Classification of Zero Divisor Graphs of Commutative Rings of Degrees 11, 12 and 13

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ABSTRACT

In 2005 Wang investigated the zero divisor graphs of degrees 5, 6, 9 and 10. In 2012 Shuker and Mohammad investigated the zero divisor graphs of degrees 7 and 8. In this paper, we consider zero divisor graphs of commutative rings of degrees 11, 12 and 13.

Key word: Zero-divisor, Ring, Zero-divisor graph.

1. Introduction

The concept of zero divisor graph of a commutative ring was introduced by Beck in [3], he let all elements of the ring be vertices of a graph. In [1] Anderson and Livingston introduced and studied the zero divisor graph whose vertices are the non-zero zero divisors.

Throughout this paper, all rings are assumed to be commutative rings with identity, and Z(R) be the set of zero divisors. We associate a simple graph Γ(R) to a ring R with vertices Z(R)*= Z(R)-{0}, the set of all non-zero zero divisors of R. For all distinct x,y ∈ Z(R)*, the vertices x and y are adjacent if and only if xy=0. In [1] Anderson and Livingston proved that for any commutative ring R, Γ(R) is connected.

In [6], Wang investigated the zero divisor graphs of degree 5, 6, 9 and 10. In [5], we consider the zero divisor graphs of degree 7 and 8. In this paper, we extend these results to consider the zero divisor graphs of commutative rings of degrees 11, 12 and 13.

2. Rings with |Z(R)*|=11

The main aim of this section is to find all possible zero divisor graphs of 11 vertices and rings correspond to them.

Recall that if R is a finite ring, then every element of R either unit or zero divisor [2]. In [6] Wang proved the following result.

Lemma 2.1: Let (R1,m1) and (R2,m2) are local rings, then |Z(R1xR2)|=|R1| |m2|+|R2| |m2|-|m1||m2|-1. ■

In [5] we extended Wang's result.
Lemma 2.2: If $(R_1,m_1)$, $(R_2,m_2)$ and $(R_3,m_3)$ are finite local rings, then $|Z(R_1R_2xR_3)|=|R_1||R_2||R_3| |m_1| |m_2| |m_3| - 1$ where $|Z(R_1R_2)|=|R_1||R_2| |m_1| + |R_2||m_2| - |m_1||m_2|$. ■

As a direct consequence to Lemma 2.2, we obtain the following:

Corollary 2.3: If $R_1$, $R_2$ and $R_3$ are finite fields, then $|Z(R_1R_2xR_3)|=|R_1||R_2||R_3|$. ■

Corollary 2.4: If $R$ is a finite ring and $R \cong R_1R_2xR_3$, then $|Z(R)^*| \geq 13$ for some local ring $R$, but not field. ■

Corollary 2.5: If $R_i$ is local not field for some $1 \leq i_1,i_2 \leq 3$, then $|Z(R)^*| \geq 27$. ■

Lemma 2.6: [6] Let $R$ be a ring and $R \cong R_1R_2xR_3$, where $R_i$ is local for $i=1,2,3$, then

1- If $|R_i| \geq 3$ for some $i_1,i_2$, then $|Z(R)| \geq 13$.
2- If $|R_i| \geq 4$ for some $i_1$, then $|Z(R)| \geq 12$.

Lemma 2.7: [6] Let $R_1R_2xR_3xR_4$, where $R_i$ is local for every $i$. Then $|Z(R)^*| \geq 14$.

Next, we prove two fundamental lemmas

Lemma 2.8: Let $R$ be a ring with $|Z(R)|=11$, then $R \cong R_1R_2$, where $R_1$ and $R_2$ are local rings.

Proof: Let $R \cong R_1 \times R_2 \ldots \times R_n$, where each $R_i$ is a local ring. If $n \geq 4$ or $n=3$ with $R_i$ not field for some $i=1,2,3$, and then we have a contradiction, by Lemma 2.7 and Corollary 2.4 respectively. It is clear that if $n=1$, then $|Z(R)|=12$ and hence, it also a contradiction so that we can investigate the case when $n=3$ and $R_i$ are fields for each $i=1,2,3$. By Corollary 2.3, if $Z(R_1R_2xR_3)=|R_1||R_2||R_3|$, then $|R_1|=|R_2|=|R_3|=11$. If $|R_1|=|R_2|$, then $|R_3|=11/3$ which is a contradiction. If $|R_1|=2$, $|R_2|=3$, then $|R_3|=5/2$, which is a contradiction. If $|R_1|=2$ and $|R_2|=4$, then by Lemma 2.6(2) $|Z(R)^*| \geq 12$, which is a contradiction. If $|R_1|$ and $|R_2|$ are finite, then by Lemma 2.6(2) $|Z(R)^*| \geq 13$, which is again a contradiction. Therefore, $n=2$ and, hence $R \cong R_1R_2$.

Lemma 2.9: Let $R$ be a ring with $|Z(R)|=11$. Then, $R \cong Z_4xZ_4$, $Z_3xZ_2[X]/(X^2)$, $Z_2[X]/(X^2)xZ_2[X]/(X^2)$, $Z_2xZ_3, Z_2xZ_3[X]/(X^2), Z_2xZ_4, Z_2xZ_2[X]/(X^3), Z_2xZ_4[X]/(2X,X^2-2), Z_2xZ_2[X,Y]/(X,Y)^3, Z_2xZ_4[X]/(X^2,2X), Z_4xZ_4, Z_2xZ_2[X]/(X^2), Z_2xZ_3, F_4xF_0$ or $Z_4xF_4$.

Proof: By Lemma 2.8, $R \cong R_1 \times R_2$, where $R_1$ and $R_2$ are local rings. If $R_1$ and $R_2$ are not fields, then $|Z(R_1R_2)|=|R_1||m_1|+|R_2||m_2| - 1$. If $|m_1|=p$, then $|R_1|=p^2$ [6, Lemma 4.8]. If $|m_1|=2$, then $|R_1|=4$, which implies that $|R_2|=6$. If $|m_2|=2$ and $|R_2|=4$ then $R_2 \cong Z_2xZ_2[X]/(X^2)$ or $Z_2xZ_3[X]/(X^2)$ or $Z_2xZ_3[X]/(X^2)$. If $|m_1|=3$, then $|R_1|=9$ which implies that $|R_2|=4$ or $|m_2|=2$, but $|m_2|=2$, therefore $R_2 \cong Z_2xZ_3$. If $|m_1|=4$, then $|R_1|=8$ so that $11=|Z(R)| \geq 47$ which is a contradiction. If $|m_1|=5$, then $R_2 \neq R_2$ is not a field, then $|R_2|=12$ or $|m_2|=3$. Let $|m_2|=2$, then $|R_2|=12$. Therefore, $|m_2|=3, |m_2|=9$ or $|m_2|=4$. $|R_2|=8$ and, hence $R \cong Z_2xZ_3, Z_2xZ_2[X]/(X^3), Z_2xZ_3, Z_2xZ_3[X]/(X^3), Z_2xZ_4[X]/(2X,X^2-2), Z_2xZ_4[X]/(X,Y)^3$ or $Z_2xZ_4[X]/(X^2,2X)$.

Let $|m_1|=3$, then $|m_2|=12-2|m_2|$, which is a contradiction. Let $|m_1|=4$, then $|m_2|=12-3|m_2|$, which is also a contradiction. Let $|m_1|=5$, then $|m_2|=12-4|m_2|$. Therefore, $|m_2|=2$ and $|R_2|=4$ so that $R \cong Z_2xZ_4$ or $Z_4xZ_3[X]/(X^2)$. Let $|m_1|=7$, then $|m_2|=12-6|m_2|$ and since $|m_2|=2$, then $|R_2|=0$ which is a contradiction. If $R_1$ and $R_2$ are fields, then applying Lemma 2.1 $|R_1R_2|=13$ and hence $|R_1|=2, |R_2|=11$ or $|R_1|=4, |R_2|=9$ or $|R_1|=5, |R_2|=8$. Therefore, $R \cong Z_2xZ_3, F_4xF_0$ or $Z_4xF_4$. ■

Now, we shall prove the main result of this section.
**Theorem 2.10:** Let \( R \) be a ring with \( |Z(R)^*| = 11 \), then, the graphs depicted in the following figures can be realized as \( \Gamma(R) \).

![Figure (1)](image1)
![Figure (2)](image2)
![Figure (3)](image3)
![Figure (4)](image4)

![Figure (5)](image5)
![Figure (6)](image6)
![Figure (7)](image7)
![Figure (8)](image8)

**Proof:** By Lemma 2.7: \( R \cong \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2[x]/(x^2) \), \( \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \), \( \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(2x, x^2-2) \), \( \mathbb{Z}_2 \times \mathbb{Z}_2[Z, Y]/(X, Y)^3 \), \( \mathbb{Z}_2 \times \mathbb{Z}_2[Z]/(x^2, 2x), \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_2[Z]/(x^3), \mathbb{Z}_5 \times \mathbb{Z}_2[X]/(2x, x^2-2) \), \( \mathbb{Z}_5 \times \mathbb{Z}_2[Z, Y]/(X, Y)^3 \). Figure (1), can be realized as \( \Gamma(\mathbb{Z}_4 \times \mathbb{Z}_4) \) or \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[Z]/(x^2)) \) or \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[Z]/(X, Y)^3) \). Figure (2), can be realized as \( \Gamma(Z_5 \times Z_5) \) or \( \Gamma(Z_5 \times Z_2[Z]/(x^2)) \). Figure (3), can be realized as \( \Gamma(Z_5 \times Z_2[Z]/(X, Y)^3) \). Figure (4), can be realized as \( \Gamma(Z_5 \times Z_2[Z]/(x^2)) \) or \( \Gamma(Z_5 \times Z_2[Z]/(X, Y)^3) \). Figure (5), can be realized \( \Gamma(Z_5 \times Z_2[Z]/(2x, x^2)) \) or \( \Gamma(Z_5 \times Z_2[Z]/(X, Y)^3) \). Figure (6), can be realized as \( \Gamma(Z_5 \times Z_1) \). Figure (7), can be realized as \( \Gamma(F_4 \times F_5) \) and Figure (8), can be realized as \( \Gamma(Z_5 \times F_1) \).

3. Rings with \( |Z(R)^*|=12 \)

The main aim of this section is to find all possible zero divisor graphs of 12 vertices and rings correspond to them.

We shall start this section with the following lemmas.

**Lemma 3.1:** Let \( R \) be a ring with \( |Z(R)^*|=12 \); if \( R \cong \mathbb{R}_1 \times \mathbb{R}_2 \times \ldots \times \mathbb{R}_n \), where \( R_i \) is a local ring for all \( i \geq 1 \), then \( n=3 \) if and only if \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4 \)

**Proof:** Let \( R \) be a ring with \( |Z(R)^*|=12 \) and let \( R \cong \mathbb{R}_1 \times \mathbb{R}_2 \times \mathbb{R}_3 \), where \( R_i \) is a local ring for all \( i=1,2,3 \). If \( R_i \) is not a field for some \( 1 \leq i \leq 3 \), then \( |Z(R)^*| \geq 13 \) which is a contradiction, so that \( R_i \) is a field for all \( 1 \leq i \leq 3 \), then by Corollary 2.3, \( |Z(R \times R_1, R_2, R_3)| = |R_1||R_2||R_3| + |R_1||R_3| + |R_2||R_3| + |R_1||R_2| = 12 \). If \( |R_1|=2 \), then \( |R_2| = 2 \), so that \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4 \). If \( |R_1|=2 \) and \( |R_2|=2 \), then \( |R_3|=3 \) which is a contradiction. If \( |R_1| \geq 3 \) and \( |R_2| \geq 3 \), then by Lemma 2.6(1) \( |Z(R)^*| \geq 13 \), which is a contradiction.

**Lemma 3.2:** Let \( R \) be a ring with \( |Z(R)^*|=12 \), if \( R \cong \mathbb{R}_1 \times \mathbb{R}_2 \times \ldots \times \mathbb{R}_n \), where \( R_i \) is a local ring for all \( i \geq 1 \), then \( n=2 \) if and only if \( R \cong \mathbb{Z}_3 \times \mathbb{Z}_1, \mathbb{Z}_3 \times \mathbb{F}_9 \) or \( \mathbb{Z}_7 \times \mathbb{Z}_7 \)

**Proof:** Let \( R \) be a ring with \( |Z(R)^*|=12 \) and let \( R \cong \mathbb{R}_1 \times \mathbb{R}_2 \times \mathbb{R}_3 \), where \( R_1 \) and \( R_2 \) are local rings. If \( R_1 \) and \( R_2 \) are not fields, then \( |Z(R \times R_2)^*| = |R_1^2||R_2| + |R_1||R_2|^2 + |R_2||R_3| = 12 \). If \( |R_1|=2 \), then \( |R_2|=2 \). So that, \( |R_2|=13/2 - |m_2| \). Since, \( |m_2| \) is an integer, then \( |R_2| \) is not an integer which is a contradiction.

If \( |m_2| \geq 3 \), then \( |R_1| \geq 8 \) and, since \( |m_2| \geq 2 \), \( |R_2| \geq 4 \), then \( 12 = |Z(R)^*| \geq 21 \) which is also a contradiction.

If \( |R_1| \) is a field and \( |R_2| \) is not a field, then \( |R_2| + |m_2||R_1||m_2|=12 \) which leads to a contradiction. If \( R_1 \) and \( R_2 \) are fields, then \( |R_1| + |R_2|=14 \) which implies that \( |R_1|=3, |R_2|=11 \) or \( |R_1|=5, |R_2|=9 \) or \( |R_1|=|R_2|=7 \). Therefore \( R \cong \mathbb{Z}_3 \times \mathbb{Z}_1, \mathbb{Z}_3 \times \mathbb{F}_9 \) or \( \mathbb{Z}_7 \times \mathbb{Z}_7 \).
Let $R$ be a ring with $|Z(R)^*|=12$. Then, $R \cong \mathbb{Z}_2xZ_2xX_4, Z_3xZ_{11}, Z_5xX_9, Z_7xZ_7, Z_{169}$ or $Z_{13}[X]/(X^2)$.

**Proof:** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i$ is a local ring. If $n \geq 4$, then by Lemma 2.7 $|Z(R)^*| \geq 14$. If $n=3$, then $R \cong \mathbb{Z}_2xZ_2xX_4$. If $n=2$, then $R \cong \mathbb{Z}_5xZ_{11}, \mathbb{Z}_5xX_9$ or $\mathbb{Z}_7xZ_7$. If $n=1$ and $R$ is a field, then $Z(R)=\{0\}$ which is a contradiction. If $R$ is a local ring, then $|Z(R)^*|=|m|-1=12$, so that $|m|=13$. Therefore, $|R|=169$, which implies that $R \cong \mathbb{Z}_{169}$ or $Z_{13}[X]/(X^2)$. ■

**Theorem 3.4:** Let $R$ be a ring with $|Z(R)^*|=12$, then the graphs depicted in the following figures can be realized as $\Gamma(R)$.

![Graphs](image)

**Proof:** By Lemma 3.3; $R \cong \mathbb{Z}_2xZ_2xX_4, Z_3xZ_{11}, Z_5xX_9, Z_7xZ_7, Z_{169}$ or $Z_{13}[X]/(X^2)$.

In Figure (1), can be realized as $\Gamma(\mathbb{Z}_2xZ_2xX_4)$. Figure (2), can be realized as $\Gamma(Z_3xZ_{11}).$ Figure (3), can be realized as $\Gamma(Z_5xX_9).$ Figure (4), can be realized as $\Gamma(Z_7xZ_7).$ Figure (5), can be realized as $\Gamma(Z_{169})$ or $\Gamma(Z_{13}[X]/(X^2))$. ■

**4. Rings with $|Z(R)^*|=13$**

The main aim of this section is to find all possible zero divisor graphs of 13 vertices and rings correspond to them.

We shall start this section with following lemma.

**Lemma 4.1:** Let $R$ be a ring with $|Z(R)^*|=13$, if $R \cong R_1 \times R_2 \times R_3$, where $R_i$ is a local ring for all $i \geq 1$, then $n=3$ if and only if $R \cong \mathbb{Z}_2xZ_3xZ_3, \mathbb{Z}_2xZ_2xX_4$ or $\mathbb{Z}_2xZ_2xZ_2[X]/(X^2)$.

**Proof:** Let $R$ be a ring with $|Z(R)^*|=13$ and let $R \cong R_1 \times R_2 \times R_3$, where $R_i$ local rings for all $1 \leq i \leq 3$. If $R_i$ is not a field, for some $1 \leq i_1, i_2 \leq 3$, then $|Z(R_1xR_2xR_3)^*| \geq 27$ which is a contradiction.

If $R_3$ is not a field and $R_1$ and $R_2$ are fields, then $|Z(R_1xR_2)|=|R_1|+|R_2|-1$ and $|Z(R_1xR_2xR_3)^*|=|R_1||R_2||m_3|+(|R_1|+|R_2|-1)(|R_3|-|m_3|)-1$, so that $|R_1||R_2||m_3|+(|R_1|+|R_2|-1)(|R_3|-|m_3|)=14$.

If $|R_1|=|R_2|=2$, then $|R_3|=14-\frac{|m_3|}{3}$ which implies that $|R_3|=4$ and $|m_3|=2$. Therefore, $R \cong \mathbb{Z}_2xZ_2xX_4$ or $\mathbb{Z}_2xZ_2xZ_2[X]/(X^2)$. If $|R_1| \geq 2$ and $|R_2| \geq 2$, and since $|R_3| \geq 4$ and $|m_3| \geq 2$, then $13=|Z(R_1xR_2xR_3)^*| \geq 2.3.2+(2+3-1)(4-2)-1=19$ which is a contradiction. If $R_i$ is a field for all $1 \leq i \leq 3$, then

$|R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3|=13$. If $|R_1|=|R_2|$, then $|R_3|=13/2$ which is a contradiction. If $|R_1|=2$, $|R_2|=3$, then $R \cong Z_{2xZ_3xZ_3}$. If $|R_1|=2$ and $|R_2|=4$, then $|R_3|=11/5$ which is a contradiction. If $|R_1|=2$ and $|R_2|=5$, then $|R_3|=5/3$ which is a contradiction. If $|R_1|=2$ and $|R_2| \geq 7$, then $|R_3| \leq 1$ which is a contradiction. If $|R_1| \geq 3$ and $|R_2| \geq 4$, then $|R_3| \leq 4/3$ which is a contradiction. ■

**Lemma 4.2:** Let $R$ be a ring with $|Z(R)^*|=13$, if $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i$ is a local ring for all $i \geq 1$, then $n=2$ if and only if $R \cong \mathbb{Z}_2xZ_3, \mathbb{F}_4xZ_21, \mathbb{F}_7xX_8$. 


Proof: Let $R$ be a ring with $|Z(R)|=13$ and let $R \cong R_1 \times R_2$, where $R_1$ and $R_2$ are local rings. If $R_1$ and $R_2$ are not fields, then

$$|Z(R_1 \times R_2)| = |R_1||m_2| + |R_2||m_1|-|m_1||m_2|=14.$$ If $|m_1|=2$, then $|R_1|=4$, so that $|R_2|=7-|m_2|$ which is a contradiction. If $|m_1||m_2| \geq 3$, then $|R_1||R_2| \geq 8$, so that $|Z(R_1 \times R_2)| \geq 3.8+8, 3.3-1=38$ which is a contradiction. If $R_1$ field and $R_2$ local not field, then $|R_1||m_2| + |R_2| - |m_1||m_2|=14$ which implies that $|R_2|=14-([R_1]-1)|m_2|$. If $|R_1|=2$, then $|R_2|=14-|m_2|$ which is a contradiction. If $|R_1|=4$, then $|R_2|=14-3|m_2|$ which is a contradiction. If $|m_1|=5$, then $|R_2|=14-4|m_2|$ which is a contradiction. If $|R_1| \geq 7$, then $|Z(R_1 \times R_2)| \geq 15$ which is a contradiction. Therefore, $R_1$ and $R_2$ are fields, which imply that $|R_1|+|R_2|=15$ and, hence $|R_1|=2$, $|R_2|=13$ or $|R_1|=4$, $|R_2|=11$ or $|R_1|=7$, $|R_2|=8$. Therefore, $R \cong Z_2 \times Z_{13}$, $F_4 \times Z_{11}$ or $Z_7 \times F_8$. ■

Lemma 4.3: Let $R$ be a ring with $|Z(R)|=13$, then $R \cong Z_2 \times Z_2 \times Z_4$, $Z_2 \times Z_2 \times Z_2[X]/(X^2)$, $R \cong Z_2 \times Z_{13}$, $F_4 \times Z_{11}$ or $Z_7 \times F_8$.

Proof: Let $R \cong R_1 \times R_2 \ldots \times R_n$, where $R_i$ is a local ring. If $n \geq 4$, then by Lemma 2.7 $|Z(R)| \geq 14$. If $n=3$, then $R \cong Z_2 \times Z_2 \times Z_4$ or $Z_2 \times Z_2 \times Z_2[X]/(X^2)$ Lemma 4.1. If $n=2$, then $R \cong Z_2 \times Z_{13}$, $F_4 \times Z_{11}$ or $Z_7 \times F_8$ Lemma 4.2. If $n=1$ and $R$ is a field, then $Z(R)={0}$ which is a contradiction. If $R$ is a local ring, then $|Z(R)|=m-1=13$, so that $|m|=14$ which is also a contradiction. ■

Theorem 4.4: Let $R$ be a ring with $|Z(R)|=13$, then the graphs depicted in the following figures can be realized as $\Gamma(R)$

- Figure (1)
- Figure (2)
- Figure (3)
- Figure (4)

Proof: By Lemma 4.3 $R \cong Z_2 \times Z_2 \times Z_4$, $Z_2 \times Z_2 \times Z_2[X]/(X^2)$, $Z_2 \times Z_{13}$, $F_4 \times Z_{11}$ or $Z_7 \times F_8$. Figure (1) can be realized as $\Gamma(Z_2 \times Z_2 \times Z_4)$ or $\Gamma(Z_2 \times Z_2 \times Z_2[X]/(X)^2)$. Figure (2) can be realized as $\Gamma(Z_2 \times Z_{13})$. Figure (3) can be realized as $\Gamma(F_4 \times Z_{11})$ and Figure (4) can be realized as $\Gamma(Z_7 \times F_8)$. 

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REFERENCES


