|$ and $|R_2| \geq 3$, then by Lemma2.6(1) $|Z(R)^*|\geq 13$, which is again a contradiction. Therefore, n=2 and, hence $R\cong R_1xR_2$. ■

Lemma 2.9: Let R be a ring with $|Z(R)^*|=11$. Then, $R\cong Z_4xZ_4$, $Z_4xZ_2[X]/(X^2)$, $Z_2[X]/(X^2)x$ $Z_2[X]/(X^2)$, Z_2xZ_9, $Z_2xZ_3[X]/(X^2)$, Z_2xZ_8, $Z_2xZ_2[X]/(X^3)$, $Z_2xZ_4[X]/(2X,X^2-2)$, $Z_2xZ_2[X,Y]/(X,Y)^3$, $Z_2xZ_4[X]/(X^2,2X)$, Z_5xZ_4, $Z_5xZ_2[X]/(X^2)$, Z_2xZ_{11}, F_4xF_9 or Z_5xF_8.

Proof: By Lemma 2.8; $R\cong R_1xR_2$, where R_1,R_2 are local rings. If R_1 and R_2 are not fields, then $|Z(R_1xR_2)^*|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1||m_2|-1=11$. If $|m_1|=p$, where p is prime, then $|R_1|=p^2$ [6, Lemma 4.8]. If $|m_1|=2$, then $|R_1|=4$ which implies that $|R_2|=6-|m_2|$, therefore $|m_2|=2$ and $|R_2|=4$ so that $R\cong Z_4xZ_4$ or $Z_4xZ_2[X]/(X^2)$ or $Z_2[X]/(X^2)x$ $Z_2[X]/(X^2)$. if $|m_1|=3$, then $|R_1|=9$ which implies that $|R_2|=4-2|m_2|$, but $|m_2|\geq 2$, therefore $|R_2|\leq 0$ which is a contradiction . If $|m_1|,|m_2|\geq 4$, then $|R_1|,|R_2|\geq 8$ so that $11=|Z(R)^*|\geq 47$ which is a contradiction. If R_1 is a field and R_2 is not a field, then $|R_2|=12-|m_2|(|R_1|-1)$. Let $|R_1|=2$,then $|R_2|=12-|m_2|$. Therefore, $|m_2|=3$, $|R_2|=9$ or $|m_2|=4$, $|R_2|=8$ and, hence $R\cong Z_2xZ_9$, $Z_2xZ_3[X]/(X^2)$, Z_2xZ_8, $Z_2xZ_2[X]/(X^3)$ $Z_2xZ_4[X]/(2X,X^2-2)$, $Z_2xZ_2[X,Y]/(X,Y)^3$ or $Z_2xZ_4[X]/(X^2,2X)$.

Let $|R_1|=3$, then $|R_2|=12-2|m_2|$, which is a contradiction. Let $|R_1|=4$: Then, $|R_2|=12-3|m_2|$, which is also a contradiction. Let $|R_1|=5$. Then, $|R_2|=12-4|m_2|$. Therefore, $|m_2|=2$ and $|R_2|=4$ so that $R\cong Z_5xZ_4$ or Z_5x $Z_2[X]/(X^2)$. Let $|R_1|\geq 7$: Then, $|R_2|=12-6|m_2|$ and since $|m_2|\geq 2$, then $|R_2|\leq 0$ which is a contradiction. If R_1 and R_2 are fields, then applying Lemma 2.1 $|R_1|+|R_2|=13$ and hence $|R_1|=2$, $|R_2|=11$ or $|R_1|=4$, $|R_2|=9$ or $|R_1|=5$, $|R_1|=8$. Therefore, $R\cong Z_2xZ_{11}$, F_4xF_9 or Z_5xF_8. ■

Now, we shall prove the main result of this section.

Theorem 2.10: Let R be a ring with $|Z(R)^*| =11$, then, the graphs depicted in the following figures can be realized as $\Gamma(R)$.

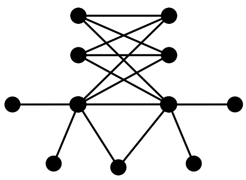


Figure (1)

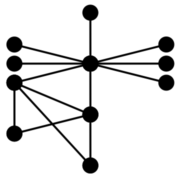


Figure (2)

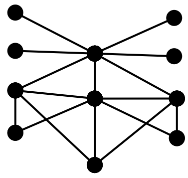


Figure (3)

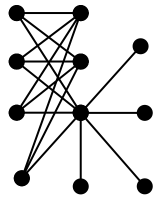
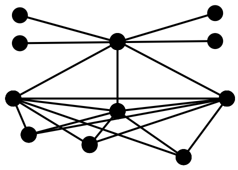


Figure (4)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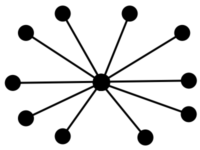


Figure (6)

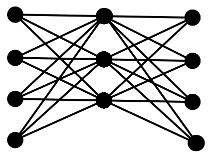


Figure (**7)**

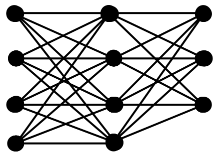


Figure (8)

Proof: By Lemma 2.7; $R\cong Z_4xZ_4$, $Z_4xZ_2[X]/(X^2)$, $Z_2[X]/(X^2)x\ Z_2[X]/(X^2)$, Z_2xZ_9, $Z_2xZ_3[X]/(X^2)$, Z_2xZ_8, $Z_2xZ_2[X]/(X^3)$, $Z_2xZ_4[X]/(2X,X^2-2)$, $Z_2xZ_2[X,Y]/(X,Y)^3$, $Z_2xZ_4[X]/(X^2,2X)$, Z_5xZ_4, $Z_5xZ_2[X]/(X^2)$, Z_2xZ_{11}, F_4xF_9 or Z_5xF_8. Figure (1), can be realized as $\Gamma(Z_4xZ_4)$ or $\Gamma(Z_4xZ_2[X]/(X^2))$ or $\Gamma(Z_2[X]/(X^2)x\ Z_2[X]/(X^2))$. Figure (2) , can be realized as $\Gamma(Z_2xZ_9)$ or $\Gamma(Z_2xZ_3[X]/(X^2))$. Figure (3) , can be realized as $\Gamma(Z_2xZ_8)$ or $\Gamma(Z_2xZ_2[X]/(X^3))$ or $Z_2xZ_4[X]/(2X,X^2-2)$. Figure (4) , can be realized as $\Gamma(Z_5xZ_4)$ or $\Gamma(Z_5xZ_2[X]/(X^2))$. Figure (5) , can be realized $\Gamma(Z_2xZ_4[X]/(2X,X^2))$ or $\Gamma(Z_2xZ[X,Y]/(X,Y)^2)$. Figure (6) , can be realized as $\Gamma(Z_2xZ_{11})$. Figure (7) , can be realized as $\Gamma(F_4xF_9)$ and Figure(8) , can be realized as $\Gamma(Z_5xF_8)$. ■

3. Rings with |Z(R)*|=12

The main aim of this section is to find all possible zero divisor graphs of 12 vertices and rings correspond to them.

We shall start this section with the following lemmas.

Lemma 3.1 : Let R be a ring with $|Z(R)^*|=12$; if $R\cong R_1xR_2x\ldots R_n$, where R_i is a local ring for all $i\geq 1$, then n=3 if and only if $R\cong Z_2xZ_2xF_4$

Proof: Let R be a ring with $|Z(R)^*|=12$ and let $R\cong R_1xR_2xR_3$ where R_i is a local ring for all i=1,2,3. If R_i is not a field for some $1\leq i\leq 3$, then $|Z(R)^*|\geq 13$ which is a contradiction, so that R_i is a field for all $1\leq i\leq 3$, then by Corollary2.3, $|Z(R_1xR_2xR_3)^*| = |R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3|=12$. If $|R_1|=|R_2|=2$, then $|R_3|=4$, so that $R\cong Z_2xZ_2xF_4$. If $|R_1|=2$ and $|R_2|=3$, then $|R_3|=13/4$ which is a contradiction. If $|R_1|\geq 3$ and $|R_2|\geq 3$, then by Lemma2.6(1) $|Z(R)^*|\geq 13$, which is a contradiction. ■

Lemma 3.2: Let R be a ring with $|Z(R)^*|=12$, if $R\cong R_1xR_2x\ldots R_n$, where R_i is a local ring for all $i\geq 1$, then n=2 if and only if $R\cong Z_3xZ_{11}$, Z_5xF_9 or Z_7xZ_7

Proof: Let R be a ring with $|Z(R)^*|=12$ and let $R\cong R_1\ xR_2$ where R_1 and R_2 are local rings. If R_1 and R_2 are not fields, then $|Z(R_1xR_2)^*|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1||m_2|-1=12$.

If $|m_1|=2$, then $|R_1|=4$. So that, $|R_2|=13/2 - |m_2|$. Since, $|m_2|$ is an integer , then $|R_2|$ is not an integer which is a contradiction.

If $|m_1|\geq 3$, then $|R_1|\geq 8$ and, since $|m_2|\geq 2$, $|R_2|\geq 4$,then $12=|Z(R)^*|\geq 21$ which is also a contradiction.

If $|R_1|$ is a field and $|R_2|$ is not a field, then $|R_2|+|m_2||R_1|-|m_2|=12$ which this leads to a contradiction. If R_1 and R_2 are fields, then $|R_1|+|R_2|=14$ which implies that $|R_1|=3,|R_2|=11$ or $|R_1|=5,|R_2|=9$ or $|R_1|=|R_2|=7$. Therefore $R\cong Z_3xZ_{11}$, Z_5xF_9 or $Z_7x\ Z_7$. ■

Lemma 3.3: Let R be a ring with $|Z(R)^*|=12$. Then, $R\cong Z_2xZ_2xF_4$, Z_3xZ_{11}, Z_5xF_9, Z_7xZ_7, Z_{169} or $Z_{13}[X]/(X^2)$

Proof: Let $R \cong R_1 \times R_2 \ldots x R_n$, where R_i is a local ring. If $n\geq 4$, then by Lemma 2.7 $|Z(R)^*|\geq 14$. If n=3 , then $R\cong Z_2xZ_2xF_4$. If n=2 , then $R\cong Z_3xZ_{11}$ Z_5xF_9 or Z_7xZ_7. If n=1 and R is a field, then Z(R)={0} which is a contradiction. If R is a local ring, then $|Z(R)^*|=|m|-1=12$, so that |m|=13. Therefore, |R|=169, which implies that $R\cong Z_{169}$ or $Z_{13}[X]/(X^2)$. ■

Theorem 3.4: Let R be a ring with $|Z(R)|^*=12$, then the graphs depicted in the following figures can be realized as Γ(R)

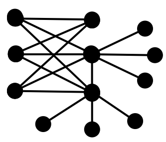


Figure (1)

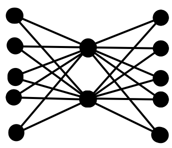


Figure (2)

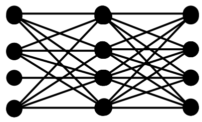


Figure (3)

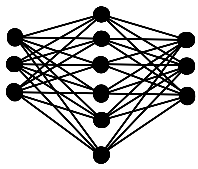


Figure (4)

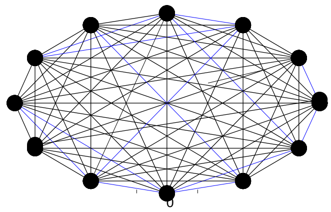


Figure (5)

Proof: By Lemma 3.3; $R\cong Z_2xZ_2xF_4$, Z_3xZ_{11}, Z_5xF_9, Z_7xZ_7, Z_{169} or $Z_{13}[X]/(X^2)$. In Figure (1), can be realized as $\Gamma(Z_2xZ_2xF_4)$. Figure (2) , can be realized as $\Gamma(Z_3xZ_{11})$. Figure (3) , can be realized as $\Gamma(Z_7xZ_7)$. Figure (4) , can be realized as $\Gamma(Z_7xZ_7)$. Figure (5) , can be realized as $\Gamma(Z_{169})$ or $\Gamma(Z_{13}[X]/(X)^2)$. ■

4. Rings with |Z(R)*|=13

The main aim of this section is to find all possible zero divisor graphs of 13 vertices and rings correspond to them.

We shall start this section with following lemma.

Lemma 4.1 : Let R be a ring with $|Z(R)^*|=13$, if $R\cong R_1xR_2x\ldots R_n$, where R_i is a local ring for all $i\geq 1$, then n=3 if and only if $R\cong Z_2xZ_3xZ_3$, $Z_2xZ_2xZ_4$ or $Z_2xZ_2xZ_2[X]/(X^2)$.

Proof : Let R be a ring with $|Z(R)^*|=13$ and let $R\cong R_1xR_2xR_3$, where R_i local rings for all $1 \leq i \leq 3$. If R_i is not a field, for some $1 \leq i_1, i_2 \leq 3$, then $|Z(R_1xR_2xR_3)^*|\geq 27$ which is a contradiction.

If R_3 is not a field and R_1 and R_2 are fields, then $|Z(R_1xR_2)|=|R_1|+|R_2|-1$ and $|Z(R_1xR_2xR_3)^*|=|R_1||R_2||m_3|+(|R_1|+|R_2|-1)(|R_3|-|m_3|)-1$, so that $|R_1||R_2||m_3|+(|R_1|+|R_2|-1)(|R_3|-|m_3|)=14$

If $|R_1|=|R_2|=2$, then $|R_3|=\frac{14-|m_3|}{3}$ which implies that $|R_3|=4$ and $|m_3|=2$. Therefore, $R\cong Z_2xZ_2xZ_4$ or $Z_2xZ_2xZ_2[X]/(X^2)$. If $|R_1|\geq 2$ and $|R_2|\geq 3$, and since $|R_3|\geq 4$ and $|m_3|\geq 2$, then $13=|Z(R_1xR_2xR_3)^*|\geq 2.3.2+(2+3-1)(4-2)-1\geq 19$ which is a contradiction. If R_i is a field for all $1 \leq i \leq 3$, then

$|R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3|=13$. If $|R_1|=|R_2|$, then $|R_3| = 13/2$ which is a contradiction. If $|R_1|=2$, $|R_2|=3$, then $|R_3|=3$ so that $R\cong Z_2xZ_3xZ_3$. If $|R_1|=2$ and $|R_2|=4$, then $|R_3|=11/5$ which is a contradiction. If $|R_1|=2$ and $|R_2|=5$, then $|R_3|=5/3$ which is a contradiction. If $|R_1|=2$ and $|R_2|\geq 7$, then $|R_3|\leq 1$ which is a contradiction. If $|R_1|\geq 3$ and $|R_2|\geq 4$, then $|R_3|\leq 4/3$ which is a contradiction. ■

Lemma 4.2 : Let R be a ring with $|Z(R)^*|=13$, if $R\cong R_1xR_2x\ldots R_n$, where R_i is a local ring for all $i\geq 1$, then n=2 if and only if $R\cong Z_2xZ_{13}$, F_4xZ_{11} or Z_7xF_8.

Proof : Let R be a ring with $|Z(R)^*|=13$ and let $R\cong R_1xR_2$, where R_1 and R_2 local rings. If R_1 and R_2 are not fields, then

$|Z(R_1xR_2)^*|=|R_1||m_2|+|R_2||m_1|-|m_1||m_2|=14$. If $|m_1|=2$, then $|R_1|=4$, so that $|R_2|=7-|m_2|$ which is a contradiction. If $|m_1|,|m_2|\geq 3$, then $|R_1|,|R_2|\geq 8$, so that $|Z(R_1xR_2)^*|\geq 3.8+8.3-3.3-1=38$ which is a contradiction. If R_1 field and R_2 local not field, then $|R_1||m_2|+|R_2|-|m_2|=14$ which implies that $|R_2|=14-(|R_1|-1)|m_2|$. If $|R_1|=2$, then $|R_2|=14-|m_2|$ which is a contradiction. If $|R_1|=3$, then $|R_2|=14-2|m_2|$ which is a contradiction. If $|R_1|=4$, then $|R_2|=14-3|m_2|$ which is a contradiction. If $|R_1|=5$, then $|R_2|=14-4|m_2|$ which is a contradiction. If $|R_1|\geq 7$, then $|Z(R_1xR_2)^*|\geq 15$ which is a contradiction. Therefore, R_1 and R_2 are fields, which imply that $|R_1|+|R_2|=15$ and, hence $|R_1|=2$, $|R_2|=13$ or $|R_1|=4$, $|R_2|=11$ or $|R_1|=7$, $|R_2|=8$. Therefore, $R\cong Z_2xZ_{13}$, F_4xZ_{11} or Z_7xF_8. ■

Lemma 4.3: Let R be a ring with $|Z(R)^*|=13$, then $R\cong Z_2xZ_2xZ_4$, $Z_2xZ_2xZ_2[X]/(X^2)$, $R\cong Z_2xZ_{13}$, F_4xZ_{11} or Z_7xF_8.

Proof: Let $R \cong R_1 \times R_2 \ldots x R_n$, where R_i is a local ring. If $n\geq 4$, then by Lemma 2.7 $|Z(R)^*|\geq 14$. If n=3 , then $R\cong Z_2xZ_2xZ_4$ or $Z_2xZ_2xZ_2[X]/(X^2)$ Lemma 4.1. If n=2 , then $R\cong Z_2xZ_{13}$, F_4xZ_{11} or Z_7xF_8 Lemma4.2. If n=1 and R is a field, then Z(R)={0} which is a contradiction. If R is a local ring, then $|Z(R)^*|=m-1=13$, so that $|m|=14$ which is also a contradiction. ■

Theorem 4.4: Let R be a ring with $|Z(R)|^*=13$, then the graphs depicted in the following figures can be realized as $\Gamma(R)$

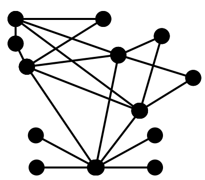


Figure (1)

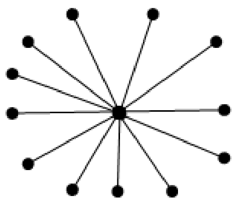


Figure (2)

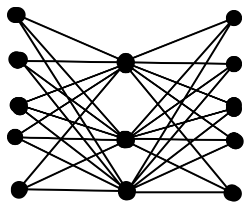


Figure (3)

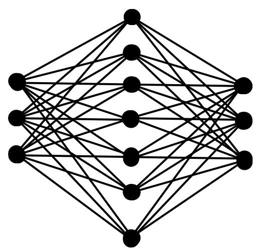
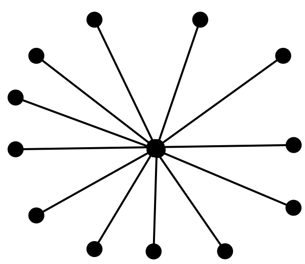


Figure (4)

Proof: By Lemma 4.3 $R\cong Z_2xZ_2xZ_4$, $Z_2xZ_2xZ_2[X]/(X^2)$, Z_2xZ_{13}, F_4xZ_{11} or Z_7xF_8. Figure (1) can be realized as $\Gamma(Z_2xZ_2xZ_4)$ or $\Gamma(Z_2xZ_2xZ_2[X]/(X)^2$. Figure (2) can be realized as $\Gamma(Z_2xZ_{13})$. Figure (3) can be realized as $\Gamma(F_4xZ_{11})$ and Figure (4) can be realized as $\Gamma(Z_7xZ_8)$.



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