On $\alpha$ - Open Sets

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ABSTRACT

In this paper, we introduce a new class of open sets defined as follows: A subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open set, if there exists a non-empty subset $O$ of $X$, $O \in \alpha O(X)$, such that $A \subseteq \overline{C}(A \cap O)$. Also, we present the notion of $\alpha$-continuous mapping, $\alpha$-open mapping, $\alpha$-irresolute mapping, $\alpha$-totally continuous mapping, $\alpha$-contra-continuous mapping, $\alpha$-contra-continuous mapping, and we investigate some properties of these mappings. Furthermore, we introduce some $\alpha$-separation axioms and the mappings are related with $\alpha$-separation axioms.

Keywords: Open totals, Topological space, Types of mappings.

$\alpha$-open

1 Introduction and Preliminaries

A Generalization of the concept of open sets is now well-known important notions in topology and its applications. Levine [7] introduced semi-open set and semi-continuous function, Njastad [8] introduced $\alpha$-open set, Askander [15] introduced i-open set, i-irresolute mapping and i-homeomorphism, Biswas [6] introduced semi-open functions, Mashhour, Hasanein, and El-Deeb [1] introduced $\alpha$-continuous and $\alpha$-open mappings, Noiri [16] introduced totally (perfectly) continuous function, Crossley [11] introduced irresolute function, Maheshwari [14] introduced $\alpha$-irresolute mapping, Beceren [17] introduced semi $\alpha$-irresolute functions, Donchev [4] introduced contra continuous functions, Donchev and Noiri [5] introduced contra semi continuous functions, Jafari and Noiri [12] introduced Contra-$\alpha$-continuous functions, Ekici and Caldas [3] introduced clopen-T1, Staum [10] introduced, ultra hausdorff, ultra normal, clopen regular and clopen normal, Ellis [9] introduced ultra regular, Maheshwari [13] introduced s-normal space, Arhangael [2] introduced $\alpha$-normal space. The main aim of this paper is to introduce and study a new class of open sets which is called $\alpha$-open set and we present the notion of $\alpha$-continuous mapping, $\alpha$-totally continuity mapping and some weak separation axioms for $\alpha$-open sets. Furthermore, we investigate some properties of these mappings. In section 2, we define $\alpha$-open set, and we investigate the relationship with, open set, semi-open set, $\alpha$-open set and i-open set. In section 3, we present the notion of $\alpha$-continuous mapping, $\alpha$-open mapping, $\alpha$-irresolute mapping and $\alpha$-homeomorphism mapping, and we investigate the relationship between $\alpha$-continuous mapping with some types of continuous mappings, the relationship between
іα-open mapping, with some types of open mappings and the relationship between іα-
irresolute mapping with some types of irresolute mappings. Further, we compare іα-
homeomorphism with i-homeomorphism. In section 4, we introduce new class of
mappings called іα-totally continuous mapping and we introduce i-contra-continuous
mapping and іα-contr-continuous mapping. Further, we study some of their basic
properties. Finally in section 5, we introduce new weak of separation axioms for іα-
open set and we conclude іα-continuous mappings related with іα-separation axioms.
Throughout this paper, we denote the topology spaces \((X, \tau)\) and \((Y, \sigma)\) simply by \(X\) and
\(Y\) respectively. We recall the following definitions, notations and characterizations. The
closure (resp. interior) of a subset \(A\) of a topological space \(X\) is denoted by \(Cl(A)\) (resp.
\(Int(A)\)).

**Definition 1.1** A subset \(A\) of a topological space \(X\) is said to be
(i) semi-open set, if \(\exists O \in \tau\) such that \(O \subseteq A \subseteq Cl(O)\) [7]
(ii) \(\alpha\)-open set, if \(A \subseteq Int(Cl(Int(A)))\) [8]
(iii) i-open set, if \(A \subseteq Cl(A \cap O)\), where \(\exists O \in \tau\) and \(O \neq X, \phi\) [15]
(iv) clopen set, if \(A\) is open and closed.
The family of all semi-open (resp. \(\alpha\)-open, i-open, clopen) sets of a topological space is
denoted by \(SO(X)\) (resp. \(aO(X), iO(X), CO(X)\)). The complement of semi-open (resp. \(\alpha\)-
open, i-open) sets of a topological space \(X\) is called semi-closed (resp. \(\alpha\)-closed, i-
closed) sets.

**Definition 1.2** Let \(X\) and \(Y\) be a topological spaces, a mapping \(f : X \to Y\) is said to be
(i) semi-continuous [7] if the inverse image of every open subset of \(Y\) is semi-open set
in \(X\).
(ii) \(\alpha\)-continuous [1] if the inverse image of every open subset of \(Y\) is an \(\alpha\)-open set in \(X\).
(iii) i-continuous [15] if the inverse image of every open subset of \(Y\) is an i-open set in \(X\).
(iv) totally (perfectly) continuous [16] if the inverse image of every open subset of \(Y\) is
clopen set in \(X\).
(v) irresolute [11] if the inverse image of every semi-open subset of \(Y\) is semi-open
subset in \(X\).
(vi) \(\alpha\)-irresolute [14] if the inverse image of every \(\alpha\)-open subset of \(Y\) is an \(\alpha\)-open subset
in \(X\).
(vii) semi \(\alpha\)-irresolute [17] if the inverse image of every \(\alpha\)-open subset of \(Y\) is semi-
open subset in \(X\).
(viii) i-irresolute [15] if the inverse image of every i-open subset of \(Y\) is an i-open subset
in \(X\).
(ix) contra-continuous [4] if the inverse image of every open subset of \(Y\) is closed set in
\(X\).
(x) contra semi continuous [5] if the inverse image of every open subset of \(Y\) is semi-
closed set in \(X\).
(xi) contra \(\alpha\)-continuous [12] if the inverse image of every open subset of \(Y\) is an \(\alpha\)-
closed set in \(X\).
(xii) semi-open [6] if the image of every open set in \(X\) is semi-open set in \(Y\).
(xiii) \(\alpha\)-open [1] if the image of every open set in \(X\) is an \(\alpha\)-open set in \(Y\).
(xiv) i-open [15] if the image of every open set in \(X\) is an i-open set in \(Y\).

**Definition 1.3** Let \(X\) and \(Y\) be a topology space, a bijective mapping \(f : X \to Y\) is said
to be i-homeomorphism [15] if \(f\) is an i-continuous and i-open.

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Lemma 1.4 Every open set in a topological space is an i-open set [15].

Lemma 1.6 Every semi-open set in a topological space is an i-open set [15].

Lemma 1.8 Every α-open set in a topological space is an i-open set [15].

2 Sets That are iα-Open Sets and Some Relations With Other Important Sets

In this section, we introduce a new class of open sets which is called iα-open set and we investigate the relationship with open set, semi-open set, α-open set and i-open set.

Definition 2.1 A subset \( A \) of the topological space \( X \) is said to be iα-open set if there exists a non-empty subset \( O \) of \( X \), such that \( A \subseteq \text{Cl}(A \cap O) \). The complement of the iα-open set is called iα-closed. We denote the family of all iα-open sets of a topological space \( X \) by \( \text{iαO}(X) \).

Example 2.2 Let \( X=\{a,b,c\}, \tau=\{\emptyset,\{b\},\{b,c\},X\} \), \( SO(X)=\text{aO}(X)=\{\emptyset,\{b\},\{a,b\},\{b,c\},X\} \), and \( i\alpha O(X)=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},X\} \). Note that \( SO(X)=\text{aO}(X)\subseteq i\alpha O(X) \).

Example 2.3 Let \( X=\{a,b,c,d\}, \tau=\{\emptyset,\{a,d\},\{b,c\},X\} \), \( i\alpha O(X)\).

Example 2.4 Let \( X=\{a,b,c\}, \tau=\{\emptyset,\{a\},\{b\},\{a,b\},X\} \), \( i\alpha O(X)\).

Lemma 2.5 Every i-open set in any topological space is an iα-open set.

Proof. Let \( X \) be any topological space and \( A \subseteq X \) be any i-open set. Therefore, \( A \subseteq \text{Cl}(A \cap O) \), where \( \exists O \in \tau \) and \( O \neq X, \emptyset \). Then, \( \exists O \in \text{aO}(X) \). We obtain \( A \subseteq \text{Cl}(A \cap O) \), where \( \exists O \in \text{aO}(X) \) and \( O \neq X, \emptyset \). Thus, \( A \) is an iα-open set.

The following example shows that iα-open set need not be i-open set.

Example 2.6 Let \( X=\{1,2,3,4\}, \tau=\{\emptyset,\{4\},X\} \), \( i\alpha O(X)=\text{aO}(X)=\{\emptyset,\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},X\} \subseteq i\alpha O(X)(\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\},X,X) \).

Remark 2.7
(i) The intersection of iα-open sets is not necessary to be iα-open set as shown in the example 2.4.
(ii) The union of iα-open set is not necessary to be iα-open set as shown in the example 2.3.

3 Mappings That are iα-Continuous and iα-Homeomorphism

In this section, we present the notion of iα-continuous mapping, iα-irresolute mapping and iα-homeomorphism mapping.

Definition 3.1 Let \( X, Y \) be a topological spaces, a mapping \( f : X \rightarrow Y \) is said to be iα-continuous, if the inverse image of every open subset of \( Y \) is an iα-open set in \( X \).

Example 3.2 Let \( X=\{a,b,c\}, \tau=\{\emptyset,\{b\},\{c\},\{b,c\},X\} \), \( i\alpha O(X)=\{\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\} \) and \( \sigma=\{\emptyset,\{a\},\{b\},X\} \). Clearly, the identity mapping \( f : X \rightarrow Y \) is an iα-continuous.

Proposition 3.3 Every i-continuous mapping is an iα-continuous.
Proof. Let \( f : X \to Y \) be an i-continuous mapping and \( V \) be any open subset in \( Y \). Since, \( f \) is an i-continuous, then \( f^{-1}(V) \) is an i-open set in \( X \). Since, every i-open set is an iα-open set by lemma 2.5, then \( f^{-1}(V) \) is an iα-open set in \( X \). Therefore, \( f \) is an iα-continuous 

Remark 3.4 The following example shows that iα-continuous mapping need not be continuous, semi-continuous, \( \alpha \)-continuous and i-continuous mappings.

Example 3.5 Let \( X=\{a,b,c\} \) and \( Y=\{1,2,3\} \), \( \tau=\{\emptyset,\{b\},X\} \), \( SO(X)=\alpha O(X)=iO(X)=\{\emptyset,\{b\},\{a,b\},\{b,c\},X\} \), \( iO(Y)=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},X\} \). A mapping \( f : X \to Y \) is defined by \( f(a)=\{2\} \), \( f(b)=\{1\} \), \( f(c)=\{3\} \). Clearly, \( f \) is an iα-continuous, but \( f \) is not continuous, \( f \) is not semi-continuous, \( f \) is not \( \alpha \)-continuous and \( f \) is not i-continuous because for open subset \( \{2\} \), \( f^{-1}\{2\} = \{a\} \notin \tau \) and \( f^{-1}\{2\} = \{a\} \notin SO(X)=\alpha O(X)=iO(X) \).

Definition 3.6 Let \( X \) and \( Y \) be a topological space, a mapping \( f : X \to Y \) is said to be iα-open, if the image of every i-open set in \( X \) is an iα-open set in \( Y \).

Example 3.7 Let \( X=\{a,b,c\} \), \( \tau=\{\emptyset,\{b\},X\} \), \( \sigma=\{\emptyset,\{a\},Y\} \), and \( iO(Y)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\} \). Clearly, the identity mapping \( f : X \to Y \) is an iα-open.

Proposition 3.8 Every i-open mapping is an iα-open.

Proof. Let \( f : X \to Y \) be an i-open mapping and \( V \) be any open set in \( X \). Since, \( f \) is an i-open, then \( f(V) \) is an i-open set in \( Y \). Since, every i-open set is an iα-open set by lemma 2.5, then \( f(V) \) is an iα-open set in \( Y \). Therefore, \( f \) is an iα-open.

Remark 3.9 The following example shows that iα-open mapping need not be open, semi-open, \( \alpha \)-open and i-open mappings.

Example3.10 Let \( X=\{1,2,3\} \), \( \tau=\{\emptyset,\{3\},X\} \), \( \sigma=\{\emptyset,\{1\},Y\} \), \( SO(Y)=\alpha O(Y)=iO(Y)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},Y\} \). A mapping \( f : X \to Y \) is defined by \( f(1)=2 \), \( f(2)=1 \), \( f(3)=3 \). Clearly, \( f \) is an i-open, but \( f \) is not open, \( f \) is not semi-open, \( f \) is not \( \alpha \)-open and \( f \) is not i-open because for open subset \( \{3\} \), \( f^{-1}\{3\} = \{3\} \notin \sigma \) and \( f^{-1}\{3\} = \{3\} \notin SO(Y)=\alpha O(Y)=iO(Y) \).

Definition 3.11 Let \( X \) and \( Y \) be a topological space, a mapping \( f : X \to Y \) is said to be iα-irresolute, if the inverse image of every iα-open subset of \( Y \) is an iα-open subset in \( X \).

Example 3.12 Let \( X=\{a,b,c\} \), \( \tau=\{\emptyset,\{b\},X\} \), \( iO(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\} \), \( \sigma=\{\emptyset,\{c\},Y\} \) and \( iO(Y)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\} \). Clearly, the identity mapping \( f : X \to Y \) is an iα-irresolute.

Proposition 3.13 Every i-irresolute mapping is an iα-irresolute.

Proof. Let \( f : X \to Y \) be an i-irresolute mapping and \( V \) be any iα-open set in \( Y \). Since, \( f \) is an i-irresolute, then \( f^{-1}(V) \) is an i-open set in \( X \). Hence, iα-open set in \( X \) by lemma 2.5. Therefore, \( f \) is an iα-irresolute.

Remark 3.14 The following example shows that iα-irresolute mapping need not be irresolute, semi \( \alpha \)-irresolute, \( \alpha \)-irresolute and i-irresolute mappings.

Example 3.15 Let \( X=\{a,b,c\} \), \( \tau=\{\emptyset,\{a\},X\} \), \( SO(X)=\alpha O(X)=iO(X)=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},X\} \), \( \sigma=\{\emptyset,\{c\},Y\} \), \( SO(Y)=\alpha O(Y)=iO(Y)=\{\emptyset,\{c\},\{a,c\},\{b,c\},Y\} \) and \( iO(Y)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\} \). Clearly,
the identity mapping \( f : X \to Y \) is an \( \alpha \)-irresolute, but \( f \) is not irresolute, \( f \) is not \( \alpha \)-irresolute, \( f \) is not semi \( \alpha \)-irresolute and \( f \) is not i-irresolute because for semi-open, \( \alpha \)-open and i-open subset \{c\}, \( f^{-1}(c) = \{c\} \notin SO(X) = aO(X) = iO(X) \).

**Proposition 3.16** Every \( \alpha \)-irresolute mapping is an \( \alpha \)-continuous.

**Proof.** Let \( f : X \to Y \) be an \( \alpha \)-irresolute mapping and \( V \) be any open set in \( Y \). Since, every open set is an \( \alpha \)-open set. Since, \( f \) is an \( \alpha \)-irresolute, then \( f^{-1}(V) \) is an \( \alpha \)-open set in \( X \). Therefore \( f \) is an \( \alpha \)-continuous. The converse of the above proposition need not be true as shown in the following example.

**Example 4.17** Let \( X = Y = \{a,b,c\} \), \( \tau = \{\emptyset, \{a\}, X\} \), \( i\alpha O(X) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\} \), \( \sigma = \{\emptyset, \{a,c\}, Y\} \) and \( i\alpha O(Y) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, Y\} \). Clearly, the identity mapping \( f : X \to Y \) is an \( \alpha \)-continuous, but \( f \) is not \( \alpha \)-irresolute because for \( \alpha \)-open set \{c\}, \( f^{-1}(c) = \{c\} \notin i\alpha O(X) \).

**Definition 3.18** Let \( X \) and \( Y \) be a topological space, a bijective mapping \( f : X \to Y \) is said to be \( \alpha \)-homeomorphism if \( f \) is an \( \alpha \)-continuous and \( \alpha \)-open.

**Theorem 3.19** If \( f : X \to Y \) is an \( \alpha \)-homomorphism, then \( f : X \to Y \) is an \( \alpha \)-homomorphism.

**Proof.** Since, every i-continuous mapping is an \( \alpha \)-continuous by proposition 3.3. Also, since every i-open mapping is an \( \alpha \)-open 3.8. Further, since \( f \) is bijective. Therefore, \( f \) is an \( \alpha \)-homomorphism. The converse of the above theorem need not be true as shown in the following example.

**Example 3.20** Let \( X = Y = \{a,b,c\} \), \( \tau = \{\emptyset, \{a\}, X\} \), \( i\alpha O(X) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\} \), \( \sigma = \{\emptyset, \{b\}, Y\} \), \( i\alpha O(Y) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, Y\} \) and \( i\alpha O(Y) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, Y\} \). Clearly, the identity mapping \( f : X \to Y \) is an \( \alpha \)-homomorphism, but it is not i-homomorphism because \( f \) is not i-continuous, since for open subset \{b\}, \( f^{-1}(b) = \{b\} \notin iO(X) \).

## 4 Mappings That are \( \alpha \)-Totally Continuous and \( \alpha \)-Contra-Continuous

In this section, we introduce new classes of mappings called \( \alpha \)-totally continuous, i-contra-continuous and \( \alpha \)-contra-continuous.

**Definition 4.1** Let \( X \) and \( Y \) be a topological space, a mapping \( f : X \to Y \) is said to be \( \alpha \)-totally continuous, if the inverse image of every \( \alpha \)-open subset of \( Y \) is clopen set in \( X \).

**Example 4.2** Let \( X = Y = \{a,b,c\} \), \( \tau = \{\emptyset, \{a\}, \{b,c\}, X\} \), \( \sigma = \{\emptyset, \{a\}, Y\} \) and \( i\alpha O(Y) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, Y\} \). The mapping \( f : X \to Y \) is defined by \( f(a) = \{a\}, f(b) = f(c) = b \). Clearly, \( f \) is an \( \alpha \)-totally continuous mapping.

**Theorem 4.3** Every \( \alpha \)-totally continuous mapping is totally continuous.

**Proof.** Let \( f : X \to Y \) be \( \alpha \)-totally continuous and \( V \) be any open set in \( Y \). Since, every open set is an \( \alpha \)-open set, then \( V \) is an \( \alpha \)-open set in \( Y \). Therefore, \( f \) is totally continuous. The converse of the above theorem need not be true as shown in the following example.

**Example 4.4** Let \( X = Y = \{a,b,c\} \), \( \tau = \{\emptyset, \{a\}, \{b,c\}, X\} \), \( \sigma = \{\emptyset, \{a\}, Y\} \) and \( i\alpha O(Y) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, Y\} \). Clearly, the identity mapping is totally \( f : X \to Y \).
continuous, but \( f \) is not \( \alpha \)-totally continuous because for \( \alpha \)-open set \( \{a,c\} \), 
\[ f^{-1}\{a,c\}=\{a,c\} \notin CO(X) \).

**Theorem 4.5** Every \( \alpha \)-totally continuous mapping is an \( \alpha \)-irresolute.

**Proof.** Let \( f: X \rightarrow Y \) be \( \alpha \)-totally continuous and \( V \) be an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous mapping, then \( f^{-1}(V) \) is clopen set in \( X \), which implies \( f^{-1}(V) \) open, it follow \( f^{-1}(V) \) \( \alpha \)-open set in \( X \). Therefore, \( f \) is an \( \alpha \)-irresolute. The converse of the above theorem need not be true as shown in the following example.

**Example 4.6** Let \( X=\{1,2,3\} \), \( \tau=\{\emptyset,\{2\},X\} \), \( \text{ia}O(X)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\} \) \( \sigma=\{\emptyset,\{1,2\},Y\} \) and \( \text{ia}O(Y)=\{\emptyset,\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\},Y\} \). Clearly, the identity mapping \( f: X \rightarrow Y \) is an \( \alpha \)-irresolute, but \( f \) is not \( \alpha \)-totally continuous because for \( \alpha \)-open subset \( \{1,3\} \), \( f^{-1}\{1,3\}=\{1,3\} \notin CO(X) \).

**Theorem 4.7** The composition of two \( \alpha \)-totally continuous mapping is also \( \alpha \)-totally continuous.

**Proof.** Let \( f: X \rightarrow Y \) be any \( \alpha \) Vtotally continuous. Let \( -o \) be any tw \( g: Y \rightarrow Z \) and \( f: X \rightarrow Y \) open in \( Z \). Since, \( g \) is an \( \alpha \)-totally continuous, then \( g^{-1}(V) \) is clopen set in \( Y \), which implies \( g^{-1}(V) \) open, it follow \( g^{-1}(V) \) \( \alpha \)-open set. Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is clopen in \( X \). Therefore, \( g \circ f: X \rightarrow Z \) is an \( \alpha \)-totally continuous.

**Theorem 4.8** If \( f: X \rightarrow Y \) be an \( \alpha \)-totally continuous and be an \( \alpha \) \( g: Y \rightarrow Z \) irresolute, then \( g \circ f: X \rightarrow Z \) is an \( \alpha \)-totally continuous.

**Proof.** Let \( f: X \rightarrow Y \) be \( \alpha \)-totally continuous and \( g: Y \rightarrow Z \) be \( \alpha \)-irresolute. Let \( V \) be \( \alpha \)-open set in \( Z \). Since, \( g \) is an \( \alpha \)-irresolute, then \( g^{-1}(V) \) is an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is clopen in \( X \). Therefore, \( g \circ f: X \rightarrow Z \) is an \( \alpha \)-totally continuous.

**Theorem 4.9** If \( f: X \rightarrow Y \) is an \( \alpha \)-totally continuous and is an \( \alpha \) \( g: Y \rightarrow Z \) continuous, then \( g \circ f: X \rightarrow Z \) is totally continuous.

**Proof.** Let \( f: X \rightarrow Y \) be continuous. Let \( g: Y \rightarrow Z \) totally and \( \beta \)-be \( \alpha \) \( f: X \rightarrow Y \) \( V \) be an open set in \( Z \). Since, \( g \) is an \( \alpha \)-continuous, then \( g^{-1}(V) \) is an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is clopen in \( X \). Therefore, \( g \circ f: X \rightarrow Z \) is totally continuous.

**Definition 4.10** Let \( X, Y \) be a topological spaces, a mapping \( f: X \rightarrow Y \) is said to be \( \alpha \)-contra-continuous (resp. i-contra-continuous), if the inverse image of every open subset of \( Y \) is an \( \alpha \)-closed (resp. i-closed) set in \( X \).

**Example 4.11** Let \( X=\{a,b,c\}, \tau=\{\emptyset,\{a\},X\} \), \( \sigma=\{\emptyset,\{c\},Y\} \) and \( \text{ia}O(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\} \). Clearly, the identity mapping \( f: X \rightarrow Y \) is an i-contra-continuous and \( \alpha \)-contra-continuous.

**Proposition 4.12** Every contra-continuous mapping is an i-contra-continuous.

**Proof.** Let \( f: X \rightarrow Y \) be contra continuous mapping and \( V \) any open set in \( Y \). Since, \( f \) is contra continuous, then \( f^{-1}(V) \) is closed sets in \( X \). Since, every closed set is an i-closed set, then \( f^{-1}(V) \) is an i-closed set in \( X \). Therefore, \( f \) is an i-contra-continuous. Similarly we have the following results.
Proposition 4.13 Every contra semi-continuous mapping is an i-contra-continuous.

Proof. Clear since every semi-open set is an i-open set.

Proposition 4.14 Every contra \( \alpha \)-continuous mapping is an i-contra-continuous.

Proof. Clear since every \( \alpha \)-open set is an i-open set.

The converse of the propositions 4.12, 4.13 and 4.14 need not be true in general as shown in the following example

Example 4.15 Let \( X=Y=\{a,b,c\} \), \( \tau=\{\emptyset,\{a\},X\} \), \( \sigma=\{\emptyset,\{a\},\{c\}\} \). Clearly, the identity mapping \( f:X\rightarrow Y \) is an i-contra continuous, but \( f \) is not contra-continuous, \( f \) is not contra semi-continuous, \( f \) is not contra \( \alpha \)-continuous because for open subset \( f^{-1}\{a\}=\{c\} \) is not closed in \( X \), \( f^{-1}\{c\}=\{a\} \) is not semi-closed in \( X \) and \( f^{-1}\{c\}=\{a\} \) is not \( \alpha \)-closed in \( X \).

Proposition 4.16 Every i-contra-continuous mapping is an i\( \alpha \)-contra-continuous.

Proof. Let \( f:X\rightarrow Y \) be an i-contra-continuous mapping and \( V \) any open set in \( Y \). Since, \( f \) is an i-contra continuous, then \( f^{-1}(V) \) is an i-closed sets in \( X \). Since, every i-closed set is an i\( \alpha \)-closed, then \( f^{-1}(V) \) is an i\( \alpha \)-closed set in \( X \). Therefore, \( f \) is an i\( \alpha \)-contra-continuous.

Remark 4.17 The following example shows that i\( \alpha \)-contra-continuous mapping need not be contra-continuous, contra semi-continuous, contra \( \alpha \)-continuous and i-contra-continuous mappings.

Example 4.18 Let \( X=Y=\{a,b,c\} \), \( \tau=\{\emptyset,\{a\},X\} \), \( \sigma=\{\emptyset,\{a\},\{c\}\} \). A mapping continuous, -contra-is an i\( \alpha \) \( f \) \( (c) = a \). Clearly, \( f(b) = b, f(a) = c, f \) is defined by \( f : X \rightarrow Y \) but \( f \) is not contra-continuous, \( f \) is not contra semi continuous, \( f \) is not contra \( \alpha \)-continuous and \( f \) is not i-contra-continuous because for open subset \( \{c\}, f^{-1}\{c\}=\{a\} \) is not closed, \( f^{-1}\{c\}=\{a\} \) is not semi-closed, \( f^{-1}\{c\}=\{a\} \) is not \( \alpha \)-closed and \( f^{-1}\{c\}=\{a\} \) is not i-closed in \( X \).

Theorem 4.19 Every totally continuous mapping is an i\( \alpha \)-contra-continuous.

Proof. Let \( f:X\rightarrow Y \) be totally continuous and \( V \) be any open set in \( Y \). Since, \( f \) is totally continuous mapping, then \( f^{-1}(V) \) is clopen set in \( X \), and hence closed, it follows i\( \alpha \)-closed. Therefore, \( f \) is an i\( \alpha \)-contra-continuous.

The converse of the above theorem need not be true as shown in the following example

Example 4.20 Let \( X=Y=\{a,b,c\} \), \( \tau=\{\emptyset,\{c\},X\} \), \( \sigma=\{\emptyset,\{a\},Y\} \) and i\( \alpha \)O(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}. Clearly, the identity mapping -contra-is an i\( \alpha \) \( f : X \rightarrow Y \) continuous, but \( f \) is not totally continuous because for open subset \( f^{-1}\{a\}=\{a\} \notin CO(X) \).

5 Separation Axioms with i\( \alpha \)-open Set

In this section, we introduce some new weak of separation axioms with i\( \alpha \)-open sets.

Definition 5.1 A topological space \( X \) is said to be
(i) i\( \alpha \)-T\(_0\) if for each pair distinct points of \( X \), there exists i\( \alpha \)-open set containing one point but not the other.
(ii) i\( \alpha \)-T\(_1\) (resp. clopen -T\(_1\) [3]) if for each pair of distinct points of \( X \), there exists two i\( \alpha \)-open (resp. clopen) sets containing one point but not the other .
(iii)i\( \alpha \)-T\(_2\) (resp. ultra hausdorff (U\( \alpha \)T\(_2\)) [10]) if for each pair of distinct points of \( X \) can be separated by disjoint i\( \alpha \)-open (resp. clopen) sets.
(iv) $\alpha$-regular (resp. ultra regular [9]) if for each closed set $F$ not containing a point in $X$ can be separated by disjoint $\alpha$-open (resp. clopen) sets.

(v) clopen regular [10] if for each clopen set $F$ not containing a point in $X$ can be separated by disjoint open sets.

(vi) $\alpha$-normal (resp. ultra normal [10], s-normal [13], $\alpha$-normal [2]) if for each of non-empty disjoint closed sets in $X$ can be separated by disjoint $\alpha$-open (resp. clopen, semi-open, $\alpha$-open) sets.

(vii) clopen normal [10] if for each of non-empty disjoint clopen sets in $X$ can be separated by disjoint open sets.

(viii) $\alpha$-T$_{1/2}$ if every $\alpha$-closed is $\alpha$-closed in $X$.

**Remark 5.2** The following example shows that $\alpha$-normal need not be normal, s-normal, $\alpha$-normal spaces.

**Example 5.3** Let $X = \{1, 2, 3, 4, 5\}$, $\tau = \{\emptyset, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, X\}$ and $\alpha O(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 3, 4, 5\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}, X\}$. Clearly, the space $X$ is $\alpha$-T$_0$, $\alpha$-T$_1$, $\alpha$-T$_2$, $\alpha$-regular, $\alpha$-normal and $\alpha$-T$_{1/2}$, but $X$ is not normal, s-normal and $\alpha$-normal.

**Theorem 5.4** If a mapping continuous mapping and the $-\alpha$-contra is an $\alpha$ $f : X \to Y$ space $X$ is an $\alpha$-T$_{1/2}$, then $f$ is an $\alpha$-contra-continuous.

**Proof.** Let $Y$ is any open set in $X$ continuous mapping and $-\alpha$-racont $\alpha$ $f : X \to Y$. Since, $f$ is an $\alpha$-contra-continuous mapping, then $f^{-1}(V)$ is an $\alpha$-closed in $X$. Since, $X$ is an $\alpha$-T$_{1/2}$, then $f^{-1}(V)$ is $\alpha$-closed in $X$. Therefore, $f$ is an $\alpha$-contra-continuous.

**Theorem 5.5** If $f : X \to Y$ is an $\alpha$-totally continuous injection mapping and $Y$ is an $\alpha$-T$_1$, then $X$ is clopen-$T_1$.

**Proof.** Let $x$ and $y$ be any two distinct points in $X$. Since, $f$ is an injective, we have $f(x) \neq f(y)$. Since, $Y$ is an $\alpha$-T$_1$, there exists $\alpha$-open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \notin U$ and $f(y) \in V$, $f(x) \notin V$. Therefore, we have $x \notin f^{-1}(U)$, $y \notin f^{-1}(U)$ and $x \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subsets of $X$ because $f$ is an $\alpha$-totally continuous. This shows that $X$ is clopen-$T_1$.

**Theorem 5.6** If $f : X \to Y$ is an $\alpha$-totally continuous injection mapping and $Y$ is an $\alpha$-T$_0$, then $X$ is ultra-Hausdorff (U$T_2$).

**Proof.** Let $a$ and $b$ be any pair of distinct points of $X$ and $f$ be an injective, then $f(a) \neq f(b)$ in $Y$. Since $Y$ is an $\alpha$-T$_0$, there exists $\alpha$-open set $U$ containing $f(a)$ but not $f(b)$, then we have $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$. Since, $f$ is an $\alpha$-totally continuous, then $f^{-1}(U)$ is clopen in $X$. Also $a \in f^{-1}(U)$ and $b \in X-f^{-1}(U)$. This implies every pair of distinct points of $X$ can be separated by disjoint clopen sets in $X$. Therefore, $X$ is ultra-Hausdorff.

**Theorem 5.7** Let $f : X \to Y$ be a closed $\alpha$-continuous injection mapping. If $Y$ is an $\alpha$-normal, then $X$ is an $\alpha$-normal.

**Proof.** Let $F_1$ and $F_2$ be disjoint closed subsets of $X$. Since, $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since, $Y$ is an $\alpha$-normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint $\alpha$-open sets $V_1$ and $V_2$ respectively. Therefore, we obtain, $F_1 \subset f^{-1}(V_1)$ and $F_2 \subset f^{-1}(V_2)$. Since, $f$ is an $\alpha$-continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are $\alpha$-open sets in $X$. Also, $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \emptyset$. Thus, for each pair of non-
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empty disjoint closed sets in $X$ can be separated by disjoint $\alpha$-open sets. Therefore, $X$ is an $\alpha$–normal.

**Theorem 5.8** If $f : X \rightarrow Y$ is an $\alpha$-totally continuous closed injection mapping and $Y$ is an $\alpha$–normal, then $X$ is ultra-normal.

**Proof.** Let $F_1$ and $F_2$ be disjoint closed subsets of $X$. Since, $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since, $Y$ is an $\alpha$–normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint $\alpha$-open sets $V_1$ and $V_2$ respectively. Therefore, we obtain, $F_1 \subseteq f^{-1}(V_1)$ and $F_2 \subseteq f^{-1}(V_2)$. Since, $f$ is an $\alpha$-totally continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets in $X$. Also, $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \emptyset$. Thus, for each pair of non-empty disjoint closed sets in $X$ can be separated by disjoint clopen sets in $X$. Therefore, $X$ is ultra-normal.

**Theorem 5.9** Let $f : X \rightarrow Y$ be a totally continuous closed injection mapping, if $Y$ is an $\alpha$–regular, then $X$ is ultra-regular.

**Proof.** Let $F$ be a closed set not containing $x$. Since, $f$ is closed, we have $f(F)$ is a closed set in $Y$ not containing $f(x)$. Since, $Y$ is an $\alpha$–regular, there exists disjoint $\alpha$-open sets $A$ and $B$ such that $f(x) \in A$ and $f(F) \subseteq B$, which imply $x \in f^{-1}(A)$ and $F \subseteq f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are clopen sets in $X$ because $f$ is totally continuous. Moreover, since $f$ is an injective, we have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}((\emptyset)) = \emptyset$. Thus, for a pair of a point and a closed set not containing a point in $X$ can be separated by disjoint clopen sets. Therefore, $X$ is ultra-regular.

**Theorem 5.10** If open mapping from a -is totally continuous injective $\alpha$ $f : X \rightarrow Y$ clopen regular space $X$ into a space $Y$, then $Y$ is an $\alpha$–regular.

**Proof.** Let $F$ be a closed set in $Y$ and $y \notin F$. Take $y = f(x)$. Since, $f$ is totally continuous, $f^{-1}(F)$ is clopen in $X$. Let $G = f^{-1}(F)$, then we have $x \notin G$. Since, $X$ is clopen regular, there exists disjoint open sets $U$ and $V$ such that $G \subseteq U$ and $x \in V$. This implies $F = f(G) \subseteq f(U)$ and $y = f(x) \in V$. Further, since $f$ is an injective and $\alpha$-open, we have $f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset$, $f(U)$ and $f(V)$ are $\alpha$-open sets in $Y$. Thus, for each closed set $F$ in $Y$ and each $y \in F$, there exists disjoint $\alpha$-open sets $f(U)$ and $f(V)$ in $Y$ such that $F \subseteq f(U)$ and $y \in f(V)$. Therefore, $Y$ is an $\alpha$–regular.

**Theorem 5.11** If $f : X \rightarrow Y$ is a totally continuous injective and $\alpha$-open mapping from clopen normal space $X$ into a space $Y$, then $Y$ is an $\alpha$–normal.

**Proof.** Let $F_1$ and $F_2$ be any two disjoint closed sets in $Y$. Since, $f$ is totally continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are clopen subsets of $X$. Take $U = f^{-1}(F_1)$ and $V = f^{-1}(F_2)$. Since, $f$ is an injective, we have $U \cap V = f^{-1}(F_1 \cap F_2) = f^{-1}(F_1 \cap F_2) = f^{-1}(\emptyset) = \emptyset$. Since, $X$ is clopen normal, there exists disjoint open sets $A$ and $B$ such that $U \subseteq A$ and $V \subseteq B$. This implies $F_1 = f(U) \subseteq f(A)$ and $F_2 = f(V) \subseteq f(B)$. Further, since $f$ is an injective $\alpha$-open, then $f(A)$ and $f(B)$ are disjoint $\alpha$-open sets. Thus, each pair of disjoint closed sets in $Y$ can be separated by disjoint $\alpha$-open sets. Therefore, $Y$ is an $\alpha$–normal.
REFERENCES


